Statistical Inference

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Lecture – 52
Likelihood Ratio Tests- II

Fortunately, for the Likelihood Ratio Test the asymptotic properties do hold; that means, asymptotic distinction of the test statistic is nice, it becomes actually the chi square. But before that let me also consider the other alternative here; what is the other alternative? That is H 4. So, I have considered here the testing problem which is a specified by H 1 that is mu less than or equal to 0 and mu greater than 0.

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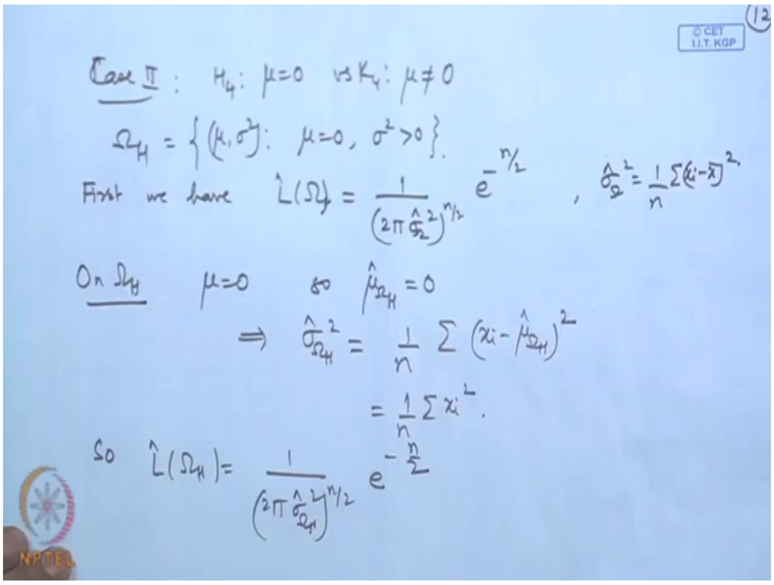

Now, let me modify and I will consider second case of the mu is equal to 0 against mu not equal to 0 that two sided alternative. In this situation we had seen the UMP unbiased test was based on the t statistic. In fact, the same thing modulus of its square root and x bar by s was greater than or equal to t n minus 1 alpha by 2. So, here what is happening? The omega H becomes mu is equal to 0 and sigma square is greater than 0; if that happens. So, now, let us see first we have L hat omega.

So, this will be as before because am not doing fresh calculations it is 2 pi sigma omega hat square to the power n by 2 e to the power minus n by 2, where sigma omega hat square was 1 by n sigma xi minus x bar whole square. Now, on omega H mu is equal to

0; so there is only one point here there is no question of further maximization; so this is equal to 0. If that happens then sigma omega H hat square which I calculated in the previous case what was happening that the value was dependent upon mu only.

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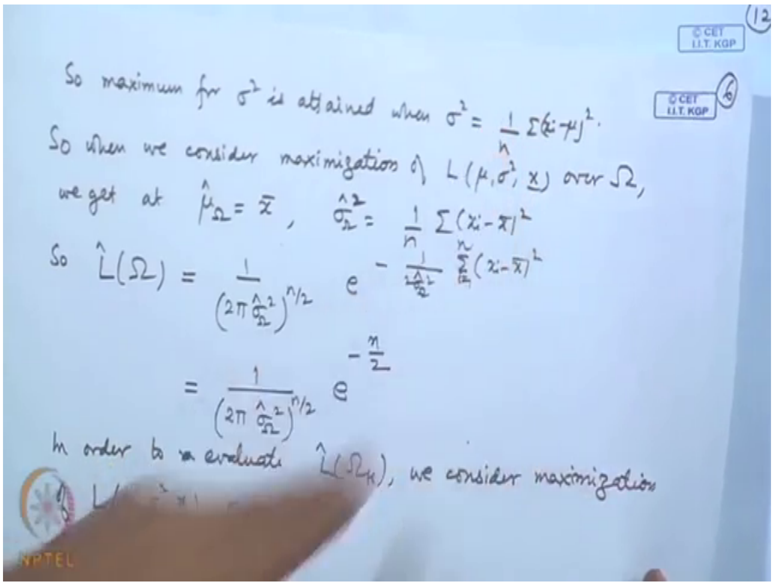

The value was dependent upon the mu; it was 1 by n sigma xi minus mu hat mu square. So, if I put mu is equal to 0; I will get the sigma omega H hat square. So, this becomes 1 by n sigma xi minus mu omega H hat square that is equal to 1 by n sigma xi square. So, if we substitute this we get L hat omega H is equal to 1 by 2 pi sigma omega H hat square to the power n by 2; e to the power minus n by 2.

