**Advanced Engineering Mathematics**
**Lecture 14**

# 1 Integral Calculus

**Line integral.** A curve in $\mathbb{R}^2$ is given by $c = \{(x,y) : x = \phi(t), y = \psi(t), a \leq t \leq b\}$. We may write this curve $C$ as $x = \phi(t), y = \psi(t), a \leq t \leq b$. Let $f(x,y)$ be a bounded function defined at every point of $C$. Consider any partition $P$ of $[a,b]$ such that $P = \{a = t_0, t_1, t_2, \cdots, t_n = b\}$. Let $x_r = \phi(t_r)$, $y_r = \psi(t_r)$ and $\xi_r \in [t_{r-1}, t_r]$ be arbitrary. consider the sum

$$S = \sum_{r=1}^{n} f[\phi(\xi_r), \psi(\xi_r)](x_r - x_{r-1}).$$

If the norm $\mu(P) \to 0$, the sum $S$ tends to a finite limit which is independent of the choice of the point $\xi_r$ and the limit is denoted by $\int_C f(x,y)\,dx$.

Working principle : $\int_C f(x,y)\,dx = \int_a^b f(\phi(t), \psi(t))\phi'(t)\,dt$

**Example 1.1.** Find $I = \int_C \frac{y^2\,dx - x^2\,dy}{x^2+y^2}$, where $C$ is the semi-circle $x = a\cos t, y = a\sin t, 0 \leq t \leq \pi$.

**Sol**.

$$\begin{aligned} I &= \int_C \frac{y^2\,dx - x^2\,dy}{x^2+y^2} = \int_C \frac{y^2\,\frac{dx}{dt} - x^2\,\frac{dy}{dt}}{x^2+y^2}\,dt \\ &= \frac{1}{a^2}\int_{t=0}^{\pi} [a^2\sin^2 t(-a\sin t) - a^2\cos^2 t(a\cos t)]\,dt \\ &= -\frac{a^3}{a^2}\int_0^{\pi} [\sin^3 + \cos^3 t]\,dt \\ &= -a\int_0^{\pi} [\sin^3 + \cos^3 t]\,dt = -\frac{4a}{5} \end{aligned}$$

**Example 1.2.** Evaluate $I = \int_C (x^2\,dx + xy\,dy)$ taken along the line segment $(1,0)$ to $(0,1)$.

**Sol**. The equation of the line passing through $(1,0)$ to $(0,1)$ is $x + y = 1 \Rightarrow dy = -dx$.

Then, $I = \int_C (x^2\,dx + xy\,dy) = \int_0^1 (x^2 dx + x(1-x)(-dx)) = \int_0^1 (2x^2 - x)\,dx = -\frac{1}{6}$.

**Double integral over a rectangle** $R$. Let $f$ be a function of two variables $x$ and $y$ over a rectangle $R : [a,b] \times [c,d]$. Let

$$P_1 = \{a = x_0, x_1, \cdots, x_n = b\},\ P_2 = \{c = y_0, y_1, \cdots, y_n = d\}$$

$$U(P,f) = \sum_i \sum_j M_{ij}(x_i - x_{i-1})(y_j - y_{j-1}),\ L(P,f) = m_{ij}(x_i - x_{i-1})(y_j - y_{j-1}),$$

where $U(P,f)$ and $L(P,f)$ are the upper sum and lower sum, respectively. $M_{ij}$ and $m_{ij}$ are the upper bound and the lower bound of $f(x,y)$ in $[x_{i-1}, x_i] \times [y_{i-1}, y_i]$, respectively.

If $\lim_{n\to\infty} U(P,f)$ exists, then it is denoted as $\overline{\iint_R} f(x,y)\,dx\,dy$ (upper integral) and similarly if $\lim_{n\to\infty} L(P,f)$ exists, then it is denoted as $\underline{\iint_R} f(x,y)\,dx\,dy$ (lower integral).

If both upper integral and lower integral exists and equal, then the value is denoted by $\iint_R f(x,y)\,dx\,dy$. This is called the double integral of $f$ over $R$.

**Example 1.3.** Evaluate $\displaystyle\iint_R (x^2+2y)\,dx\,dy$ where $R=[0,1]\times[0,2]$.

**Sol**.

$$\int_{y=0}^{2}\left[\int_{x=0}^{1}(x+2x)\,dx\right]dy = \int_0^2\left[\frac{x^3}{3}+2xy\right]_0^1 dy = \int_0^2\left[\frac{1}{3}+2y\right]_0^1 dy = \left[\frac{y}{3}+y^2\right]_0^2 = \frac{14}{3}.$$