Statistical Inference
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Lecture – 47
Unbiased Test for Normal Populations – I

Yesterday, we have discussed in the previous lecture UMP unbiased tests for the multi parameter exponential family of distributions let me just recollect the discussion.

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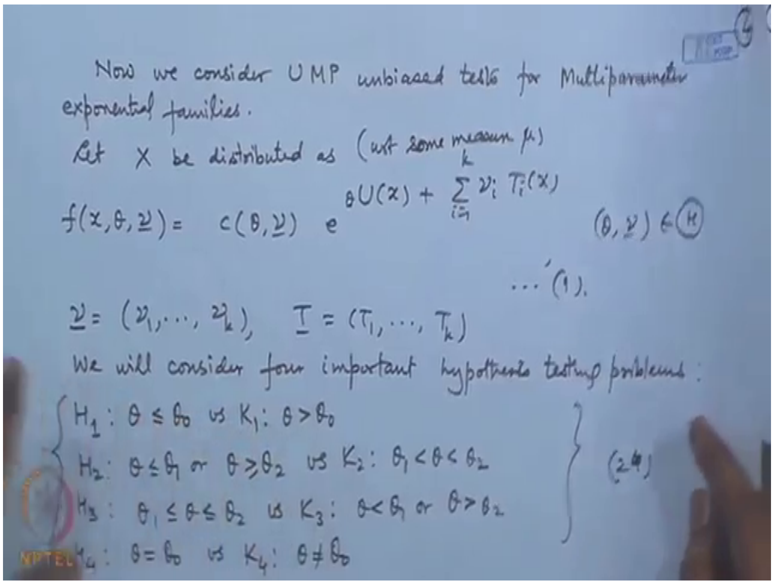

We had considered a distribution of the type e to the power theta U x plus sigma nu i T i x. Here we are able to derive the UMP unbias test for 4 types of hypothesis called H 1 H 2 H 3 H 4 versus K 1 K 2 K 3 K 4 respectively for theta and here nu 1 nu 2 nu K. We are considered as the nuisance parameter of course, we had considered here that the parameter space is K plus 1 dimension and convex also.

A peculiar nature of these tests was that these tests were conditional.

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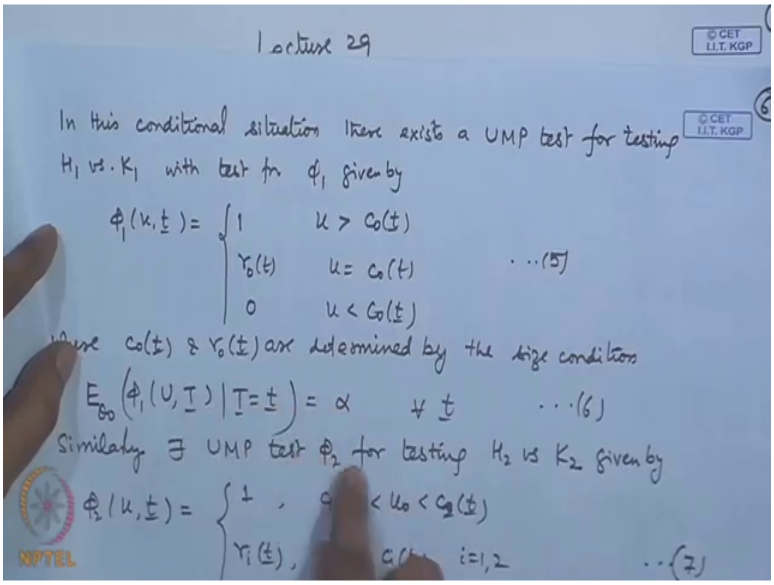

Let me show the form here for at least one of them. So, if you recollect the form the tests were of the form phi 1. So, I am calling it condition on u greater than C naught t and here this term is gamma naught t and this C naught and gamma naught are determined from the conditional thing. A extension of this was done later on where we called this as the unconditional tests also.

However the derivation of this we demonstrated by application to comparison of two binomial proportions two Poisson arrival rates and so, on and in also in testing for independence and contingency table in each of these cases we saw that we have to actually determine the conditional distribution of u given t.

Now in the case of continuous distributions for example if we consider normal distributions gamma distributions etcetera then these conditional distributions are not so, easy. Because in the discrete case we are able to write down the conditional distribution in terms of probability and we are able to apply the formula for the condition probability that is probability of a given b is equal to probability of a intersection b divided by probability of b, and we are able to actually derive the exact form of the test.

