Statistical Inference
Prof. Somesh Kumar
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture – 49
Unbiased Test for Normal Populations – III

In the previous lecture, I have started discussing test how to obtain the UMP Unbiased Test for the variance of a Normal Population when both the parameters are unknown. So the model, let me recollect here.

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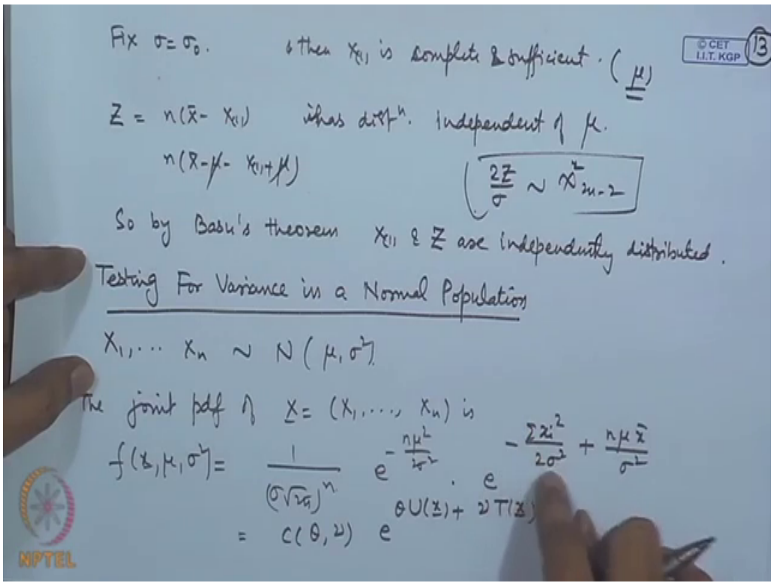

We considered a random sample from normal mu sigma square distribution. The joint density function of X 1, X 2, X n, I am expressing in the form of a two parameter exponential population. C theta nu e to the power theta U x plus nu T x where, I am defining theta as minus 1 by 2 sigma square, U is equal to sigma X i square, nu is equal to n mu by sigma square and T is equal to X bar.

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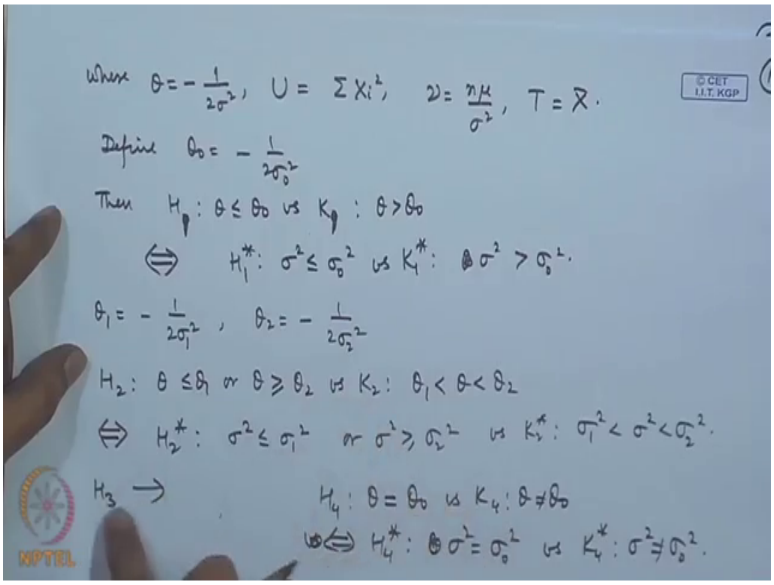

I have shown that the four testing problems H 1, H 2, H 3 and H 4 they are equivalent to testing about sigma square. So, and they are having the same form that is because theta naught theta is equal to minus 1 by 2 sigma square. This is an increasing function of sigma square. Therefore, all the inequalities or equalities are maintained. That is, theta less than or equal to theta naught is equivalent to sigma square is less than or equal to sigma naught square; if I define theta naught to be equal to minus 1 by 2 sigma naught square.