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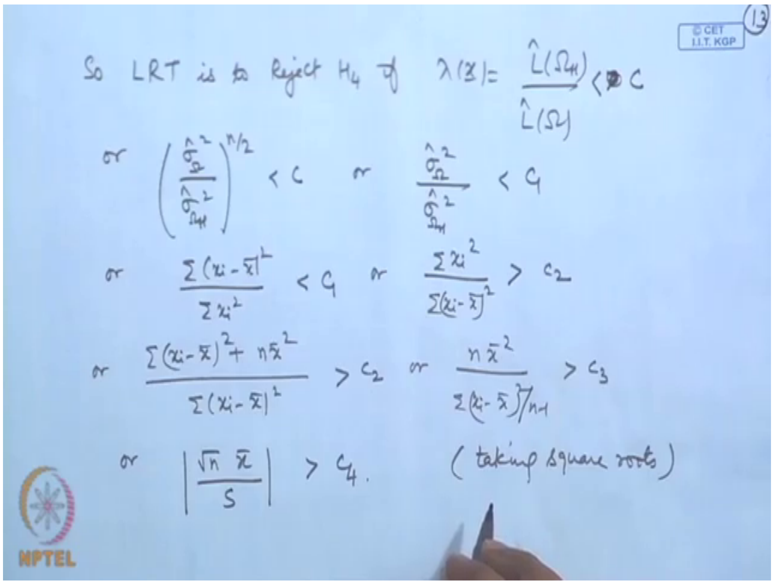

So, if I take the likelihood ratio test this is to reject H 4 if lambda x that is equal to L hat omega H by L hat omega is greater than sorry. It is less than C. So, this is equivalent to sigma omega hat square by sigma omega H hat square to the power n by 2 less than C or sigma omega hat square by sigma omega H hat square less than say C 1. Or if I substitute the values here I get sigma xi minus x bar square by sigma xi square less than C 1 or if I take the reciprocal is greater than say C 2.

And as before this I can write as sigma xi minus x bar square plus nx bar square divided by sigma xi minus x bar whole square greater than C 2 or nx bar square divided by sigma xi minus x bar square divided by n minus 1 greater than some C 3 or square root n x bar divided by S modulus greater than say C 4 that is I have taken the square roots.

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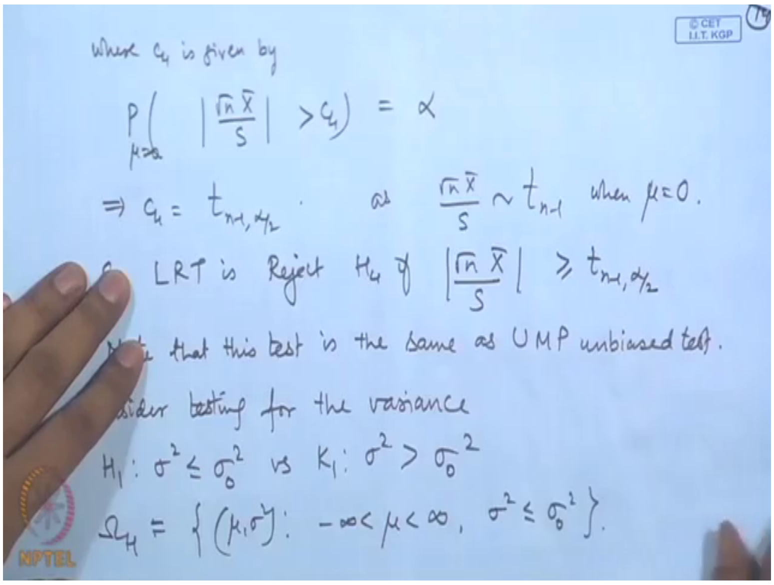

Now, to determine C 4 where C 4 is given by probability of modulus root n x bar by s greater than C 4 at mu equal to 0 is equal to alpha. This means C 4 is nothing, but t n minus 1 alpha by 2 because root n X bar by S follows t distribution on n minus 1 degrees of freedom when mu is equal to 0.

Now, so the test has become; so likelihood ratio test is reject H 4 if root n X bar by S modulus is greater than or equal to t n minus 1 alpha by 2; which is actually the UMP unbiased test for. Note that this test is the same as UMP unbiased test. So, which the point which I mentioned, that in many situations the likelihood ratio test leads to the same theory as in the UMP unbiased tests.

