Statistical Inference
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Lecture - 55
Likelihood Ratio Tests -V

In the last two lectures, I have introduced the concept of Likelihood Ratio Tests. We consider derivation of the likelihood ratio test for some one sample problems and especially for the parameters of normal distribution when we have a sample from one normal distribution. Then later on I introduced two normal populations and we had some independent random samples from both of them.

We considered the test for comparing the means and in that I had given the test for mu 1 less than or equal to mu 2 the null hypothesis. Now, today I will firstly derive the test when we are having the null hypothesis as equality and the alternative hypothesis as the inequality.

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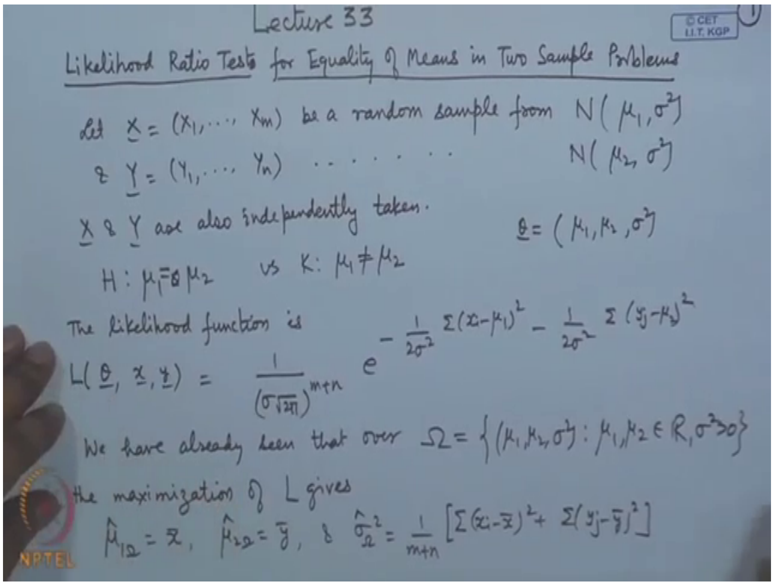

So, here let us consider, so let me reintroduce the model here let X 1, X 2, X m be a random sample from normal mu 1 sigma square distribution and Y is equal to Y 1, Y 2, Y n is a random sample from normal mu 2 sigma square. And X and Y are also independently taken. This is the model I had introduced yesterday. Yesterday we had considered the hypothesis mu 1 less than or equal to mu 2. Now, today I will consider mu

1 less than are mu 1 is equal to mu 2, against mu 1 not equal to mu 2. Another important point to notice here is that in the likelihood ratio test the alternative hypothesis does not play a role. Whereas, in the Neyman Pearson Theory the alternative hypothesis has to be specified in order to discuss the power, because we are talking about the most powerful, uniformly most powerful are the UMP unbiased tests. So, all the time power is a consideration.

In the likelihood ratio test we are only looking at the null hypothesis parameter set and the full parameter set. Of course, by elimination what is happening is that, when we are taking the full parameter space the complimentary space of the null hypothesis space is the alternative hypothesis parameter space. Therefore, indirectly it is playing a role and you have already seen yesterday that the tests which we are deriving using the likelihood ratio method is they are actually the same as the UMP unbiased tests for the parameters of normal distributions. Now we consider this particular hypothesis. Now as before let me write down the likelihood function here.

So, mu 1, mu 2, sigma square these are the parameters here and ah. So, the likelihood function L, theta, x, y that is equal to one by sigma root 2 pi to the power m plus n e to the power minus 1 by 2 sigma square sigma x i minus mu 1 square minus 1 by 2 sigma square, sigma y j minus mu 2 whole square. So, we have already seen that over the full parameter space omega that is mu 1, mu 2, sigma square where mu 1 mu 2 are real and sigma square is positive. Over this the maximization of L gives mu 1 omega hat is equal to x bar, mu 2 omega hat is equal to y bar and sigma omega hat square that is equal to 1 by m plus n sigma x i minus x bar square plus sigma y j minus y bar whole square.