Similarly, if I define theta 1 is equal to minus 1 by 2 sigma 1 square and theta 2 is equal to minus 1 by 2 sigma 2 square, then theta less than or equal to theta 1 or theta greater than or equal to theta 2 is equivalent to sigma square less than or equal to sigma 1 square or sigma square greater than or equal to sigma 2 square. And similarly, theta 1 less than theta less than theta 2 is equivalent to sigma 1 square less than sigma square less than sigma 2 square and so on.

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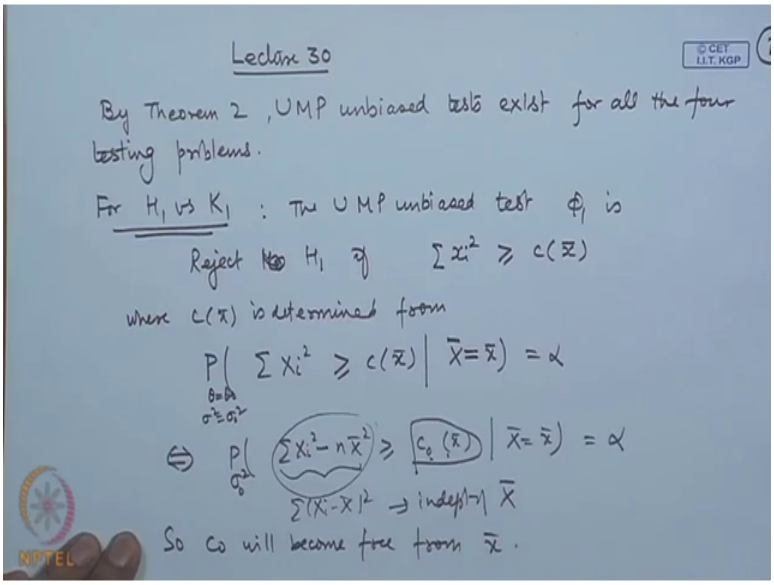

Therefore the, by theorem 2 which I discussed yesterday, UMP unbiased tests, UMP unbiased tests exist for all the four testing problems. So, let me take up H 1 versus K 1. So now, we are dealing with the continuous distributions.

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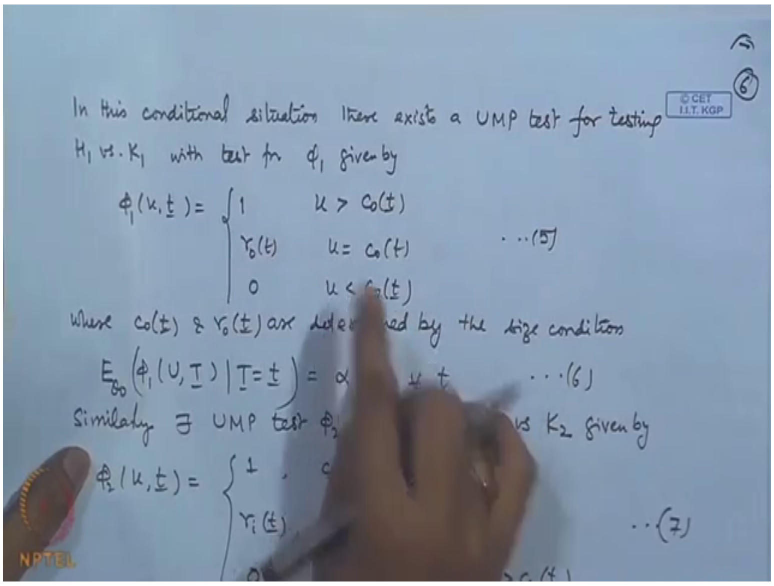

Therefore in the test function, the term gamma naught, this term I need not write because the probability of this u is equal to C naught t will be 0. So, we do not write this rather, we incorporate it in either this part or here. So, if I take this to be 0 for example, then this will be incorporated here.

So, the UMP unbiased test, say phi 1 is reject H naught sorry H 1 if U that is, sigma x i square is greater than or equal to a function of t that is x bar; and this C x bar is determined from probability of sigma X i square greater than or equal to C x bar given X bar is equal to say small x bar is equal to alpha when, theta is equal to theta naught or say sigma square is equal to sigma naught square.

