Statistical Inference
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Lecture - 56
Likelihood Ratio Tests - VI

Let me again repeat the model. We are considering the two sample problem I had taken variances to be common.

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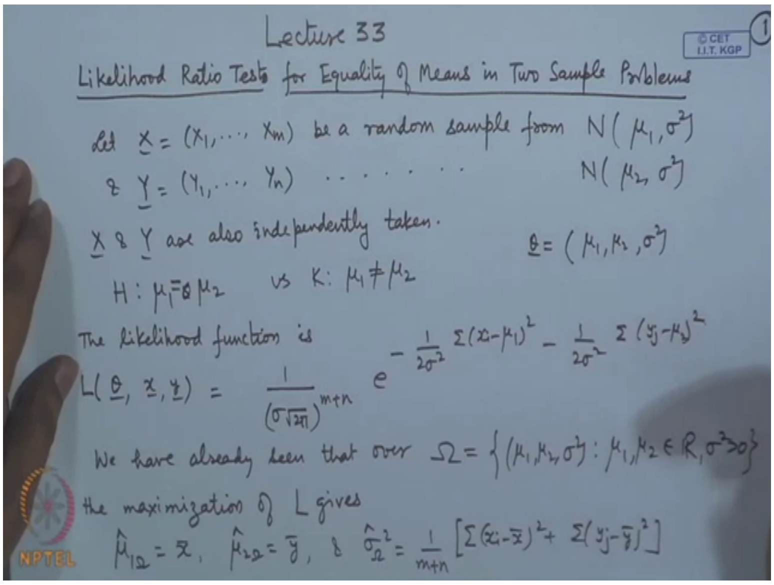

Now, in general situations, we do not know whether the variances will be common or not. So, in order to test about the comparison of the means we firstly need to check whether the variances are the same or not. So, let us consider the testing for the variances.

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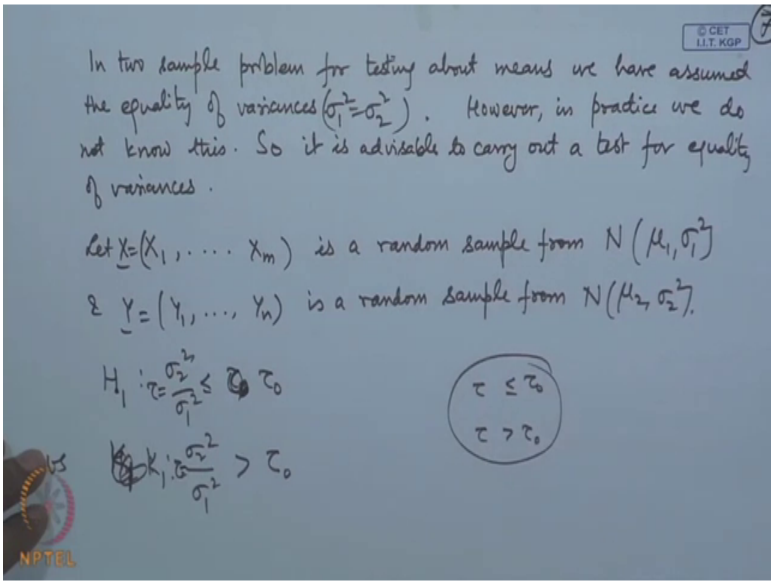

In two sample problems for testing about means, we have assumed the equality of variances. That is sigma 1 square is equal to sigma 2 square; however, in practice we do not know this.

So, it is advisable to carry out a test for equality of variances. So, I will provide this here. We have the model that is X 1, X 2, X m that is X is a random sample from normal mu 1 sigma 1 square. And Y is equal to Y 1, Y 2, Y n is a random sample from normal mu 2 sigma 2 square. For comparison of the variances; I consider the hypothesis of the nature say H 1 sigma 2 square by sigma 1 square less than or equal to some number say tau naught; tau let us say and H 2 sigma 2 square by sorry K 2; K 1 that is the alternative hypothesis.

Sigma 2 square by sigma 1 square is greater than tau say. Let me call this as tau and this as tau naught value this is tau. So, we want to check whether tau is less than or equal to tau naught and our tau is greater than tau naught; that means, my likelihood ratio then I am considering I will represent in the terms of this ratio tau.

