**Statistical Inference**
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**Lecture – 41**
**UMP Unbiased Tests - I**

So, we have demonstrated in the last lecture that there are several problems. For example, when the distributions have monotone likelihood ratio or if the distributions are in the one parameter exponential family there are one sided testing problems or certain two sided testing problems for which UMP Unbiased Tests, a UMP tests can be derived.

However, we also demonstrated that there are certain cases where UMP test cannot be found. In fact, we have demonstrated that they do not exist. Now if that, happens then we can simply impose an additional condition which is called a condition of unbiasness and then, we may try to find out the UMP test among the unbiased tests. So, these are called UMP unbiased test.

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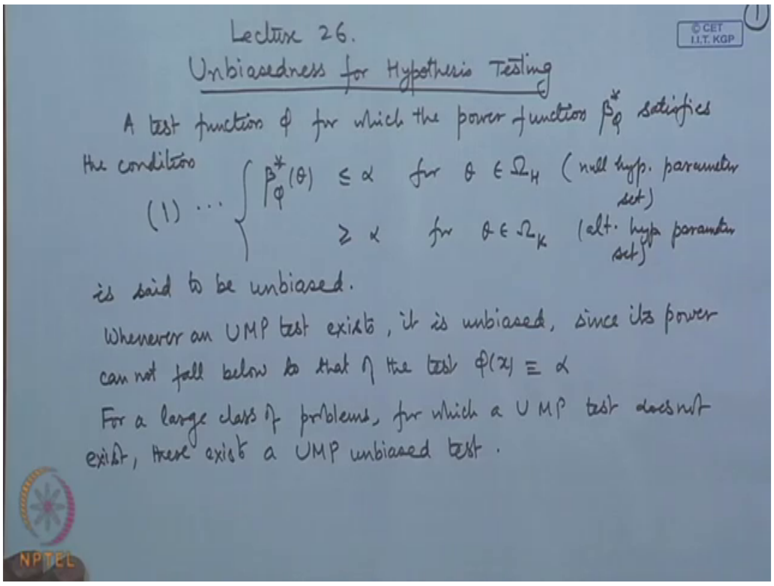


So, we may say this condition a test function phi for which the power function beta star phi satisfies the condition. Beta phi star theta is less than or equal to alpha for theta belonging to omega H that is the null hypothesis parameter set and it is greater than or equal to alpha for theta belonging to omega K, that is the alternative hypothesis parameter set. Then this set, then this test function is said to be unbiased. Let me call this condition 1.

Now, let us see what is this. For example, we have considered theorem 2 when the family see the monotone likelihood ratio were there and for the testing problem for theta less than or equal to theta naught against theta greater than theta naught. So, what we showed there that beta phi theta is actually increasing function. So, at theta is equal to theta naught the probability of type 1 error is the maximum and therefore, beta phi star theta was less than or equal to alpha for theta belonging to omega H that is theta less than or equal to theta naught and thereafter it was greater than or equal to. So, it was an example of an unbiased test.

So, whenever an UMP test exists, it is unbiased since its power cannot fall below to that of the test say phi x is equal to alpha that is the, that is we always reject with probability alpha whatever the x and we accept with probability 1 minus alpha. Then this test has expectation equal to alpha that is the power function is actually equal to alpha. So, for a large class of problems for which a UMP test, it does not exist there exists a UMP unbiased test. So, we may consider theta less than or equal to theta naught theta is equal to theta naught again theta naught equal to theta naught and also the cases of the nuisance parameters.

In many of these cases we will be actually demonstrating the existence of the UMP unbiased tests. So, we give a definition here which is called similar test.