Now let us consider testing for the variance; let us consider say H 1 sigma square say less than or equal to sigma naught square versus K 1 sigma square greater than sigma naught square. What will be required is the behavior of the likelihood function that I wrote. So, once again let us go back to the behavior of the likelihood function which I wrote in the sheet number 5 yeah this was the behavior.

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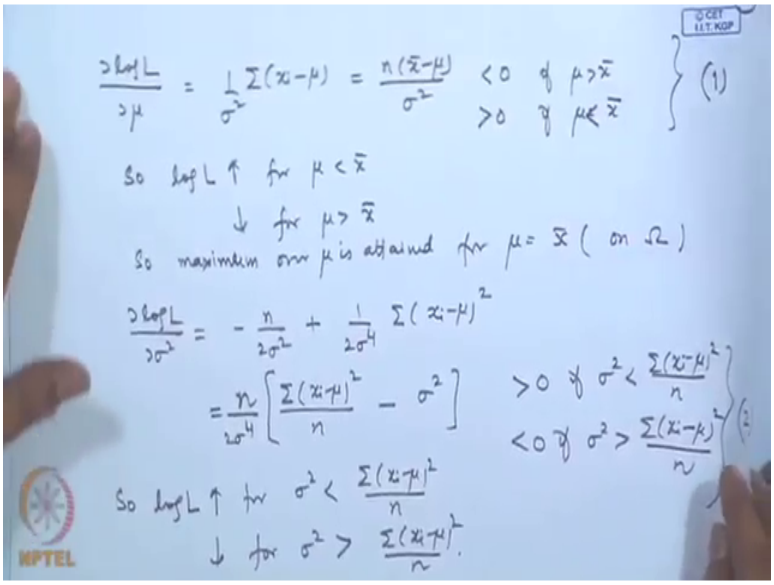

We had written the derivative with respect to mu and derivative with respect to sigma square in the equations 1 and 2. So, here you see if I am considering the omega H; in the omega H mu is on the whole real line and sigma square is less than or equal to sigma naught square.

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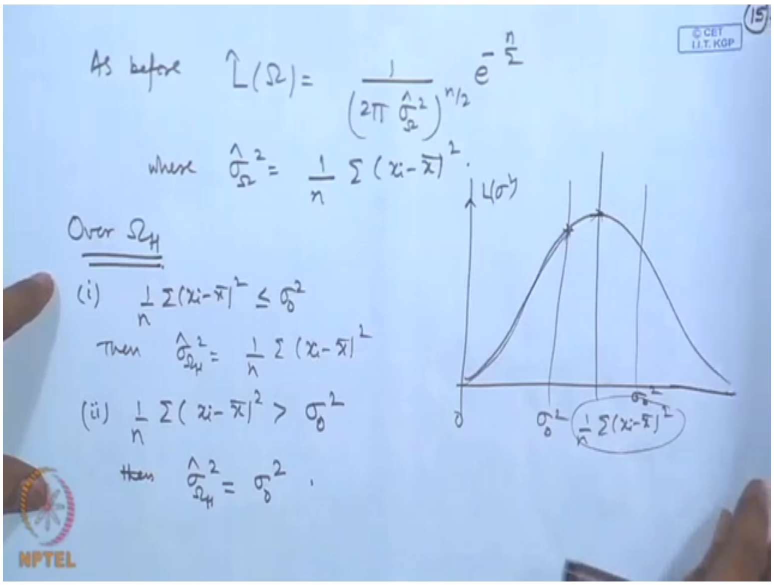

So, as before if I consider L hat omega that does not change 2 pi sigma omega hat square to the power n by 2 e to the power minus n by 2; where this sigma omega hat square was 1 by n sigma xi minus x bar square.

However, if I am considering over omega H then we look at this behavior here that that is the behavior of log L with respect to sigma square. We have seen it that; so this is 0 this point is 1 by n sigma xi minus x bar whole square. So, this is increasing up to this point and decreasing there after.

So, there can be two cases if 1 by n sigma xi minus x bar square is less than or equal to sigma naught square; that means, sigma naught square is say here. Then the maximum will be as before at this point; then sigma omega H square will be 1 by n sigma xi minus x bar whole square. Whereas, the other case can be that 1 by n sigma xi minus x bar square is greater than sigma naught square; that means, this value is coming here.

So, if am looking at this likelihood function with respect to sigma square; then it is increasing and thereafter we do not consider because the maximization ranges from 0 to sigma naught square; so this value is the maximum. Then sigma omega H hat square that is equal to sigma naught a square.