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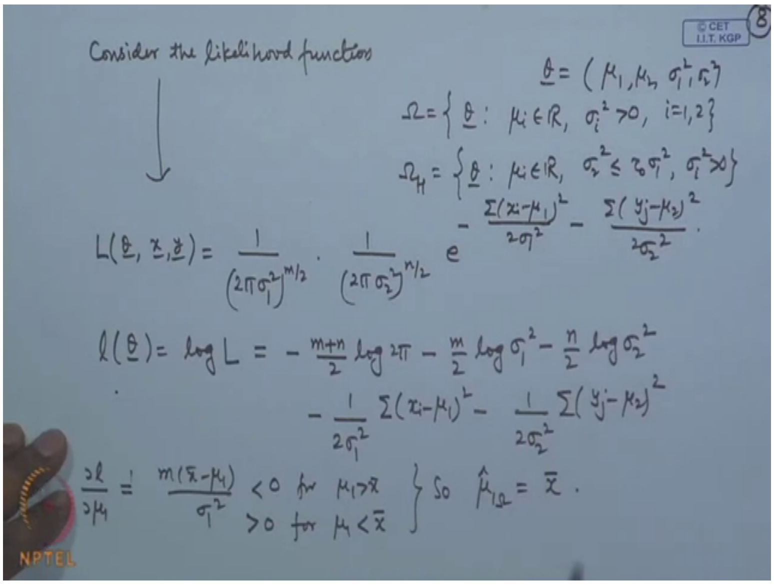

So, consider the likelihood function. Now note here my parameter is now four dimension. Four dimensional parameter set is there mu 1, mu 2, sigma 1 square, sigma 2 square. The null hypothesis parameter set is this mu i's are real and sigma i square is positive for i is equal to 1, 2 . And in the null hypothesis parameter set we are having the restriction that sigma 2 square is less than or equal to tau naught sigma 1 square.

And of course, both are positive. So, this additional restriction has come here. So, the likelihood function, now l theta x, y that is equal to 1 by 2 pi sigma 1 square to the power m by 2 1 by 2 pi sigma 2 square to the power n by 2 e to the power minus sigma x i minus mu 1 square by 2 sigma 1 square minus sigma y j minus mu 2 square by 2 sigma 2 square. So, we consider the log likelihood, that is equal to minus m plus n by 2 log of 2 pi minus m by 2 log of sigma 1 square minus n by 2 log of sigma 2 square minus 1 by 2 sigma 1 square sigma x i minus mu 1 square minus 1 by sigma 2 square sigma y j minus mu 2 square.

So, this is a full set up with four parameters in the two normal populations and they are independent therefore, the solution is straightforward here that is if I considered del l by del mu 1 that will give me m x bar minus mu 1 by sigma 1 square less than 0 for mu 1 greater than x bar greater than 0 for mu 1 less than x bar. So, mu 1 omega hat that will be equal to x bar.

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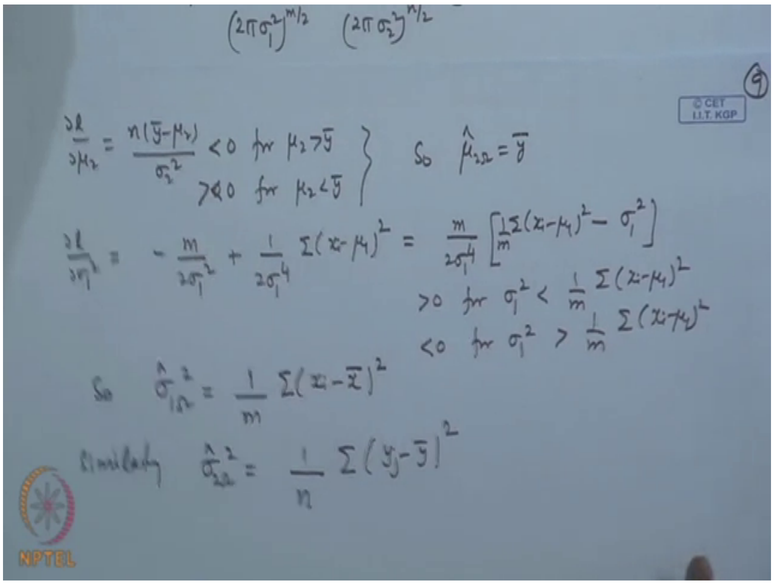

Similarly, if I consider say del l by del mu 2 then that will be n y bar minus mu 2 divided by sigma 2 square. Once again it is less than 0 for mu 2 greater than y bar less than and it is greater than 0 for mu 2 less than y bar.

So, mu 2 hat omega that is equal to y bar. Let us also consider maximization with respect to sigma 1 square and sigma 2 square. So, let us look at this term del l by del sigma 1 square that is equal to minus m by 2 sigma 1 square plus 1 by 2 sigma 1 to the power 4 sigma x i minus mu 1 square that is equal to m by 2 sigma 1 to the power 4 1 by m sigma x i minus mu 1 square minus sigma 1 square. So, this is greater than 0 for sigma 1 square less than 1 by m sigma x i minus mu 1 square it is less than 0 for sigma 1 square greater than 1 by m sigma x i minus mu 1 square.