Now note here, this meets the conditional distribution of sigma X i square even given X bar which is slightly inconvenient. However, here we can apply a trick here. This is equivalent to saying probability of sigma X i square minus n X bar square greater than or equal to some other function say C naught x bar given X bar is equal to x bar is equal to alpha at sigma naught square.

Now, this sigma X i minus X bar whole square, this is independent of X bar. If this is independent of X bar, this term will become free from; so C naught will become free from x bar. Now this is what we had required here.

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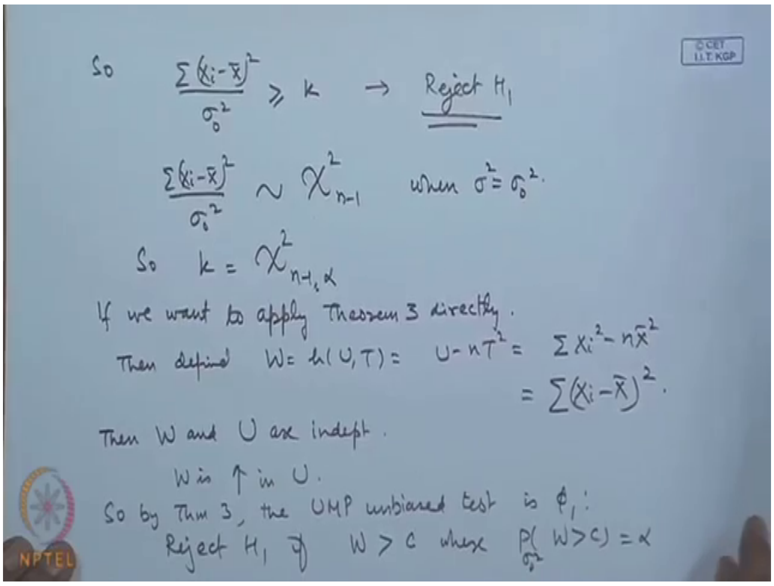

So, we can say sigma X i minus X bar whole square by sigma naught square greater than or equal to say k. So, since sigma X i minus X bar whole square by sigma naught square this follows chi square distribution on n minus 1 degrees of freedom when sigma square is equal to sigma naught square.

So, this k will become equal to upper 100 alpha percent point of chi square distribution on n minus 1 degrees of freedom. So, we have got an exact test. So, this is the reject H 1. We are getting the level alpha test. This is UMP unbiased test; if we want to apply theorem say, 3 directly, then define W is equal to h UT is equal to U minus n T square that is equal to sigma X i square minus n X bar square that is sigma X i minus X bar whole square.

Then W and U are independent. So, for sigma square is equal to sigma naught square also they will be independent and W is increasing in U. So by theorem 3, the UMP unbiased test is phi 1 reject H 1 if W is greater than some C where, probability of W greater than C at sigma naught square is equal to alpha; which is the same.

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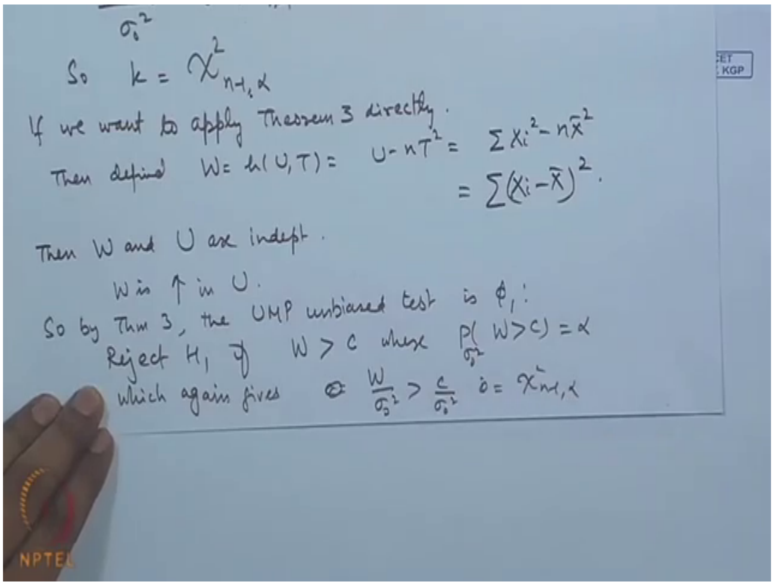

Which again gives c that is, W by sigma naught square greater than C by sigma naught square that is equal to chi square n minus 1 alpha.