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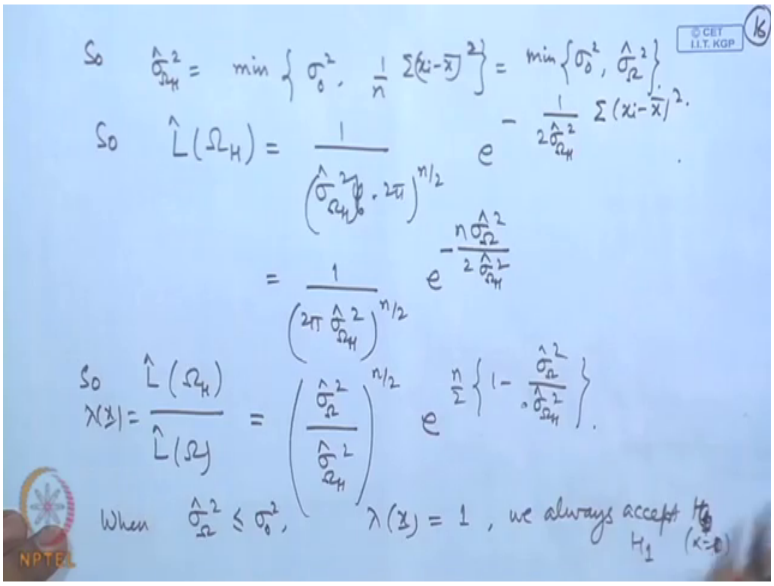

So, what we have got that sigma omega H hat square is actually minimum of sigma naught square and 1 by n sigma xi minus x bar square; that is minimum of sigma naught square and sigma omega hat square. So, L hat omega H; that is 1 by sigma omega H square sorry in to 2 pi to the power n by 2; e to the power minus 1 by 2 sigma omega H hat square; sigma xi minus x bar whole square; that is equal to 1 by 2 pi sigma omega H hat square to the power n by 2 e to the power minus sigma n sigma omega hat square divided by twice sigma omega H hat square.

So, L hat omega H divided by L hat omega; if I take this ratio it is becoming sigma omega hat square divided by sigma omega H hat square to the power n by 2; e to the power n by 2, 1 minus sigma omega hat square divided by sigma omega H hat square. Now there will be two cases if sigma omega hat square is less than sigma naught square then sigma omega H and sigma omega is equal. So, this will become 0 this will become 1. So, when sigma omega hat square is less than or equal to sigma naught square; this lambda x this ratio is 1. So, we always accept H 4 sorry this is H 1; we always accept H 1 that is alpha is equal to 0.

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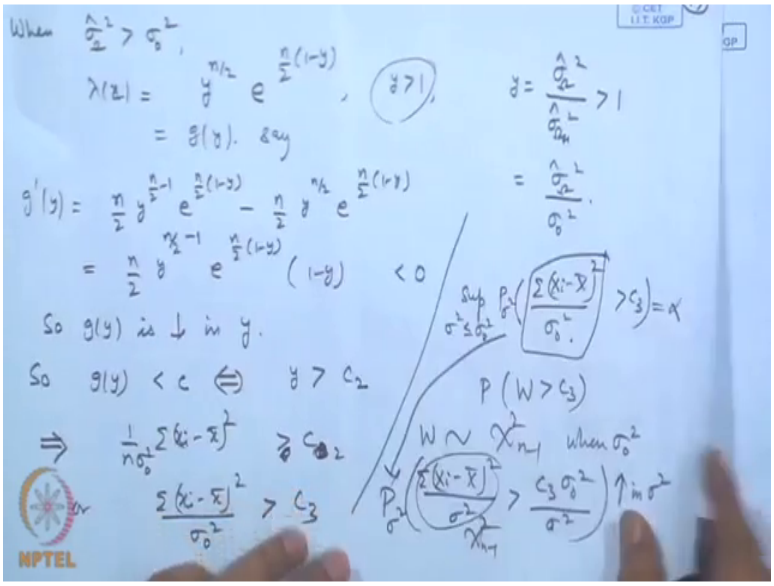

When sigma omega hat square is greater than sigma naught square then lambda x is equal to this value I write it in the form y to the power n by 2 e to the power n by 2 into 1 minus y where y is greater than 1; y is sigma omega hat square divided by sigma omega H square that is equal to actually; in this particular case when sigma square is greater than sigma naught square in that case this value is turning out to be sigma omega hat square divided by sigma naught square.