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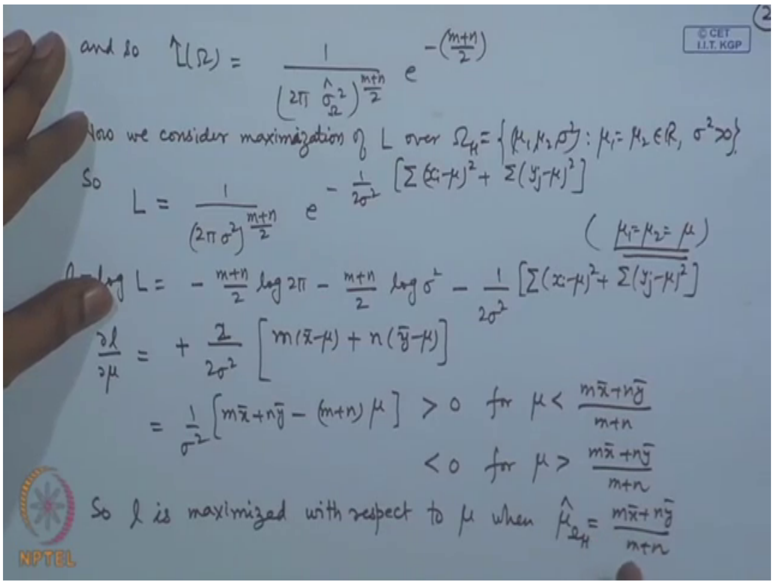

So, if we use this L hat omega that is turning out to be 1 by 2 pi sigma omega hat square to the power m plus n by 2 e to the power minus m plus n by 2. Now, we consider maximization of L over omega H that is mu 1, mu 2 sigma square: mu 1 is equal to mu 2 and sigma square is positive. So, now let us write down the likelihood function in the modified form because here now mu 1 is equal to mu 2. So, L is now 1 by 2 pi sigma square to the power m plus n by 2 e to the power minus 1 by 2 sigma square sigma x i minus mu square plus sigma y j minus mu square.

That means, I have taken here mu 1 is equal to mu 2 is equal to mu. So, log of L that is equal to minus m plus n by 2 log of 2 pi minus of m plus n by 2 log of sigma square minus 1 by 2 sigma square sigma of x i minus mu square plus sigma of y j minus mu square. So, let us consider the maximization of this with respect to mu and sigma square. So, derivative with respect to mu that will give me minus 1 by 2 sigma square; then here if I differentiate this and this term I will get minus 2. So, this will become plus and I will get n x bar minus mu plus sorry this is m x bar minus mu plus n y bar minus mu. That we can say m x bar plus n y bar minus m plus n mu and this 1 by sigma square will be outside.

So, easily you can see this is greater than 0 for mu less than m x bar plus n y bar divided by m plus n and less than 0 for mu greater than m m x bar plus n y bar divided by m plus n. So, l is maximized with respect to mu when mu is, so we call it mu hat omega H is equal to m x bar plus n y bar divided by m plus n. So, this is the maximization of l with respect to mu. When the null hypothesis mu 1 is equal to mu 2 is true.

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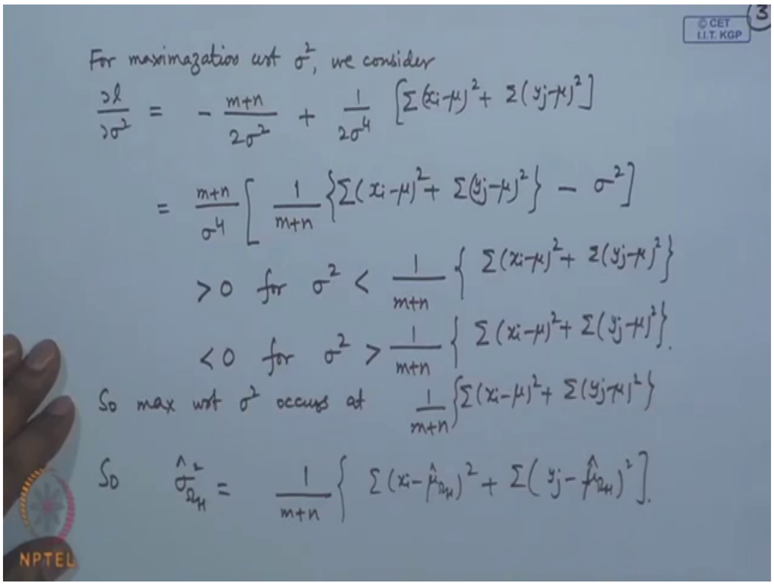

Now, based on this I can consider the maximization with respect to sigma square for maximization with respect to sigma square we consider del l by del sigma square that is equal to minus m plus n by 2 sigma square plus 1 by 2 sigma to the power 4 sigma x i minus mu square plus sigma y j minus mu square. And this you can follow from looking at this term here, that if I have differentiated this term with respect to sigma square. So, this term gives me minus m plus n by 2 sigma square and this term gives me 1 by 2 sigma to the power 4 into this term. So, once again we can write down this as m plus n by sigma to the power 4. 1 by m plus n sigma x i minus mu square plus sigma y j minus mu square minus sigma square.