So, the maximization with respect to sigma 1 square is then occurring at 1 by m sigma x i minus mu 1 square. Now mu 1 is maximized at x bar. So, it is 1 by m sigma x i minus x bar whole square. Now similarly if I look at the maximization with respect to sigma 2 square; that will become 1 by n sigma y j minus y bar square. Now these values I substitute in likelihood function. In this likelihood function if I substitute the maximizing values I get.

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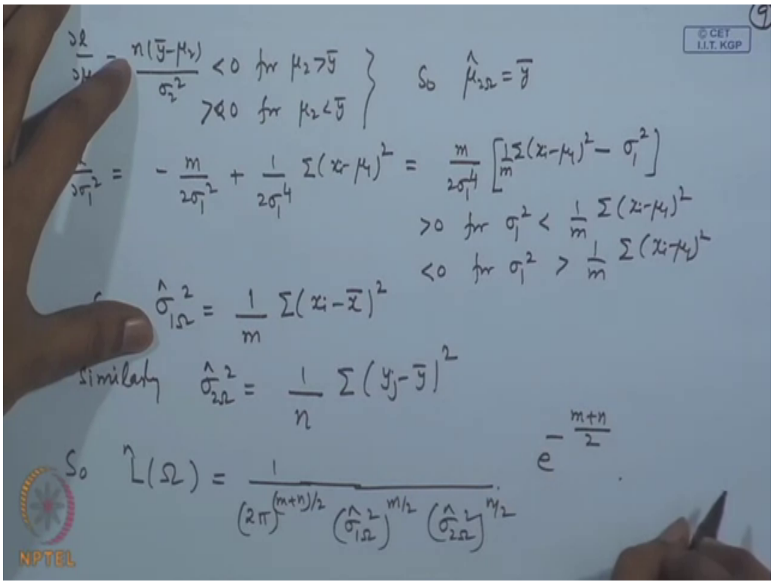

So, l hat omega that becomes equal to 1 by 2 pi to the power m plus n by 2 sigma 1 omega Hat square to the power m by 2 sigma 2 omega Hat square to the power n by 2 e to the power minus m plus n by 2.

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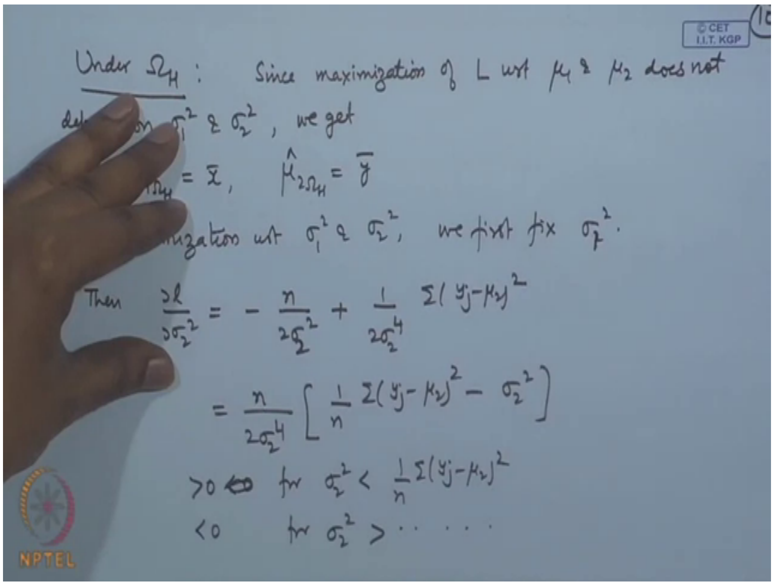

Under omega H. Now let us consider under omega H we are having now if you look at the maximization with respect to mu 1 and mu 2 there is no dependence on sigma 1 square and sigma 2 square. So, it will not change . Since maximization of L with respect to mu 1 and mu 2 does not depend on sigma 1 square and sigma 2 square, we get L hat sorry mu 1 hat omega H as x bar mu 2 omega H hat is y bar; however, if I consider for sigma 1 square and sigma 2 square, then here we are having the region sigma 2 square is less than or equal to tau naught sigma 1 square in the alternative hypothesis set; that is sigma 2 square less than or equal to tau naught sigma 1 square.