In the case of a continuous, this conditional distribution may not be so, easy. Therefore, we apply a method by which we can modify these conditions u greater than or u less than some constant which is dependent upon t to something some another statistic let us call it say w so, that this condition becomes free from t. That means, this new w variable which

I am saying it could be a function of u and t should be defined in such a way that first of all it should be an increasing function or monotonic function so, that the conditions of inequality remain same or they get reversed.

And second thing is that the independence. So, fortunately there is a method and I will explain that method now in this lecture here.

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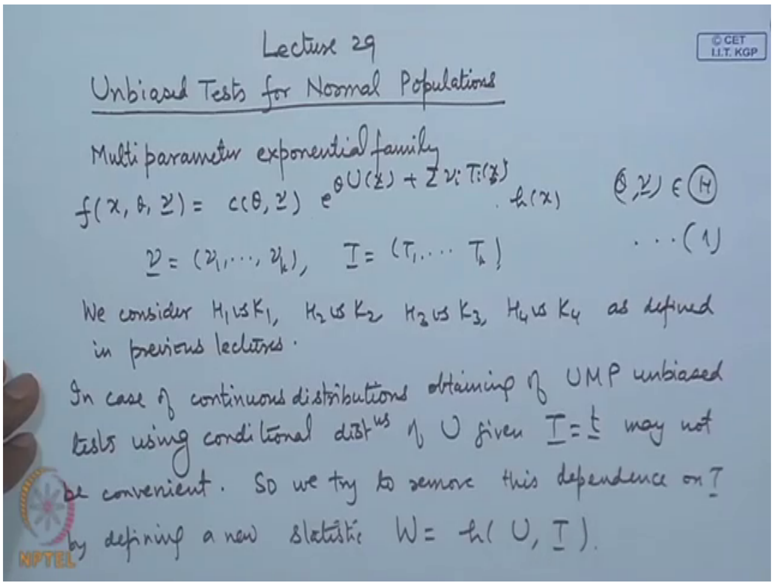

So, let us consider multi parameter exponential family; multi parameter exponential family. So, the density is of the form let me write from the yesterdays discussion f of x theta nu that is equal to some constant times e to the power theta U x plus sigma mu i T i x and of course, there will be some term here which is consisting of x here. Here nu is equal to nu 1 nu 2 nu K and t is equal to T 1 T 2 T k and this theta nu belongs to certain parameter space say script theta.

So, I call this density 1 as before I consider hypothesis testing problems H 1 versus. So, we consider H 1 versus K 1, H 2 versus K 2, H 3 versus K 3 h 4 versus K 4 as defined in previous lectures.

In case of continuous distributions obtaining of UMP unbiased tests using conditional distributions of U given T is equal to t may not be convenient. So, we try to remove this dependence on T by defining a new statistic let us call it say w is equal to a function of U and T.

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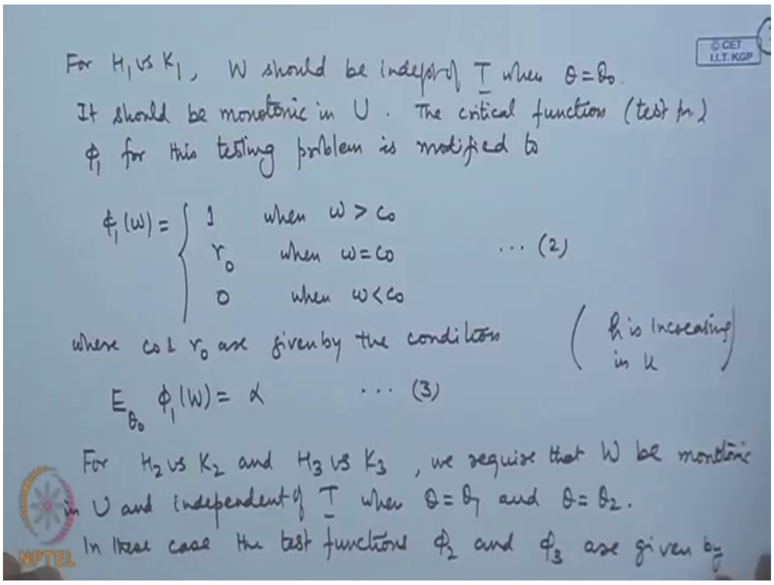

