Advanced Engineering Mathematics
Lecture 11

1 Integral Calculus

Comparison test type 1. If f and g be two positive functions on $[a, b]$ and $f(x) \leq g(x)$ $\forall x \in [a, b]$, then
i) $\int_a^b f(x)$ converges if $\int_a^b g(x)$ converges.
ii) $\int_a^b g(x)$ diverges if $\int_a^b f(x)$ diverges.

Comparison test type 2. If f and g be two positive functions on $[a, \infty)$ and $f(x) \leq g(x)$ $\forall x \in [a, \infty)$, then
i) $\int_a^\infty f(x)$ converges if $\int_a^\infty g(x)$ converges.
ii) $\int_a^\infty g(x)$ diverges if $\int_a^\infty f(x)$ diverges.

Limit form test. If f and g be two positive functions on $[a, b]$ and $f(x) \leq g(x)$ $\forall x \in [a, b]$ and $\lim_{x \to a^+} \frac{f(x)}{g(x)} = l(\neq 0, < \infty)$, then $\int_a^b f(x)$ and $\int_a^b g(x)$ converge or diverge together at a.

Example 1.1. $I = \int_0^1 \frac{e^{-x}}{x^{1-n}}$

Sol. Here 0 is the only point of infinite discontinuity of the integrand if $n < 1$. Let $f(x) = \frac{e^{-x}}{x^{1-n}}$ and $g(x) = \frac{1}{x^{1-n}}$, then

$$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} \frac{\frac{e^{-x}}{x^{1-n}}}{\frac{1}{x^{1-n}}} = \lim_{x \to 0^+} e^{-x} = 1.$$

Now, $\int_0^1 \frac{dx}{x^{1-n}}$ converges if $1 - n < 1 \Rightarrow n > 0$. Hence the given integral I converges if $n > 0$.

Example 1.2. $I = \int_0^\infty \frac{\cos x}{1 + x^2} dx$

Sol. The right end point b is infinite. Choose $f(x) = \frac{\cos x}{1+x^2}$ and $g(x) = \frac{1}{1+x^2}$. Then $f(x) = \frac{\cos x}{1+x^2} \leq \frac{1}{1+x^2} = g(x)$ $\forall x \in [0, \infty)$. Now, $I_1 = \int_0^\infty g(x) dx = \int_0^\infty \frac{dx}{1 + x^2} dx$ is convergent. Therefore, by comparison test type 1, $I = \int_0^\infty f(x) dx$ is convergent.

Beta function. $B(m, n) = \beta(m, n) = \int_0^1 x^{m-1}(1 - x)^{n-1} dx$. Hence, $B(m, n)$ is proper integral if $m \geq 1, n \geq 1$. Improperness is attained when $m, n < 1$.

$$\int_0^1 x^{m-1}(1 - x)^{n-1} dx, \quad m < 1, n < 1$$

we will obtain $B(m, n)$ converges if and only if $m, n > 0$.

Gamma function. $\Gamma = \int_0^\infty e^{-x} x^{n-1} dx$ converges if and only if $n > 0$. Here, 0 is a point of infinite discontinuity if $n < 1$.

$$\Gamma = \int_0^\infty e^{-x} x^{n-1} dx = \int_0^1 e^{-x} x^{n-1} dx + \int_1^\infty e^{-x} x^{n-1} dx$$

we can show that $\Gamma = \int_0^\infty e^{-x}x^{n-1}dx$ is convergent if only if $n > 0$.

Properties of Beta and Gamma functions.

1. $\Gamma(n) = (n-1)\Gamma(n-1)$ when $n > 1$

$$\begin{aligned}\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx &= [-x^{n-1}e^{-x}]_0^\infty + (n-1)\int_0^\infty e^{-x}x^{n-2}dx \\ &= (n-1)\Gamma(n-1) \\ \Rightarrow \Gamma(n) &= (n-1)\Gamma(n-1) \\ &= (n-1)(n-2)\cdots 2.1 \\ &= (n-1)!\end{aligned}$$

2. $B(m,n) = B(n,m)$

$$\begin{aligned}B(m,n) &= \int_0^1 x^{m-1}(1-x)^{n-1}dx \\ &= \int_0^1 (1-y)^{m-1}y^{n-1}dx \qquad\qquad [Let, (1-x) = y] \\ &= B(n,m)\end{aligned}$$