So, naturally you can see this is greater than 0 for sigma square less than 1 by m plus n sigma x i minus mu square plus sigma y j minus mu square and it is less than 0 for sigma square greater than this value. So, maximization with respect to sigma square occurs at this quantity, that is 1 by m plus n sigma x i minus mu square plus sigma y j minus mu square. So, we can say that sigma omega H hat square that is equal to 1 by m plus n sigma x i minus mu omega H hat square plus sigma y j minus mu hat omega H square. Now mu hat omega H that we evaluated just now, that is equal to m x bar plus n y bar divided by m plus n. So, this value we substitute here.

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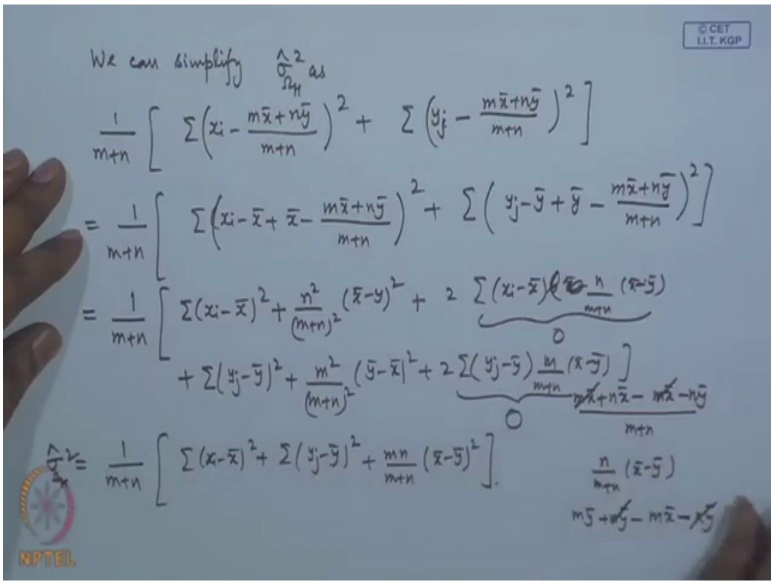

We can simplify sigma omega H hat square as 1 by m plus n sigma x i minus m x bar divided by n y bar divided by m plus n y j minus m x bar plus ny bar divided by m plus n square. In this first term I add and subtract x bar here and in the second one I add and subtract y bar. So, this becomes 1 by m plus n sigma x i minus x bar plus x bar minus m x bar plus ny bar divided by m plus n; y j minus y bar plus y bar minus m x bar plus n y bar divided by m plus n square. So, that is equal to 1 by m plus n. And this term now I expand. So, I will get sigma x i minus x bar square plus now this second term if I simplify I get m x bar plus n x bar minus m x bar minus n y bar divided by m plus n.

So, m x bar gets cancelled out and I get n divided by m plus n x bar minus y bar and this will be squared. So, I get n square by m plus n square x bar minus y bar square. Then there is a cross product term twice sigma x i minus x bar into x bar minus that is the same term n by m plus n x bar minus y bar. Now, this term is nothing but 0. Because sigma x i minus x bar is 0. Similarly, I expand the second term. If I expand the second term I will get sigma y j minus y bar square plus now if I look at this second term I can write my bar plus ny bar minus m x bar minus ny bar. So, this gets cancelled out. So, this becomes m square divided by m plus n whole square y bar minus x bar whole square. And, once again this cross product term sigma y j minus y bar into m by m plus n x bar minus y bar gets this gets cancelled out.

Now, the term that is remaining I can actually simplify this. This I can write as 1 by m plus n sigma x i minus x bar whole square plus sigma y j minus y bar whole square, plus m n by m plus n, x bar minus y bar whole square. That is the term that I will be getting after simplification of this term here. So, if I substitute this value so this is my sigma omega H hat square. So, in the likelihood function if I substitute this value that is 1 by sigma omega H hat square and here I substitute this value. So, here this is obtained after substituting mu 1 mu 2 is equal to mu I will get.

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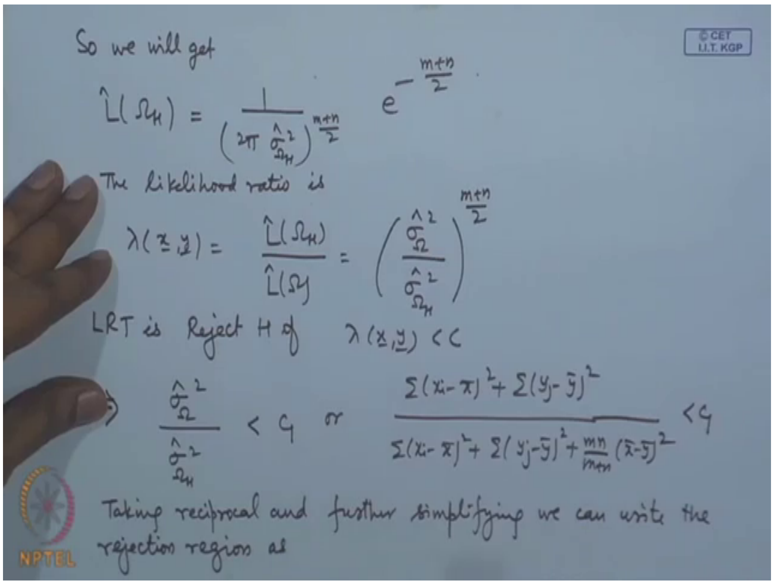