Now, the conditions that we need to impose are as follows for H 1 versus K 1, W should be independent of T when theta is equal to theta naught. Remember here the hypothesis H 1 here the condition was based on theta naught if you remember the exact condition here. Let me recollect from yesterdays lecture expectation theta naught here similarly if you see phi 2 and phi 3 here the condition is on theta 1 and theta 2. And similarly for h 4 the condition is on again theta naught the condition is on theta naught. So, we have to look at the independence of W from T at these points

So, W should be independent of theta of T when theta is equal to theta naught it should be monotonic in U the critical function or the test function phi 1 for this testing problem is modified to phi 1 of w is equal to 1 when w is greater than C naught it is equal to gamma naught when w is equal to C naught it is equal to 0 when w is less than C naught where C naught and gamma naught are given by the condition expectation of phi 1 w is equal to alpha.

One thing we should notice here I have maintained the same sign like here U was greater than C naught and here it is w is greater than C naught. If we do that then we have assumed that h is increasing in u if h is decreasing in u then this will get reversed here. For H 2 versus K 2 and H 3 versus K 3 we require that W be monotonic in U and independent of t when theta is equal to theta 1 and theta is equal to theta 2. So, in these cases the test functions phi 2 and phi 3 are given by phi 2 w is equal to 1, then c 1 is less than w less than c 2 it is equal to gamma when w is equal to c i, i is equal to 1 2 and it is equal to 0.

If w is less than c 1 or w is greater than c 2 and c i is and gamma is are determined by expectation of phi 2 W at theta 1 and expectation of theta 2 phi 2 W is equal to alpha.

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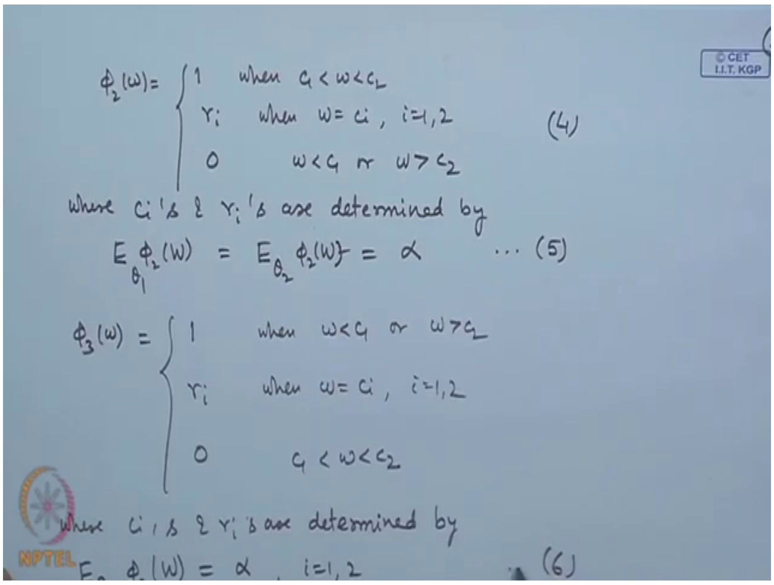

Similarly phi 3 is given by the reciprocal of this one that is it will become 1 when w is less than c 1 or w is greater than c 2 it is equal to gamma when w is equal to c i for i is equal to 1 2 and it is equal to 0 if c 1 is less than w is less than c 2. c is and gamma is are determined by the size conditions expectation theta i 3 w is equal to alpha for i is equal to 1 2.

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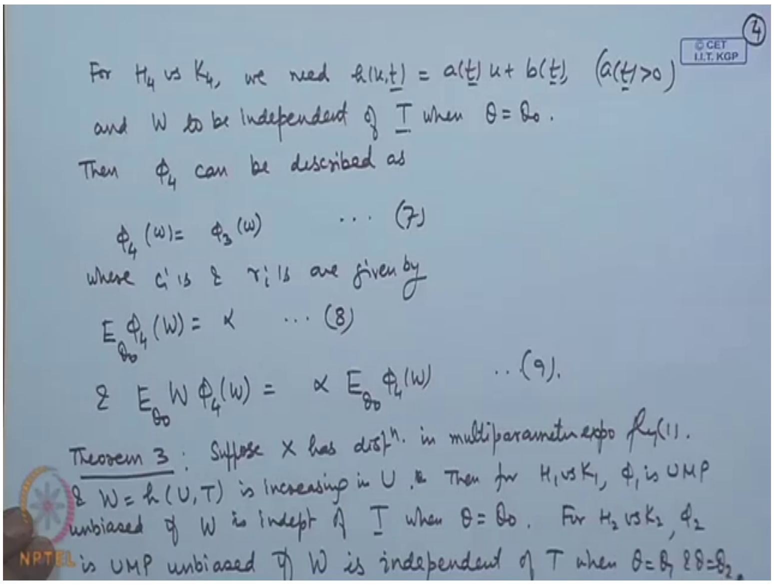

