Advanced Engineering Mathematics
Lecture 15

1 Integral Calculus

Example 1.1. Evaluate $I = \int_R x\sin(x+y)\,dx\,dy$, where $R := [0,\pi]\times[0,\frac{\pi}{2}]$.

Sol. Using $\left[\int_{x=a}^{b}\left\{\int_{y=c}^{d} f(x,y)\,dy\right\}dx = \int_{y=c}^{d}\left\{\int_{x=a}^{b} f(x,y)\,dx\right\}dy\right]$, we have

$$
\begin{aligned}
I &= \int_{y=0}^{\frac{\pi}{2}}\int_{x=0}^{\pi} x\sin(x+y)\,dx\,dy\\
&= \int_{x=0}^{\pi}\Big[\int_{y=0}^{\frac{\pi}{2}} x\sin(x+y)\,dy\Big]\,dx\\
&= \int_{0}^{\pi}\Big[-x\cos(x+y)\Big]_0^{\frac{\pi}{2}}\,dx\\
&= \int_0^{\pi}[x\sin x + x\cos x]\,dx\\
&= \pi - 2
\end{aligned}
$$

Double integral over a region. If f is a continuous function in a domain D which is bounded by the curve $y=\phi(x), y=\psi(x), x=a, y=b$, where ϕ and ψ are continuous on $[a,b]$ and $\phi(x)\le\psi(x)$ then,

$$\iint_D f(x,y)\,dx\,dy = \int_{x=0}^{b}\Big[\int_{y=\phi(x)}^{\psi(x)} f(x,y)\,dy\Big]\,dx.$$

Example 1.2. Evaluate $\iint_R x^3y^2\,dx\,dy$ over the circle $C: x^2+y^2\le a^2$.

Sol. The circle C is

$$x^2+y^2\le a^2 \Rightarrow\ y^2\le a^2-x^2 \Rightarrow\ -\sqrt{a^2-x^2}\le y\le\sqrt{a^2-x^2}.$$

When $y=0$, then $x^2\le a^2\Rightarrow\ -a\le x\le a$.

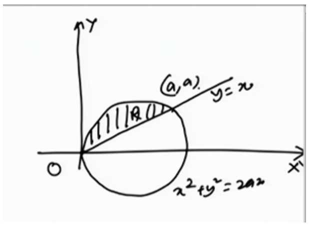
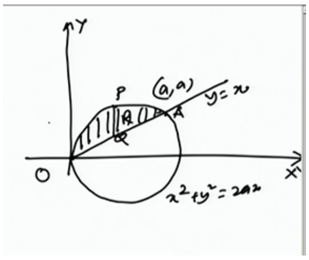
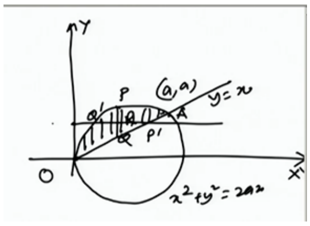
$$
\begin{aligned}
\iint_R x^3y^2\,dx\,dy &= \int_{x=-a}^{a} x^3\Big\{\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2dy\Big\}dx\\
&= \frac{2}{3}\int_{-a}^{a} x^3(a^2-x^2)^{\frac{3}{2}}dx\\
&= 0
\end{aligned}
$$

Example 1.3. Evaluate $I = \iint_R y\,dx\,dy$ over the part of the plane bounded by the line $y=x$ and the parabola $y=4x-x^2$.

Sol. Putting $y=x$ in $y=4x-x^2 \Rightarrow\ x=4x-x^2\Rightarrow\ x(x-3)=0\Rightarrow\ x=0,3$. The line joining the parabola meets at $(0,0)$ and $(3,3)$. Any line parallel to y-axis cuts the boundary of the region into two points, say P and Q. Thus,

$$I = \iint_R y\,dx\,dy = \int_{x=0}^{3} dx\int_x^{4x-x^2} y\,dy = \frac{1}{2}\int_0^3\Big[(4x-x^2)^2-x^2\Big]dx = \frac{54}{5}.$$

Example 1.4. Evaluate $I = \iint_R \cos(x+y)\,dx\,dy$ over thedomain closed by $x=0, y=\pi, y=x$.

Sol.

$$\begin{aligned} I &= \int_{x=0}^{\pi}\int_{y=x}^{\pi} \cos(x+y)\,dx\,dy \\ &= \int_{x=0}^{\pi} dx\Big\{\int_{y=x}^{\pi} \cos(x+y)\,dy\Big\} \\ &= \int_{x=0}^{\pi} \Big[\sin(x+y)\Big]_{y=x}^{\pi}\,dx \\ &= \int_{x=0}^{\pi} (-\sin x - \sin 2x)\,dx \\ &= -2 \end{aligned}$$

Change of the order of the integration.

Example 1.5. Change the order of integration in the double integral $I = \int_0^a \Big\{\int_x^{\sqrt{2ax-x^2}} f(x,y)\,dy\Big\}dx$.

Sol. The given domain of the integration is described by a line which starts from $x=0$ i.e, y-axis and moving parallel to y-axis terminates at $x=a$. Further the extremities of the moving line lie on the parts of the line $y=x$ and the circle x^2+y^2 in the first quadrant. when we change the order of integration, the same region is described by a line moving parallel to x-axis instead of y-axis. The line $y=x$ and the circle $x^2+y^2=2ax$ cut at the points $(0,0)$ and (a,a). Any line parallel to x-axis cuts the domain into two points $P'Q'$. We have

$$x^2-2ax+y^2=0 \Rightarrow\ x=a\pm\sqrt{a^2-y^2}.$$

Then,

$$I=\int_{y=0}^{a} dy\int_{x=a-\sqrt{a^2-y^2}}^{y} f(x,y)\,dx.$$

Example 1.6. $\displaystyle\int_0^1 dx\int_x^{\sqrt{x}} f(x,y)\,dy \to \int_0^1 dy\int_{y^2}^{y} f(x,y)\,dx$

Example 1.7. $\displaystyle\int_{-1}^1 dx\int_0^{\sqrt{1-x^2}} f(x,y)\,dy \to \int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx.$