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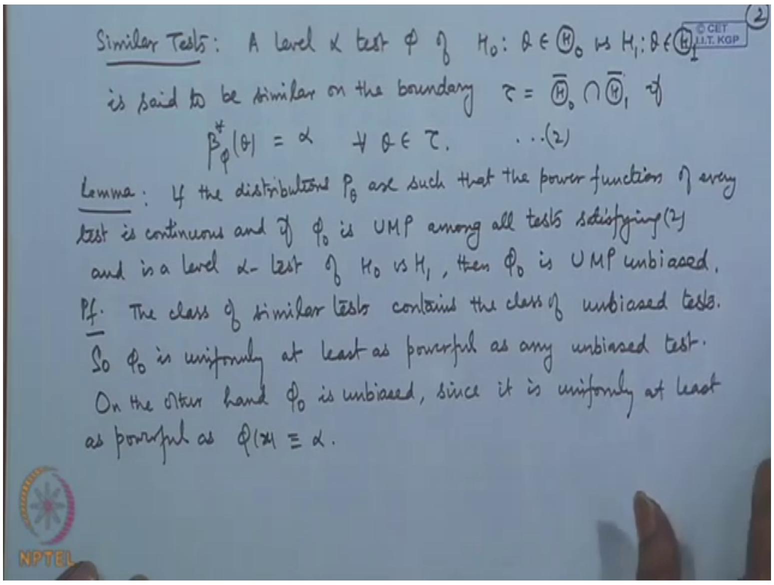


So, these are helpful in deriving UMP unbiased test. So, we call what is known as similar tests. What are similar test? So, a level alpha test phi of the hypothesis testing problem H naught theta belonging to theta naught against H 1 theta belonging to say theta 1 is said to be similar on the boundary. So, boundary of this theta naught and theta 1, let us denote it by some say tau that is equal to theta naught closer intersection theta 1 closer; if beta phi theta star is equal to alpha for all theta belonging to tau. So, this is called a similar test.

We have the following lemma here. If the distributions P theta are such that the power function of every test is continuous and if phi naught is UMP among all tests which satisfy two this condition of similarity. And is a level alpha test of H naught versus H 1, then phi naught is UMP unbiased. So, this is a very very useful result. So, the similarity is a very useful concept. So, first of all what we are saying if you look at this one, then we needed here that the power function should be less than or equal to the level for the values that is the probability of type 1 error basically. And the power should be greater than or equal to alpha that is what we are actually saying for the unbiasness condition.

So, in order to achieve this we are imposing a condition of the continuity on the test function and a similarity condition that is on the boundary the value alpha should be achieved. So, then what we are saying is that in these cases a UMP test will actually and a UMP test if it is similar test then certainly, it will be UMP unbiased. So, let us look at the proof of this which is quite simple. The class of similar tests contains the class of unbiased tests. So, fine art is uniformly at least as powerful as any unbiased test.

On the other hand, phi naught is unbiased since it is uniformly at least as powerful as phi x is equal to alpha. So, this proves that it is a UMP unbiased this class of similar tests contains the class of unbiased test. So, phi naught is uniformly at least as powerful as any unbiased test ok. On the other hand, phi naught is unbiased since it is uniformly at least as powerful as phi x is equal to alpha. Therefore, phi naught will be UMP unbiased. Now, let us consider applications to one parameter exponential families here.

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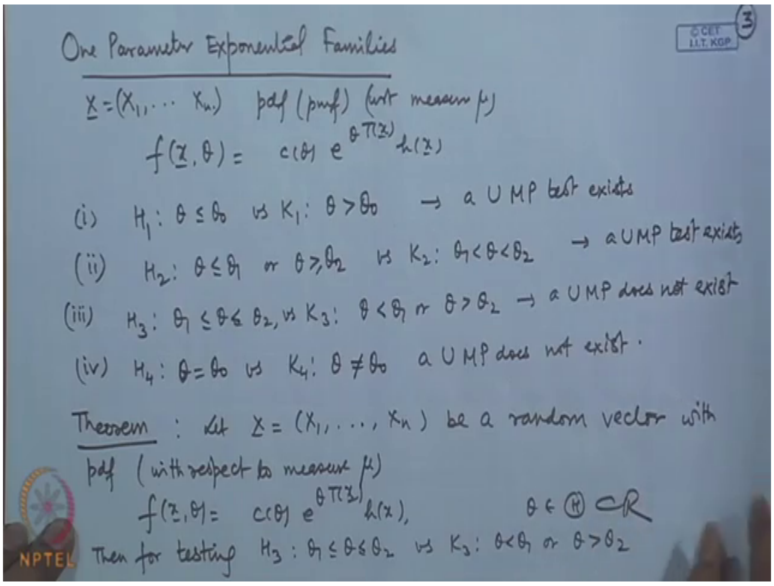


So, one parameter exponential families; so, as usual we are considering the X 1, X 2, X n as a random sample. So, we write it as X. So, pdf or pmf with respect to a measure mu we are writing down f x theta is equal to c theta e to the power say theta T x h x. So, we are having the following situation. If I am considering the hypothesis testing problem, I will name this as say H 1 theta less than or equal to theta naught against say K 1 theta greater than theta naught the situation here is the UMP test exists because q theta that is theta here is strictly monotone. So, the situation here is that a UMP test exists.