So, this will play a role here. So, for maximization with respect to sigma 1 square and sigma 2 square, we first say fix sigma 1 square say ok. Then we are saying sigma 2 hat square is less than or equal to. So, if I consider the derivative of the likelihood function with respect to sigma 2 square, I get minus n by 2 sigma 1 sigma 2 square plus 1 by 2 sigma 2 to the power 4 sigma y j minus mu 2 square. That I can write as n by 2 sigma 2 to the power 4 1 by n sigma y j minus mu 2 minus sigma 2 square. So, this is less than 0; it is greater than 0 for sigma 2 square less than 1 by n sigma y j minus mu 2 square and it is less than 0 for sigma 2 square greater than this number.

.So, the behavior of the likelihood function with respect to sigma 2 square.

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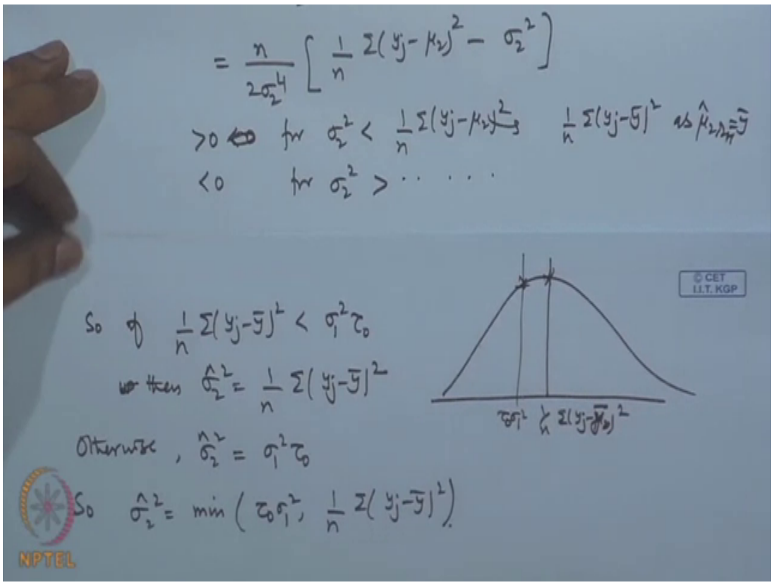

That means if I consider the plotting of the likelihood function with respect to sigma 2 square. Then at the point 1 by n sigma y j minus mu 2 square this is the maximization point. Now if this quantity 1 by n sigma y j minus this is occurring at y bar square because I have substituted mu 2 hat is equal to y bar. So, here I can put mu 2 as 1 by n sigma y j minus y bar square as mu 2 hat omega H that is equal to y bar. So, if 1 by n sigma y j minus y bar square is less than sigma 1 square tau naught then sigma 2 hat square that is equal to 1 by n sigma y j minus y bar square.

Otherwise suppose tau naught sigma 1 square is here. If that is happening, then sigma 2 hat square this maximization is then occurring at this point that is equal to sigma 1 square tau naught. So, we are saying sigma 2 hat square is equal to actually minimum of tau naught sigma 1 square and 1 by n sigma y j minus y bar square.

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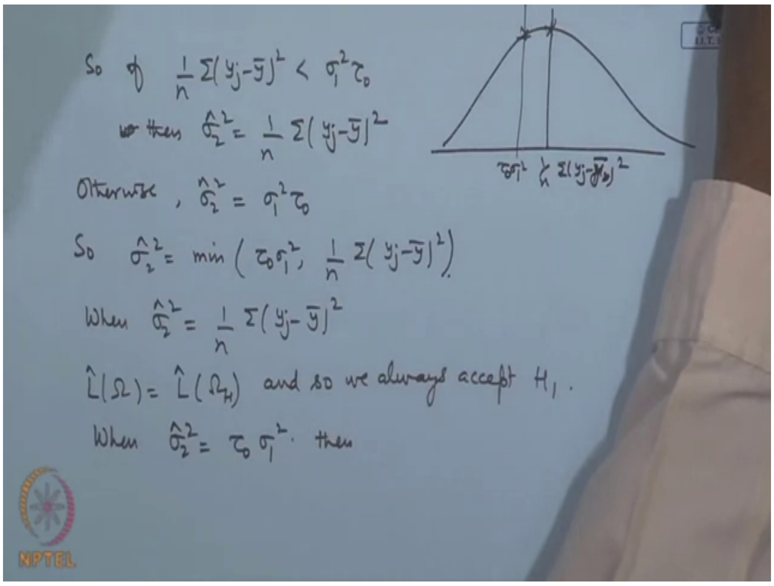

So, when sigma 2 omega hat square that is equal to 1 by n sigma y j minus y bar square. I will get L hat omega and L hat omega H same. And so, we always accept H 1. Because the likelihood ratio is supposed to be between 0 and 1. So, if it is equal to 1 we always accept H. Now the other case when sigma 2 hat square is equal to tau naught sigma 1 square.