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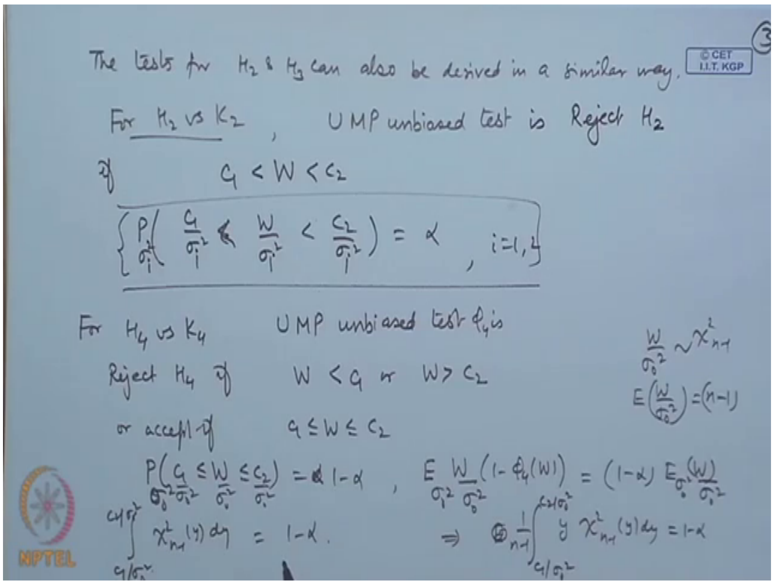

The tests for H 2 and H 3 can also be derived in a similar way. Let me write for one of them. For H 2 versus K 2 problem, UMP unbiased test is reject H 2 if C 1 is less than W is less than C 2 and you will have a probability of C 1 by sigma 1 square less than or equal to W by sigma 1 square less than C 2 by sigma 1 square is equal to alpha. Then sigma 1 square is the true parameter value and also when sigma 2 square is also coming.

So, these two conditions will give the value of C 1 and C 2 for H 4 versus K 4 problem. Now here, UMP unbaised test is reject H 4 if W is less than C 1 or W is greater than C 2 or accept if C 1 is less than or equal to W less than or equal to C 2. You will have probability of C 1 less than or equal to W less than or equal to C 2 at sigma naught square is equal to alpha.

And there will be another condition that is expectation of W 1 minus phi 4. Let me call it phi 4. Phi 4 W at sigma naught square is equal to 1 minus alpha expectation sigma naught square W. Now in this W, you consider the division by sigma naught square. So for example, here this will become sigma naught square, this becomes sigma naught square, this becomes sigma naught square. So here also you consider division. So, W by sigma naught square follows chi square n minus 1.

So, expectation of W by sigma naught square is equal to n minus 1. So this condition, second condition will become then expectation of well you can write integral 1 by n minus 1 y of chi square n minus 1 y dy. This is the density function of a chi square variable on n minus 1 degrees of freedom from C 1 by sigma naught square to C 2 by sigma naught square is equal to 1 minus alpha and this condition is actually equal to chi square n minus 1 density from C 1 by sigma naught square to C 2 by sigma naught square is equal to 1 minus alpha.

So, these two conditions will give the value of C 1 and C 2. I have demonstrated here that we can apply theorem 3; that means, we can suitably define the function W such that we are getting the UMP unbiased test for the variance testing. Let us also consider now the testing for the mean; testing for mean in a normal population.