Now, for the testing problem H 4 versus K 4, now here if you remember for phi 4 we had two conditions. The second condition was involving product of U with phi 4. So, now, if you translate we need another condition that U should be H should be linear function

So, let me write it here we need h ut to be a linear function of u of course, again I am taking a 2 to be 0 positive if it is negative and the region will get reversed and w to be independent of t, when theta is equal to theta naught. Then this phi 4 can be described as phi 4 omega is equal to say phi 3 omega where c i is and gamma is are given by expectation of phi 4 w is equal to alpha at theta naught and expectation of w phi 4 w is equal to alpha times expectation of phi 4 w at theta naught.

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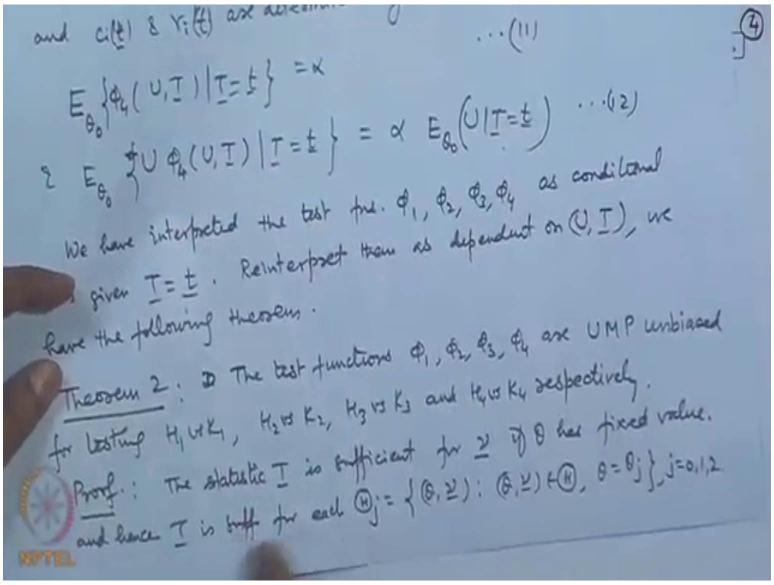

So, note here unlike the theorem 2 which I gave yesterday, we had there all the constants depending upon t that is they were calculating from the conditional distribution of u given t. But here you see that none of them is dependent upon t, the expressions that I have written they have become free from t. So, this is the advantage of this technique and in order to imply this technique, we need to suitably define this function H for various hypothesis testing problems.

So, I summarize all these results in the following theorem, let me call it theorem 3. Suppose X has distribution in multi parameter exponential family 1, and W is equal to h of U T is increasing in U and increasing in U. Then for H 1 versus K 1 phi 1 is UMP unbiased if W is independent of T when theta is equal to theta naught for H 2 versus K 2 phi 2 is UMP unbiased. If W is independent of t then theta is equal to theta 1 and theta is equal to theta 2.

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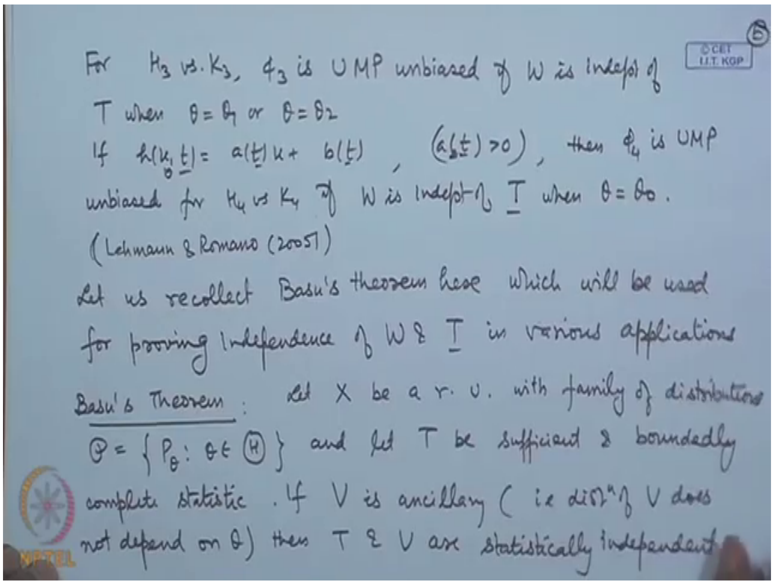

For H 3 versus K 3 phi 2 is UMP unbiased if W is independent of T then theta is equal to theta 1 or theta is equal to theta 2. If h of u t is of the form a t times u plus b t where a t is positive, then phi 4 is UMP unbiased for H 4 versus K 4, if W is independent of t when theta is equal to theta naught.