In that case with respect to sigma 1 square when I consider the likelihood function let us look at the term once again.

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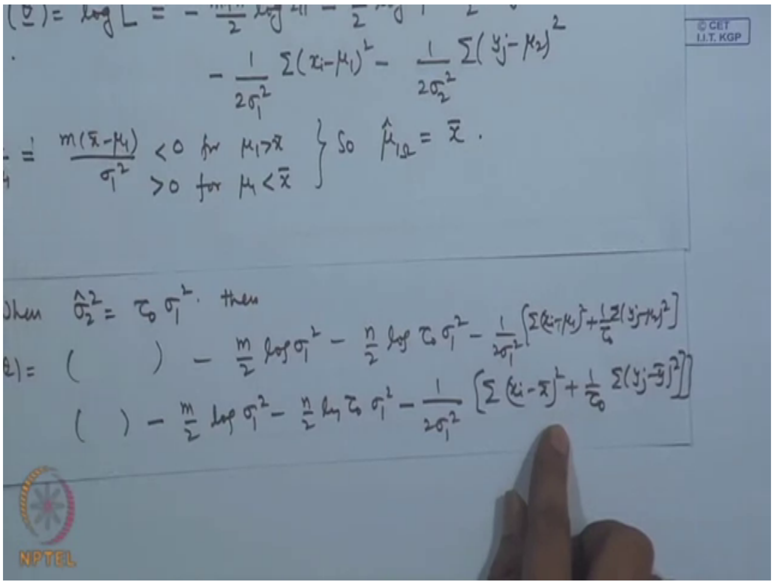

Let me look at this term [laughter]; small l. So, small l theta that is equal to now this will become. So, these terms will be there minus m by 2 log of sigma 1 square minus n by 2 log of tau naught sigma 1 square minus 1 by 2 sigma 1 square sigma x i minus mu 1 square plus 1 by tau naught sigma y j minus mu 2 square. Now we have already got the maximizing values of this. So, I will put it here x bar here. So, this term then I get minus m by 2 log of sigma 1 square minus n by 2 log of tau naught sigma 1 square minus 1 by 2 sigma 1 square sigma x i minus x bar square plus 1 by tau naught sigma y j minus y bar whole square ok.

That means, the maximization problem with respect to sigma 1 square now it has been reduced to maximization of this term. So, let us consider that now.

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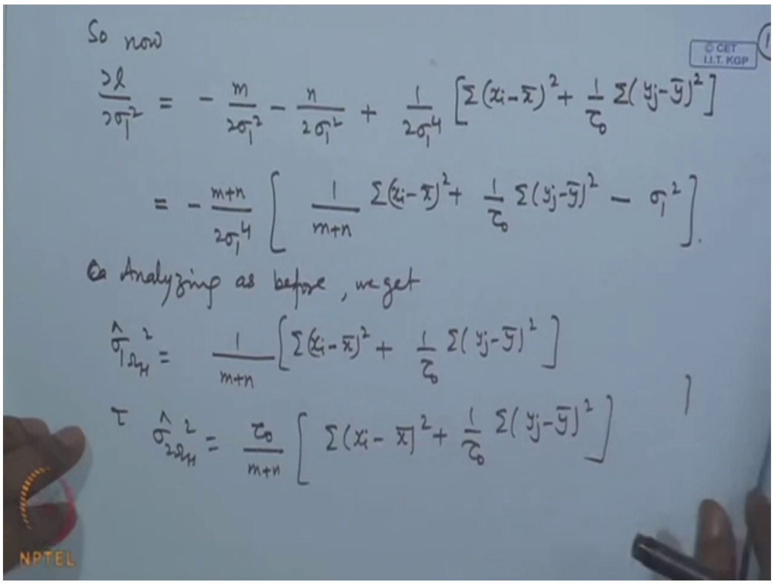

So now del l by del sigma 1 square that will give me let us look at this thing. So, minus m by 2 sigma 1 square minus n by 2 sigma 1 square. And this term will not play a role plus 1 by 2 sigma 1 to the power 4 sigma x i minus x bar square plus 1 by tau naught sigma y j minus y bar square. So, that I can write as minus m plus n by twice sigma 1 to the power 4 1 by m plus n sigma x i minus x bar square plus 1 by tau naught sigma y j minus y bar square minus sigma 1 square.