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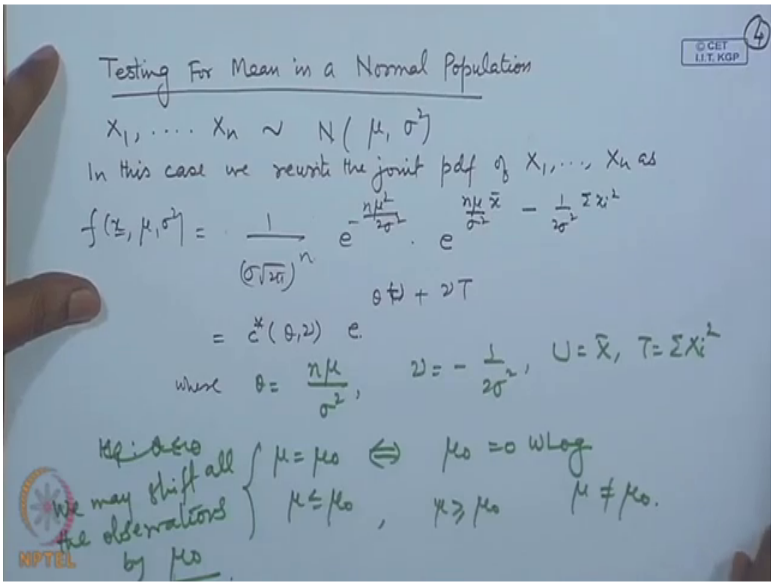

Let us go back to the original expression that I wrote for the joint density of normal distribution of X 1, X 2, X n here. Here I took theta to be minus 1 by 2 sigma square and nu to be n mu by sigma square and therefore, I was able to test for sigma square. If I want to test for mu then I have to change the role of theta and nu here. So, then I write it here. So, model is the same; that means, X 1, X 2, X n follows normal mu sigma square.

Now in this case, we rewrite the joint pdf of X 1, X 2, X n as e to the power minus n mu square by 2 sigma square. E to the power n mu x bar by sigma square minus 1 by 2 sigma square sigma x i squared and this I call c star theta nu, e to the power theta T, sorry theta nu plus nu T. Now here, I have changed the rules. Here, theta is equal to n mu by sigma square, nu is equal to minus 1 by 2 sigma square, U is equal to X bar, T is equal to sigma X i square.

Ah Let me restrict attention to H 1 and H 4. So, H 1 would be theta less than or equal to 0. Ok if I want to test for say mu is equal to mu naught, then this is equivalent to that I can take mu not to be 0 without loss of generality. If I take mu less than or equal to mu naught r and so on. Mu greater than mu naught or mu not equal to mu naught because we can shift all the observations by mu naught.

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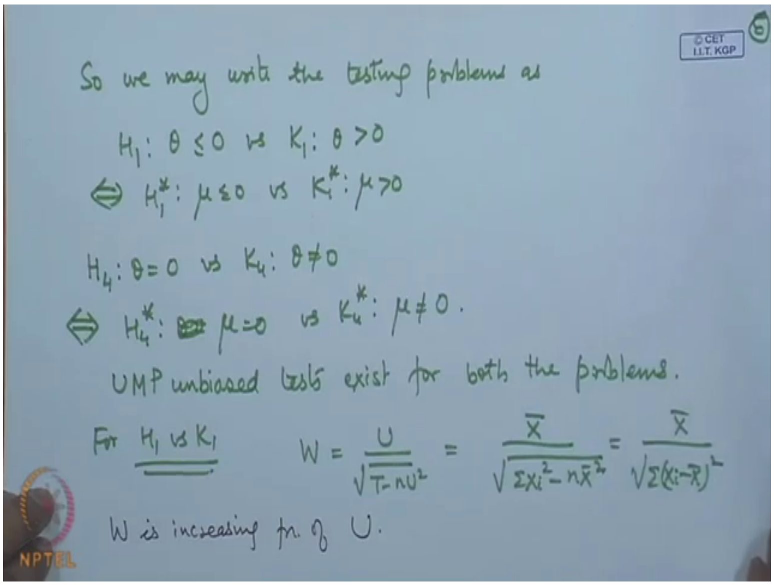

So my testing problem can be then written as. So, we may write the testing problems as; so for example, if I am looking at say H 1 theta less than or equal to 0 versus K 1 theta greater than 0 then this is equivalent to say H 1 star mu less than or equal to 0 versus K 1 star mu greater than 0. Similarly, if I consider say H 4 theta is equal to 0 versus K 4 theta is not equal to 0 then, this is equivalent to H 4 star theta, sorry mu is equal to 0 versus K 4 star mu is not equal to 0.

