Advanced Engineering Mathematics
Lecture 17

1 Integral Calculus

Example 1.1. Show that $\iint_D \sqrt{4a^2 - x^2 - y^2}\, dx\, dy = \frac{4}{9}(3\pi - 4)a^3$, taken over the upper half of the circle $x^2 + y^2 = 2ax$.

Sol. The given circle is $x^2 + y^2 = 2ax \Rightarrow (x - a)^2 + y^2 = a^2$. Center $= (a, 0)$ and radius$=a$. Putting $x = r\cos\theta$ and $y = r\sin\theta$ in $x^2 + y^2 = 2ax$ then,

$$r^2\cos^2\theta + r^2\sin^2\theta = 2ar\cos\theta \Rightarrow r^2 = 2ar\cos\theta \Rightarrow r = 2a\cos\theta$$

Thus for the given upper half of the circle our θ varies from 0 to π and r varies from 0 to $2a\cos\theta$. Then

$$\begin{aligned} I &= \iint_D \sqrt{4a^2 - x^2 - y^2}\, dx\, dy \\ &= \int_0^{\pi}\int_0^{2a\cos\theta} \sqrt{4a^2 - r^2}\, r\, dr\, d\theta \qquad \text{(Substitute } 4r^2 - a^2 = t \implies -2rdr = dt) \\ &= \int_0^{\pi}\Big[\int_{4a^2}^{4a^2\sin^2\theta} \sqrt{t}\,\frac{dt}{-2}\Big]\, d\theta \\ &= \frac{4}{9}(3\pi - 4)a^3 \end{aligned}$$

Example 1.2. Using the transformation $x+y = u$ and $y = uv$, find the value of $\int_0^1 dx \int_0^{1-x} e^{\frac{y}{x+y}}\, dy$.

Sol. $x + y = u, y = uv$, Now,

$$\begin{aligned} y = 1 - x &\Rightarrow x + y = 1 \Rightarrow u = 1, \\ y = 0 &\Rightarrow uv = 0 \Rightarrow u = 0, v = 0 \end{aligned}$$

Again, $x = 0 \Rightarrow y = u \Rightarrow u = uv \Rightarrow v = 1$ and $x = u - uv, y = uv$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 1 - v & -u \\ v & u \end{vmatrix} = u - uv + uv = u$$

$$\begin{aligned} I = \int_0^1\int_0^1 e^{\frac{uv}{u}}\, |J|\, du\, dv &= \int_0^1\int_0^1 e^v u\, du\, dv \\ &= \int_0^1 u\, du \int_0^1 e^v\, dv = \frac{e-1}{2} \end{aligned}$$

Integartion in $\mathbb{R}^3$. Consider a curve C in $\mathbb{R}^3 : x = f(t), y = g(t), z = h(t), a \le t \le b$. Let P, Q, R be three functions of x, y, z defined on domain $D \subset \mathbb{R}^3$ containing C. Let f, g, h posses continuous derivatives on (a, b). Also suppose P, Q, R are continuous on D. Then the line integral $I = \int_C (P\, dx + Q\, dy + R\, dz)$ exists and $I = \int_C (P\, dx + Q\, dy + R\, dz) = \int_{t=a}^{b} (Pf' + Qg' + Rh')\, dt$

Alternatively $\vec{F} = (P, Q, R)$ and $\vec{r} = (x, y, z) \Rightarrow d\vec{r} = (dx, dy, dz)$ then line integral $I = \vec{F}\cdot d\vec{r} = \int_{t=a}^{b} \Big(\vec{F}\cdot\frac{d\vec{r}}{dt}\Big)dt$

Example 1.3. Evaluate $I = \int_C (x^2y^3\,dx + dy + z\,dz)$ where C is the circle $x^2+y^2 = R^2, z = 0$.