So, if we carry out the analysis as we have been doing. So, analyzing as before we get here; sigma 1 omega H hat square that is equal to 1 by m plus n sigma x i minus x bar square plus 1 by tau naught sigma y j minus y bar square. And sigma 2 omega H hat square that is tau naught times this. So, it becomes tau naught divided by m plus n and the same term here x i minus x bar square plus 1 by tau naught sigma y j minus y bar square.

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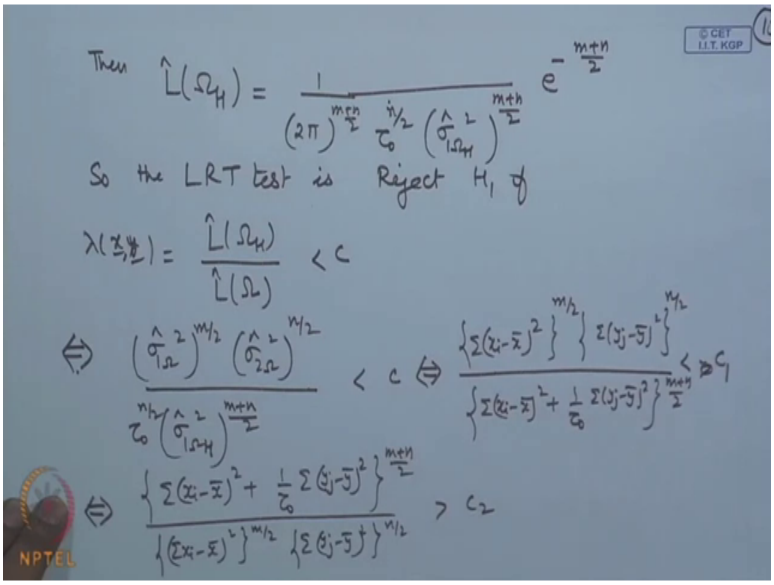

So, these values then I substitute in the likelihood function to get L hat omega H as 1 by 2 pi to the power m plus n by 2.

And then I will be getting. So, if you look at the original form of the likelihood function here you see here sigma 2 square I am saying sigma 1 square tau naught. So, this term and this term will get combined and tau naughts power will come here. So, I will get tau naught to the power n by 2 and then these 2 will get combined. So, I will get sigma 1 omega H hat square to the power m plus n by 2 in the denominator. And then in the exponent part here also what is happening sigma 2 square is sigma 1 square tau naught. So, this term I get outside then I get sigma x i minus x bar whole square plus 1 by tau naught sigma y j minus y bar whole square and 1 by tau naught is coming. So, this term will get simply cancelled out and I will get e to the power minus m plus n by 2.

So, let us look at the expressions that we derived in the page 11 I have written the expression for this value here e on page 9 yeah L hat omega. So, you look at this L hat omega is this. So, this coefficient is common in L hat omega H and L hat omega. And here I am getting sigma 1 omega hat square and sigma 2 hat omega square and here it is same term sigma 1 hat omega H square to the power m plus n by 2; this e to the power minus m plus n by 2 will also get cancelled out when I take the ratio.

So, the likelihood ratio test then will give me. So, the likelihood ratio test is reject H 1 if lambda x y that is equal to L hat omega H by L hat omega less than c. This is equivalent to sigma one hat omega square to the power m by 2 sigma 2 omega hat square to the power n by 2 divided by sigma 1 hat omega square to the power m plus n by 2 and tau naught to the power n by 2 less than say c. I have cancelled out the constant terms from here.

Once again this does not play any role here. And these terms then I simplify what I do. Firstly, because in the sigma 1 omega hat, H; this is H here. In this term I am getting both sum of squares. Here only sum of squares of the first term is coming. So, this is equivalent to I can say sigma x i minus x bar square to the power say m by 2 sigma y j minus y bar square to the power say n by 2 divided by sigma x i minus x bar square plus 1 by tau naught sigma y j minus y bar square to the power m plus n by 2 greater than sorry less than say c some. Coefficient will get cancelled out because for example, 1 by m was here 1 by n is here. So, let me call it c 1 here.

So, this is then equivalent to I take the reciprocal. I can express it as sigma x i minus x bar square plus 1 by tau naught sigma y j minus y bar square to the power m plus n by 2 divided by sigma x i minus x bar square to the power m by 2 sigma y j minus y bar square to the power n by 2 greater than say c 2.