So, let me UMP unbiased tests will exist ok. UMP unbiased tests exist for both the problems. So now, let us consider say for H 1 versus K 1 problem. For this problem, I define W is equal to U by square root T minus nU square. That is, X bar divided by root sigma X i square minus n X bar square. So this is nothing but, X bar divided by square root sigma X i minus X bar whole square.

If you see this carefully, then W is increasing function of U and if we consider the distribution of W and T. The distribution of T when mu is equal to 0 then that is free from mu; and if I consider the distribution of W, that is also free from when mu is equal to 0 it is free from. So, let me just write it here.

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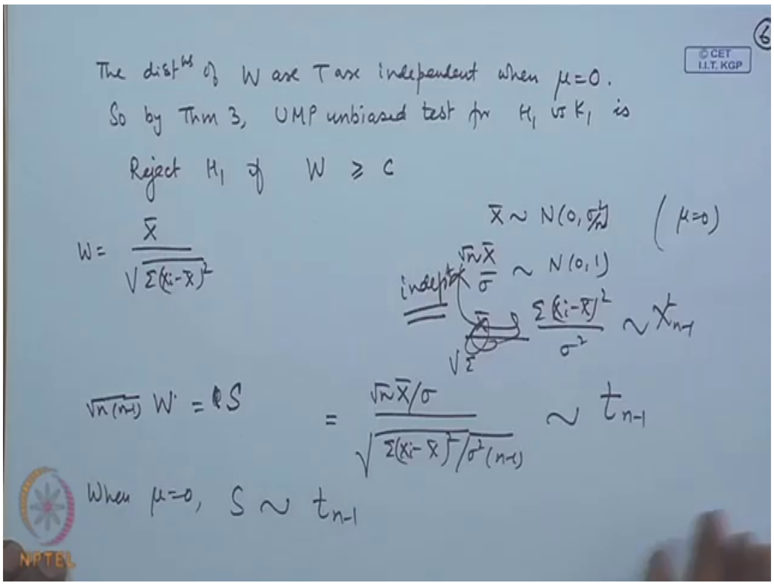

The distributions of W and T are independent when mu is equal to 0. So by theorem 3, UMP unbiased test for H 1 versus K 1 is reject H 1 if W is greater than or equal to C. Now, in order to look at the distribution of W. Cv W is X bar divided by root sigma X i minus X bar whole square. So, we may consider here X bar follows normal 0 sigma square when mu is equal to 0. So, X bar by sigma follows normal 0, 1. If I look at X bar divided by a root sorry, sigma X i minus X bar whole square by sigma square that follows chi square n minus 1 and these two are independent. These two are independent.

So, if I look at the ratio X bar by sigma divided by square root sigma X i minus X bar whole square by sigma square into n minus 1, that will follow t distribution on n minus 1 degrees of freedom. Now, there is a small mistake here this will be divided by n. So here I will have to put a square root n. So, this is square root n; that means, it is equal to square root n into n minus 1 W. So, we can modify, we can define this as say let me call it say S. So, when mu is equal to 0, S follows t distribution on n minus 1 degrees of freedom.

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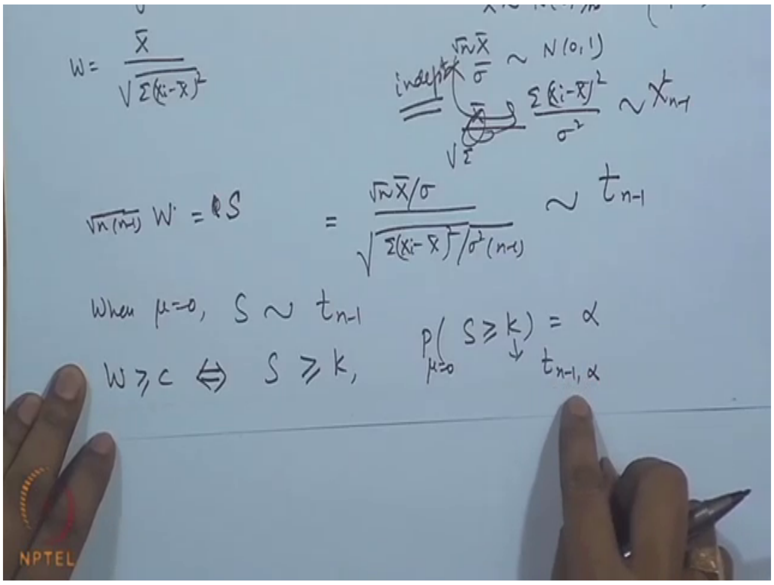