For the proofs one may look at the Lehmann and Romanos book as we have been following this theory from this text here. Now in order to have the independence of w and t we will usually require certain result which we can use for independence. Now one of the important results for approving independence now remember here either u or t they are sufficient or. In fact, if I were to look at this in the full version then u and t is complete and sufficient.

If we fix theta then t is sufficient and conversely if we fix nu is then u is sufficient. So, what happens that certain independence is there if I can use Basus theorem. So, sufficiency and then we should have completeness or bounded completeness and then we should have an celerity. Let me recollect the Basus theorem here which will be used for proving independence of w and t in various applications.

As I mentioned I will be briefly discussing mainly discussing the applications for testing problems in normal distributions. So, the Basus theorem statement let me repeat here, let X be a random vector with family of distributions P theta, theta belonging to theta and let T be sufficient and boundedly complete a statistic.

If V is ancillary that is distribution of V does not depend on theta then T and V are having independent distributions.

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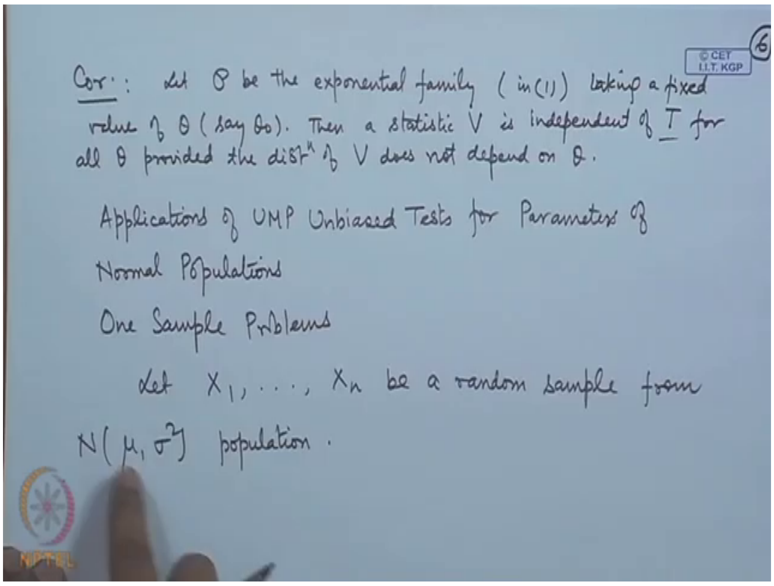

So, what is important is that we should have one sufficient and complete or sufficient and boundedly completed statistic, and another one should have a distribution free from the parameters. If that happens then T and V are statistically independent.

So, in general we will try that, that W should be complete and sufficient and then T should become ancillary or vice versa. As a corollary we have the following important result let P be the exponential family in 1 and taking a fixed value of say theta, then a statistic V is independent of T for all theta provided the distribution of V does not depend on theta.

If you look at the family in one if I fix the value of theta say theta naught, then sufficient statistic will become T 1 T 2 T k. Now if I have another statistic V whose distribution will not depend upon theta then certainly V and T will be independent. So, we will try to use this now let us consider applications to applications of UMP unbias tests for parameters of normal populations.

Ah To start with let us consider one sample problems, since we are having X 1 X 2 X n a random sample from normal mu sigma square distribution. Remember here I have discussed this normal distribution earlier also, I have considered testing for the mean of a normal distribution, testing for the variance in a normal distribution, but the crucial difference was that when I was testing for the mean I had considered variance to be known.

And accordingly the tests which were either UMP for H 1 and H 2 and for H 3 and H 4 it was u m p unbiased we were had obtained. Similarly for sigma square when I was doing the testing the nu was taken to be known and i have taken without loss of generality to be 0 and once again we had the UMP test for H 1 and H 2 and UMP unbiased test for H 3 and H 4.