So, we will get L hat omega h as equal to 1 by 2 pi sigma omega H hat square to the power m plus n by 2 e to the power minus m plus n by 2. now we have obtained the maximization over the null hypothesis parameter set. We have also derived L hat omega that is the maximization over the full parameter set. So, the likelihood ratio that is lambda x, y here that is L hat omega H by L hat omega that will become sigma omega hat square divided by sigma omega H hat square to the power m plus n by 2.

So, the likelihood ratio test is reject H if lambda x, y is less than C. Now, that will be equivalent to that sigma omega hat square divided by sigma omega H hat square is less than some C 1. And, then we can substitute this values here sigma omega hat square was obtained as it was 1 by m plus n sigma x i minus x bar whole square plus sigma y j minus y bar whole square.

So, this term is sigma x i minus x bar square plus sigma y j minus y bar square divided by sigma omega H and this we calculated just now. So, that is again equal to sigma x i minus x bar whole square plus sigma y j minus y bar whole square plus m n by m plus n x bar minus y bar square. I am saying it is less than C 1. So, I can take a reciprocal of this inequality taking reciprocal and further simplifying we can write the rejection region as, so when I take reciprocal this comes in the numerator and I divide by this full term I get 1 then I take it to the other side.

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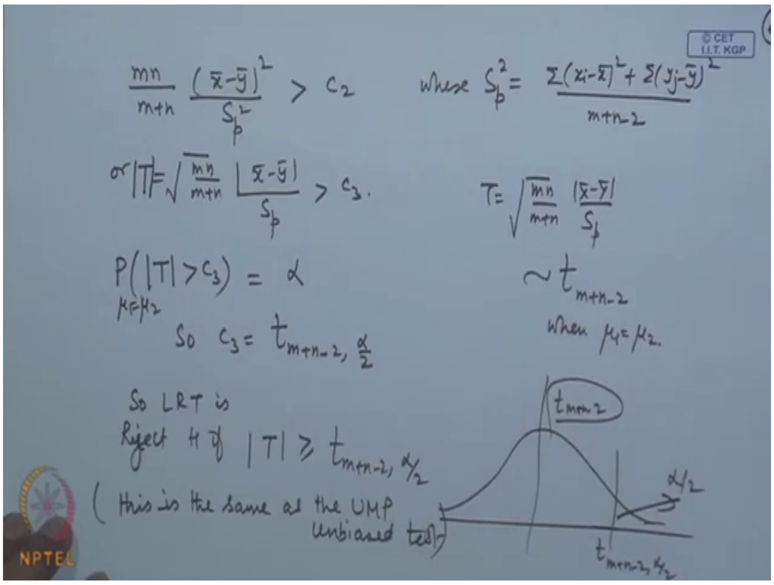

So, I get the term as m n by m plus n, x bar minus y bar square divided by S p square greater than some C 2 where, S p square is the pooled sample variance. That is sigma x i minus x bar square plus sigma y j minus y bar square divided by m plus n minus 2. So, so if I take the square root I get, square root m n by m plus n modulus of x bar minus y bar divided by S p greater than say C 3. Now, we will require let me give this term as T. Probability P of T greater than C 3. When mu 1 is equal to mu 2, it should be equal to alpha.

Now in yesterdays lecture I have given the distribution of this that is ah. So, this is modulus T. This term is T that is root m n by m plus n modulus of x bar minus y bar S p. So, this follows t distribution on m plus n minus 2 degrees of freedom, when mu 1 is equal to mu 2. So, C 3 we can take to be t m plus n minus 2 alpha by 2. That is the upper 100 alpha by 2 percent point.

So, this is t m plus n minus 2 alpha by 2 point. This probability is alpha by 2 if this is the curve of t distribution on m plus n minus 2 degrees of freedom that is the density function of this. So, you can see here likelihood ratio test is reject H if modulus of T is greater than or equal to t m plus n minus 2 alpha by 2. And this is same as the UMP unbiased test for this problem. So, once again this likelihood ratio procedure leads to excellent tests or you can say optimal tests as we have derived using the Neyman Pearson Theory.