So the region W greater than or equal to C is equivalent to S greater than or equal to some K where probability of S greater than or equal to K when mu is equal to 0 should be equal to alpha; that means, K value is nothing, but t n minus 1, alpha. That is the upper 100 alpha percent point of the t distribution on n minus 1 degrees of freedom. So, the exact test has been derived.

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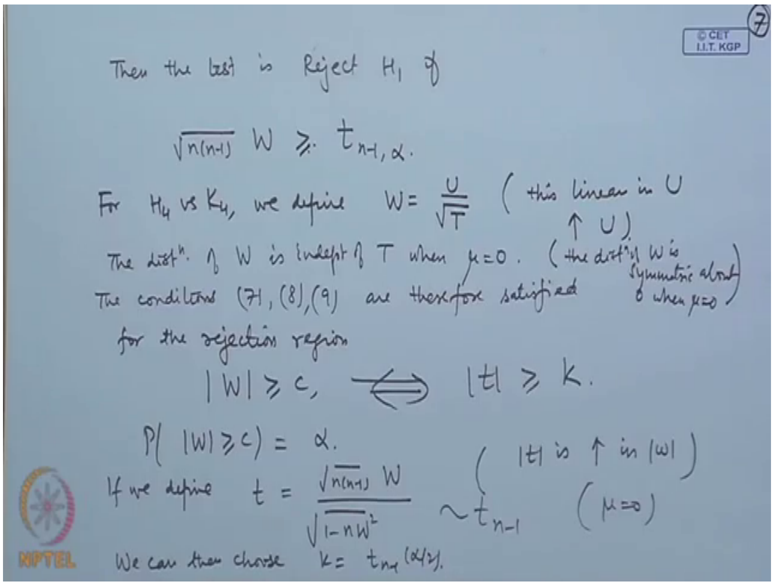

If I use the notation which I have developed here then, the test is reject H 1 if root n into n minus 1 W is greater than or equal to t on n minus 1, alpha. So, this is the exact test.

We may include equality or we may not include any equality here at this point, it does not make any difference.

If you look at this function here that I have defined, U divided by square root T minus n U square. This is not a linear function of T; a linear function of U; although it is increasing in U but it is not linear. So, if I want to test the hypothesis, H 4 versus K 4, I cannot use this. So, further I define another function for H 4 versus K 4, we define W is equal to U divided by square root of T.

Now, this is linear in T, linear in U, increasing in U and the distribution of W is independent of T when mu is equal to 0. So, the conditions that I wrote in H 4 versus K 4 test for the test function phi 4. Let us recollect those conditions. The conditions I wrote as the form of the phi 4 function the condition, that expectation of phi 4 should be equal to alpha at theta naught also, expectation of theta naught W phi 4 W. There should be alpha times expectation of phi 4 W. These conditions 7, 8, 9 they should be satisfied here.

So, the conditions 7, 8, 9 are therefore, satisfied. Ok we note here, the distribution of W is symmetric about 0 when mu is equal to 0. That is important here. The condition 7, 8,-9 are therefore, satisfied for the rejection region modulus W greater than or equal to C and probability of modulus W greater than or equal to C is equal to alpha. So, if we define say a small t is equal to square root n into n minus 1 W divided by square root 1 minus nW square then, modulus t is increasing in modulus W.

So, this region is then equivalent to modulus t greater than or equal to K and the distribution of t is nothing but t distribution on n minus 1 degrees of freedom when mu is equal to 0. So, we can choose then this K as t n minus 1 alpha by 2. This is the famous t test which was initially derived by Gossett in 1907. Of course, he did not consider it as a UMP umbiased test, he considered it as a likelihood ratio test only. But here, we are able to derive the exact test when sigma is unknown. So, this is testing for the mean.