Sol. Putting $x = R\cos t, y = R\sin t, z = 0 \Rightarrow \frac{dx}{dt} = -R\sin t, \frac{dy}{dt} = R\cos t, \frac{dz}{dt} = 0$

$$
\begin{aligned}
I &= \int_{t=0}^{2\pi} [R^5 \cos^2 t \sin^2 t(-R\sin t) + R\cos t + 0 \cdot 0]\,dt \\
&= -R^6 \int_{t=0}^{2\pi} \cos^2 t\, \sin^4 t\,dt + \int_{t=0}^{2\pi} \cos t\,dt \\
&= -4R^6 \int_{t=0}^{\frac{\pi}{2}} \cos^2 t\, \sin^4 t\,dt \\
&= -4R^6 \times \frac{3\times 1}{6\times 4\times 1} \times \frac{\pi}{2} \\
&= -\frac{\pi}{8}R^6
\end{aligned}
$$

Example 1.4. Evaluate $I = \int_C (xy\,dx + yz\,dy + zx\,dz)$ where C is curve given by $x = t, y = t^2, z = t^3$, where $-1 \le t \le 1$

Sol.

$$
\begin{aligned}
I &= \int_C (xy\,dx + yz\,dy + zx\,dz) \\
&= \int_{-1}^{1} (t.t^2.1 + t^2.t^3.2t + t^3.t.3t^2)\,dt \\
&= \int_{-1}^{1} (t^3 + 2t^6 + 3t^6)\,dt \\
&= \left[\frac{t^4}{4} + \frac{5t^7}{7}\right]_{-1}^{1} =
\end{aligned}
$$

Integral of bounded function over a parallelopiped $R = [a,b]\times[c,d]\times[e,f]$.
Let P_1, P_2, P_3 are the partitions of three intervals,

$$
\begin{aligned}
U(P,f) &= \sum_{t=1}^{p}\sum_{s=1}^{n}\sum_{r=1}^{m} M_{rst}(x_r - x_{r-1})(y_s - y_{s-1})(z_t - z_{t-1}) \\
U(P,f) &= \sum_{t=1}^{p}\sum_{s=1}^{n}\sum_{r=1}^{m} m_{rst}(x_r - x_{r-1})(y_s - y_{s-1})(z_t - z_{t-1})
\end{aligned}
$$

where, $M_{rst} =$ Upper bound of f on R and $m_{rst} =$ Lower bound of f on R.
$\lim_{||P||\to 0} U(P,f) = \lim_{||P||\to 0} L(P,f)$. Then f is integrable over R.

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \int_a^b dx \int_c^d dy \int_e^f f(x,y,z)\,dz$$

Calculation of integral in a bounded domain bounded by D. If f is continous on a domain D bounded by $z = \phi(x,y), z = \psi(x,y); y = g(x), y = h(x); x = a, x = b$. Then,

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \int_a^b \int_{g(x)}^{h(x)} \int_{\phi(x,y)}^{\psi(x,y)} f(x,y,z)\,dx\,dy\,dz$$

Example 1.5. Evaluate, $I = \iiint (x+y+z+1)^2\,dx\,dy\,dz$, over the region $x \ge 0, y \ge 0, z \ge 0, x+y+z \le 1$.

Sol. The region is bounded by $z = 0, z = 1 - x - y$. It's projection on D in $xy-$plane a triangle bounded by $x = 0, y = 0, y = 1 - x$. Then

$$\begin{aligned}
I &= \iiint (x + y + z + 1)^2 \, dx \, dy \, dz \\
&= \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} (x + y + z + 1)^2 \, dx \, dy \, dz \\
&= \frac{1}{3} \int_{x=0}^{1} \int_{y=0}^{1-x} \Big[(x + y + z + 1)^3 \Big]_0^{1-x-y} dx \, dy \\
&= \frac{1}{3} \int_{x=0}^{1} \int_{y=0}^{1-x} \Big[8 - (x + y + 1)^3 \Big] \, dx \, dy \\
&= \frac{1}{3} \int_{x=0}^{1} \Big[8y - \frac{1}{4}(x + y + 1)^4 \Big]_0^{1-x} dx \\
&= \frac{1}{3} \int_{x=0}^{1} \Big[8(1 - x) - \frac{1}{4}(16 - (x + 1)^4) \Big] \, dx \\
&= \frac{31}{60}
\end{aligned}$$