Let us consider say theta less than or equal to theta 1 or theta greater than or equal to theta 2 against say theta 1 less than theta less than theta 2. In this case also a UMP test exists. So, in the lecture 24, we have given the form of these tests and in the previous lecture 25, I have given several applications of these two tests and the form of the UMP test has been derived. Let us consider say theta 1 less than or equal to theta less than or equal to theta 2 against K 3 theta less than theta 1 or theta greater than theta 2. This 3 is actually the dual of two here a UMP test does not exist. And if we consider theta is equal to theta naught against, this is also alternative is two sided here a UMP test does not exist.

In the previous lecture, I have demonstrated through the double exponential distribution that for such a problem a UMP test does not exist. We have shown that UMP test which is having the maximum power for theta greater than or equal to theta naught has a smaller power than another test for theta greater than theta naught and vice versa. Therefore, UMP test does not exist. So, we have the following result that is UMP and bias tests do exist here. So, this is stated in the following theorem, for detail proofs you may look at the book of Laman or Rohatgi. Let x so, when we consider the random sample we generally write the joint density.

So, I am calling it as a random vector with probability density with respect to some measure mu. So, f x theta is equal to c theta e to the power theta T x h x where theta belongs to say theta subset of R. Then for testing H 3 that is this particular four third case theta 1 less than or equal to theta less than or equal to theta 2 versus K 3 theta less than theta 1 or theta greater than theta 2.

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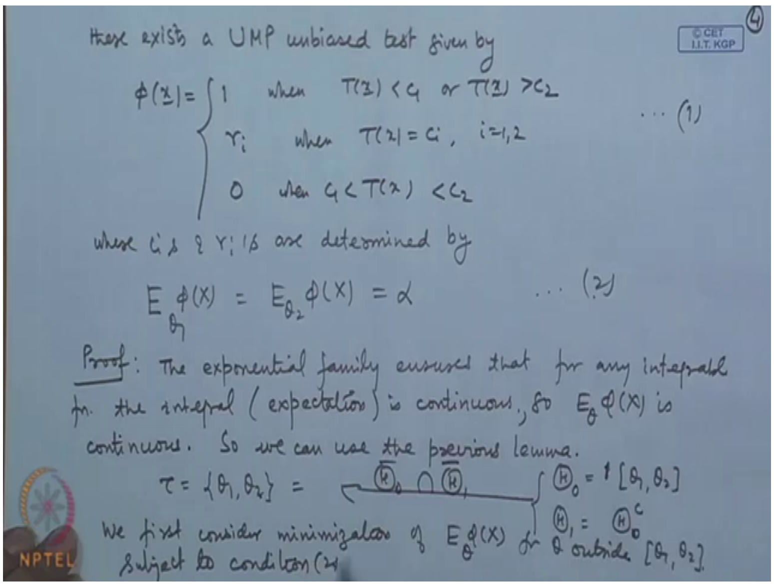

There exists a UMP unbiased test given by phi x is equal to 1 when T x is less than c 1 or T x is greater than C 2. It is equal to gamma i when T x is equal to C i for i is equal to 1, 2 and it is equal to 0. If C 1 is less than T x less than C 2 where Ci's and gamma i s is are determined by expectation of phi X under theta 1 and under theta 2 this should be equal to alpha. Let us have a comparison with the test which I gave in the lecture number 24 just to appreciate the dual problem here. We had considered this Q is strictly monotone.

