Advanced Engineering Mathematics
Lecture 16

1 Integral Calculus

Example 1.1. Change the order of the integration $\int_0^a dx \int_0^x f(x,y)\, dy$.

Sol. The given region of integration may be thought of as consisting of lines parallel to the X-axis starting from $x = y$ to $x = a$.

$$\begin{aligned} I &= \int_0^a dx \int_0^x f(x,y)\, dy \\ &= \int_0^a dy \int_y^a f(x,y)\, dx \end{aligned}$$

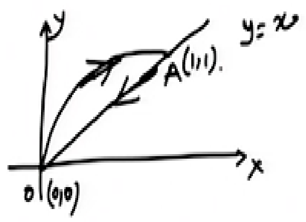
Green's theorem in $\mathbb{R}^2$. Let P, Q be two single valued functions such that $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial y}$ exist and are continuous on a simply connected domain bounded by a closed curve C. Then,

$$\oint_C (Pdx + Qdy) = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\, dx\, dy$$

where, $\oint_C$ denotes that C is closed and it is described in the $+ve$ direction.

Example 1.2. Evaluate by Green's theorem the integral $\int_C (x^2 y\, dx + xy^2\, dy)$ taken along the closed path formed by $y = x$ and $x^2 = y^3$in the first quadrant.

Sol. The curves intersect at $O(0,0)$ and $A(1,1)$. The integral is of form $\int_C (Pdx + Qdy)$ where $P = x^2 y$ and $Q = xy^2$.



$$I = \int_C (x^2 y\, dx + xy^2\, dy) = \iint_D (y^2 - x^2)\, dx\, dy = \int_{x=0}^1 \int_{y=x}^{x^{\frac{2}{3}}} (y^2 - x^2)\, dx\, dy$$

$$\Rightarrow I = \int_0^1 dx \int_x^{x^{\frac{2}{3}}} (y^2 - x^2)\, dy = \int_0^1 \Big[\frac{1}{3}(x^2 - x^3) - x^2(x^{\frac{2}{3}} - x)\Big]\, dx = \frac{1}{198}$$

Example 1.3. Evaluate the integral $\int_C [(2xy - x^2)\, dx + (x+y)^2\, dy]$, where C is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$

Change of variable in double integral.

Let a function $f(x,y)$ be continuous in a region D. Let the variables x, y in the double integral $\iint_D f(x,y)\, dx\, dy$ be changed to u, v by means of relation $x = \phi(u,v)$, $y = \psi(u,v)$ where ϕ and ψ have continuous first order partial derivative in a certain region D_1 in the uv-plane. Then, it can be shown that

$$\iint_D f(x,y)\,dx\,dy = \iint_{D_1} f(\phi(u,v),\psi(u,v))\,|J|\,du\,dv,$$

where $|J| = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$, the Jacobian of the transformation.

Let $I = \iint_D f(x,y)\,dx\,dy$, changing to polar coordinate $x = r\cos\theta,\ y = r\sin\theta$, then integration converts to $I = \iint_{D_A} f(r\cos\theta, r\sin\theta)|J|\,dr\,d\theta$, where Jacobian

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

This is how we convert integration of Cartesian coordinate to Polar coordinate.

Example 1.4. Evaluate $I = \iint \sin\pi(x^2+y^2)\,dx\,dy$ over the interior of the circle $x^2+y^2 = 1$.

Sol. let us take $x = r\cos\theta, y = r\sin\theta \Rightarrow x^2+y^2 = r^2$. Now

$$\begin{aligned} I &= \iint \sin\pi(x^2+y^2)\,dx\,dy \\ &= \int_{\theta=0}^{\pi}\int_0^1 \sin\pi r^2\,r\,dr\,d\theta \\ &= \int_{\theta=0}^{\pi} d\theta \int_0^1 \frac{1}{2}(\sin\pi t)\,dt \quad (\text{taking } t = r^2) \\ &= \int_0^{2\pi} \Big[-\frac{\cos\pi t}{2\pi}\Big]_0^1\,d\theta \\ &= 2 \end{aligned}$$