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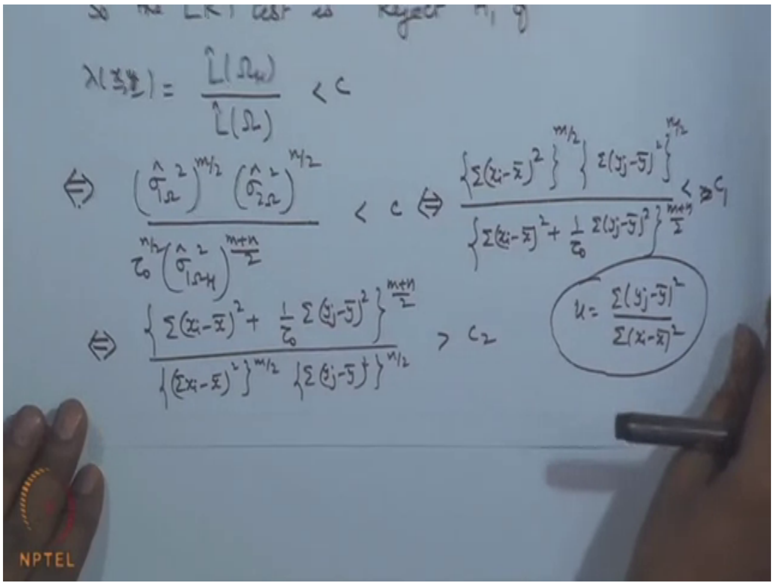

Now, let me define u is equal to sigma y j minus y bar square divided by sigma x i minus x bar square. So, in terms of this ratio this condition can be written as.

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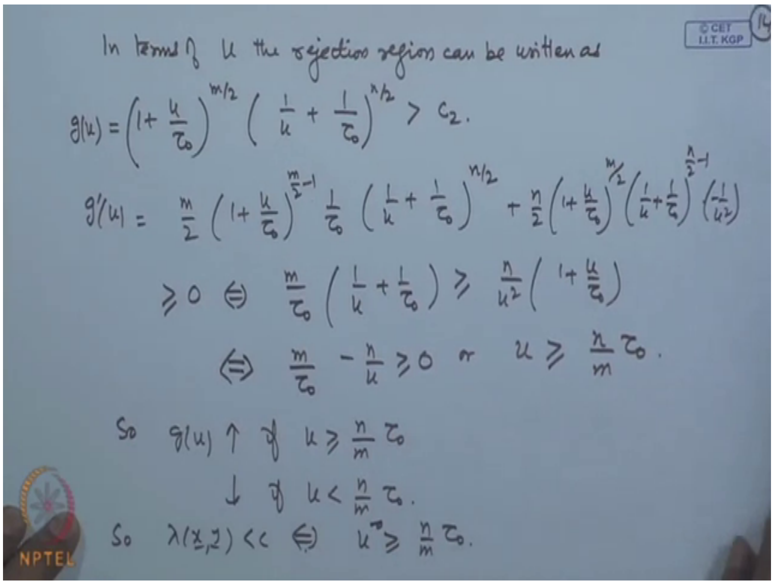

In terms of u the rejection region can be written as 1 plus u by tau naught to the power say m by 2 1 by u to the plus 1 by tau naught to the power n by 2 greater than say c 2. Now let us consider this term as say g of u.

Then let us look at the g prime u that is equal to m by 2 1 plus u by tau naught to the power m by 2 minus 1 into 1 by tau naught into 1 by u plus 1 by tau naught to the power n by 2 plus n by 2 1 plus u by tau naught to the power m by 2 1 by u plus 1 by tau naught to the power n by 2 minus 1 minus 1 by u square. So, this is greater than or equal to 0 if and only if. So, this will require certain simplification; I can take common 1 plus u by tau naught to the power m by 2 minus 1 and 1 plus; u plus 1 by tau naught to the power n by 2 minus 1. If I take this common and adjust the terms the condition is reducing to simply m by tau naught into 1 by u plus 1 by tau naught greater than or equal to n by u square 1 plus u by tau naught. Which is equivalent to m by tau naught minus n by u greater than or equal to 0 or u is greater than or equal to n by m tau naught.

So, this function is actually having increasing nature. This g u function; so, g u is increasing if u is greater than or equal to n by n tau naught and it is decreasing if u is less than n by m tau naught but what is tau naught? That is sigma y j minus y bar whole square divided by sigma x i minus x bar whole square. So, the condition lambda x y less than c this is the equivalent to u greater than or equal to n by m tau naught.