Let us look at this function I call it say g of y then what is g prime y? If I look at the derivative of this the derivative of this is n by 2 y to the power n by 2 minus 1 e to the power n by 2 into 1 minus y minus n by 2 y to the power n by 2; e to the power n by 2 into 1 minus y, that is equal to n by 2 y to the power n by 2 minus 1; e to the power n by 2 into 1 minus y into 1 minus y.

So, this is less than 0 because y is greater than 1 what I have taken this to be y. And I have considered the case when sigma naught square is greater than sorry sigma naught square is less than sigma omega hat square; so this quantity is greater than one. So,; so this is always less than 0; so, what we are saying is that g y is decreasing in y. If it is decreasing in y then the region g y less than C; this is equivalent to saying y is greater than C 2.

Now y is 1 by n sigma xi minus x bar whole square divided by sigma naught square greater than or equal to C or sigma xi minus x bar whole square by sigma naught square greater than say sorry C 3; I may say this is equal to C 3 here. Now we should have supremum of probability sigma Xi minus X bar square by sigma naught square greater than C 3; this is for sigma square less than or equal to sigma naught square, this should be equal to alpha.

I want this probability to be equal to alpha. Now let us look at this follows this is probability of W greater than C 3 where W follows chi square distribution on n minus 1 degrees of freedom; when sigma naught square is (Refer Time: 16:52). So, if I write here sigma xi minus x bar square by sigma square greater than C 3 sigma naught square by sigma square, this probability is equal to this.

Now this is a chi square n minus 1 variable; since I am considering the region sigma square less than or equal to sigma naught square; this value is greater than 1. So, if I increase sigma square; if I increase sigma square this value will increase ah, this value will decrease why? Right now sigma square is less than sigma naught square sigma square is less than sigma naught square. So, if I increase sigma square this value will decrease; if this value decreases the probability of the whole region will increase; so, this is increasing in sigma square.

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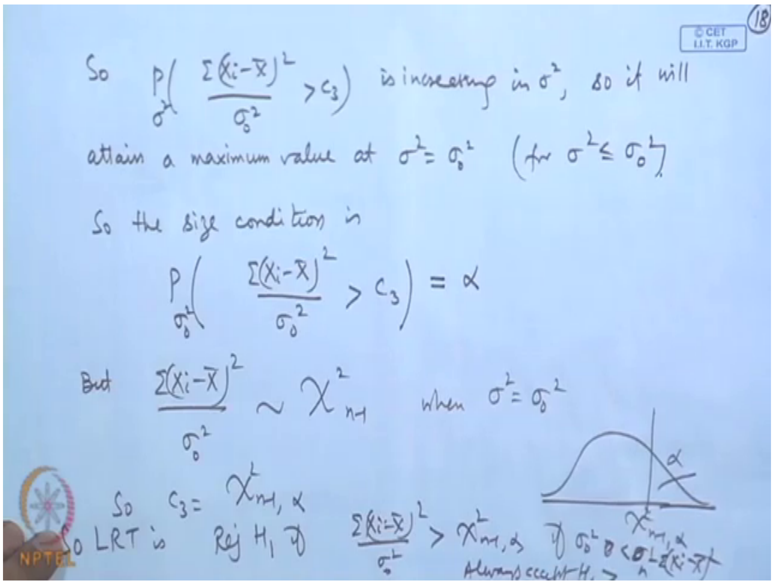

So, this probability sigma xi minus x bar Whole Square by the sigma naught square greater than C 3 is increasing in sigma square. So, it will attain a maximum value at sigma square is equal to sigma naught square for sigma square less than or equal to sigma naught square.

So, the size condition is probability of sigma Xi minus X bar whole square sigma naught square greater than C 3, when sigma naught square is the true parameter value is alpha. But sigma Xi minus X bar Whole Square by sigma naught square follows chi square distribution on n minus 1 degrees of freedom when sigma square is equal to sigma naught square.

So, C 3 is equal to chi square n minus 1 alpha that is the upper 100 alpha percent point of chi square distribution on n minus 1 degree of freedom. So, likelihood ratio test is reject H 1 if sigma Xi minus X bar whole square by sigma naught square is greater than chi square n minus 1 alpha. This is when sigma naught square is greater than is less than sigma omega that is 1 by n sigma Xi minus X bar Whole Square and if it is greater then always accept H 1; always accept H 1.

As again you can see this is similar to the UMP unbiased test which we derived in the previous lecture. In the next lecture I will continue derivation of the likelihood ratio test for various problems related to normal populations and also some other discrete and continuous distributions.