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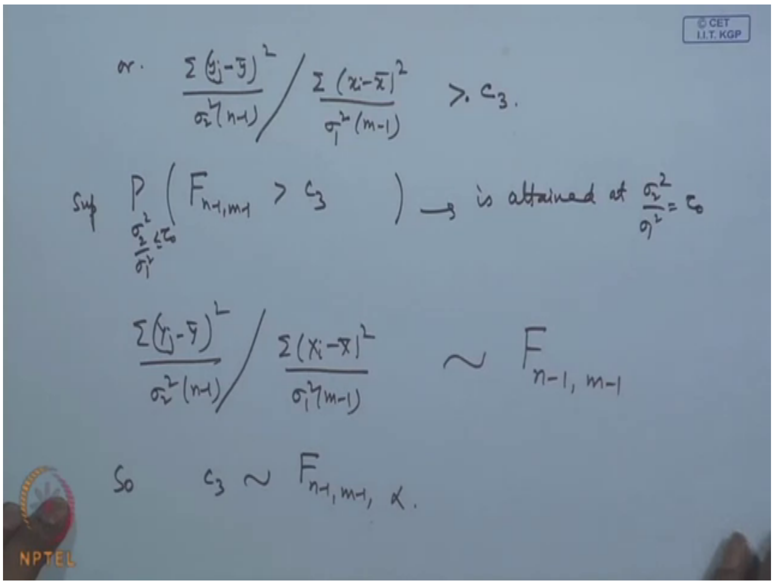

Or we can write it in the terms of sigma y j minus y bar square divided by sigma 2 square by n minus 1 divided by sigma x i minus x bar whole square divided by sigma 1 square m minus 1 that is greater than say some c 3.

Now, when I consider the probability of this region for sigma 2 square by sigma 1 square less than or equal to tau naught. Then supremum of this is attained at. So, you look at the distribution of this. This has sigma y j minus y bar whole square by sigma 2 square n minus 1 divided by sigma x i minus x bar whole square by sigma 1 square m minus 1. This follows F distribution on n minus 1; m minus 1 degrees of freedom.

So now, what we are saying is F n minus 1 m minus 1 greater than c 3. So, what we look at this that I adjust this term and I get here the maximum is attained at sigma 2 square by sigma 1 square is equal to tau naught. So, c 3 point then we take as F of n minus 1 m minus 1 alpha.

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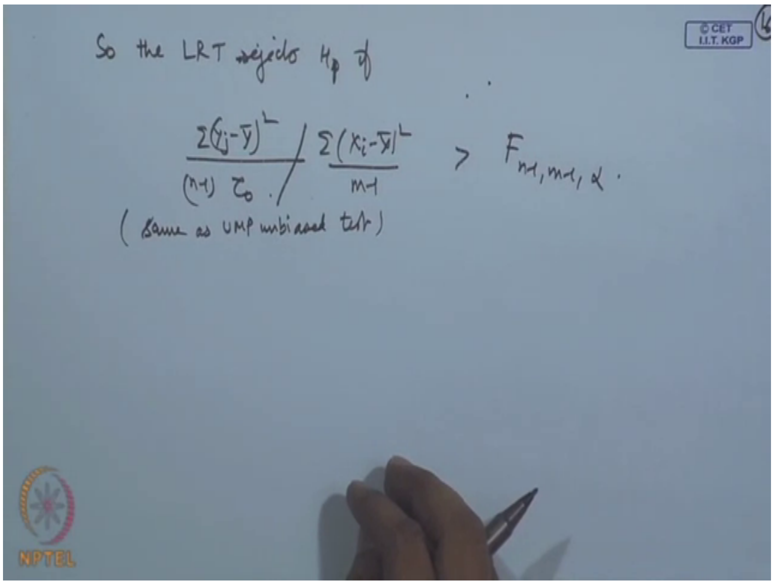

So, the test is; so, the likelihood ratio test rejects H naught; H 1 if sigma y j minus y bar square divided by n minus 1 divided by sigma x i minus x bar whole square divided by m minus 1 is greater than F n minus 1 m minus 1 alpha.

Here tau naught will be coming sigma 2 square by sigma 1 square. Yeah 1 by tau naught will be coming here because here I should have sigma 2 square by sigma 1 square that is equal to tau naught yeah. So, same as the same as the UMP unbiased test that I derived that day. I will carry out the analysis for the equality test also. That is when we want to test that sigma 2 square by sigma 1 square is equal to delta naught or tau naught or against not equal to. And there of course, I can take the case where tau naught is equal to 1. So, I will be discussing this case in the following lecture.

Moreover there is another important problem in testing of hypothesis. I have discussed two normal populations. Now in place of two normal population; we may have k normal population where k is can be 3, 4 and so on. And we may again like to test about the equality of means. So, the ump unbiasness theory does not work, there because we cannot write the mu 1 is equal to mu 2 etcetera in a form where which can be working as a multi parameter exponential family; however, for these situations a likelihood ratio test can be derived. So, this is the problem of testing homogeneity of means in a one way analysis of variance model. So, in the next lecture I will be discussing that also.