So, you had monotone likelihood ratio and T x which is satisfied here for theta less than or equal to theta 1 or theta greater than or equal to theta 2 which I am actually describing as H 2. A UMP test was given here. Now in H 3 we are having simply the dual of this. In H 3 we are having the dual of this H 2 here; however, the test is UMP unbiased here. Let me briefly sketch the proof of this although I have been skipping the proofs of the theorems. In fact, for the detailed proofs one can look at the book of Laman and Gomano R Rohatgi and Saleh etcetera. However, a few of the proofs I will just simply sketch here.

So, here the distribution is in the exponential family. In the exponential family if I have an integral function the integral or the expectation you can say, it is continuous function. So, we can use this thing that is because to apply the lemma what we wanted here is that power function of every test is continuous. So, power function here will be expectation of phi X here. So, this should be continuous now in order to have that what we can do is we can use the condition that we are having the exponential family here. So, the exponential family ensures that for any integrable function the integral or expectation is continuous. So, expectation of theta phi X is continuous.

So, we can apply the previous lemma. So, according to the previous lemma, we let us consider the boundary. What was the boundary? Boundary was consisting of theta 1 and theta 2 that is you had theta actually the notation that I have used here is theta naught and theta 1 for the null and alternative hypothesis set. So, here theta naught was in this particular problem the theta 1 to theta 2 and theta 1 is actually complement of theta naught.

So, if I consider this tau that is theta naught closer intersection theta 1 closer, then this is going to be equal to theta 1 theta 2. So, now, let us consider we first consider minimization of expectation phi X for theta outside then travel theta 1 theta 2 subject to the condition 2.

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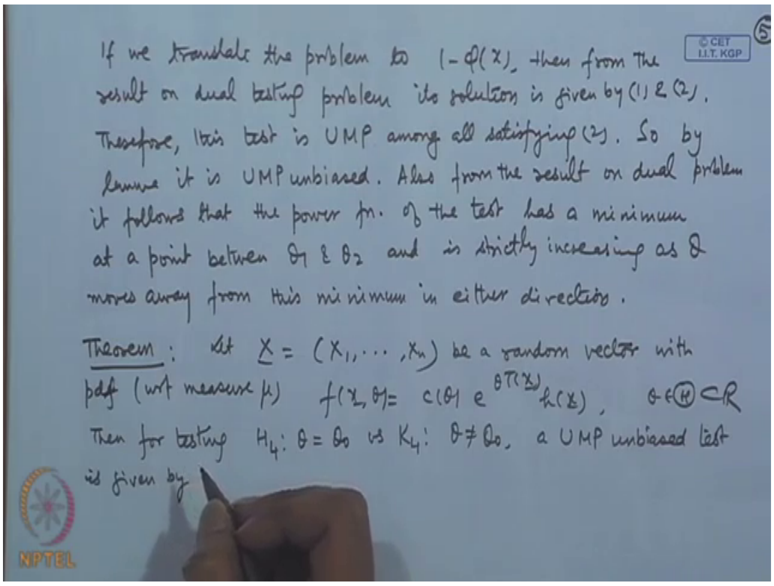


Now, if we translate the problem to 1 minus phi X function, then from the result on dual testing problem its solution is given by 1 and 2. Therefore, this test is UMP among all satisfying 2. So, by it is UMP unbiased. Also from the resultant on dual problem, it follows that the power function of the test has a minimum at a point between theta 1 and theta 2 and is strictly increasing mps as theta moves away from this minimum in either direction.

So, basically what you have seen here is that this result is actually following straightforwardly from the concept of similarity and the dual problem that we consider we are the null hypothesis was two sided here the alternative is two sided. So, UMP does not exist, but UMP unbiased can be found here. Now, I will also consider and of course, you can see here the test functions form is exactly 1 minus the form of the test for the dual problem here condition is also the same here. Now let us consider the point null hypothesis and the alternative is two sided.

Now, in that case one of the conditions gets modified. Let me state it in the following theorem. Let X is equal to X 1, X 2 X n. So, again the same conditions are there. Let me just for convenience I am restating it be a random vector with probability density function with respect to some measure mu that is f x theta is equal to c theta to the power theta T x h x and theta is of course, lying in a parameter space which is a subset of the real line.

Then for testing H 4 theta is equal to theta naught against K 4 theta is not equal to theta naught a UMP unbiased test is given by you note here that actually the form is given by the test which I have stated for the previous problem. However, the size condition will get modified. So, let me give it here.

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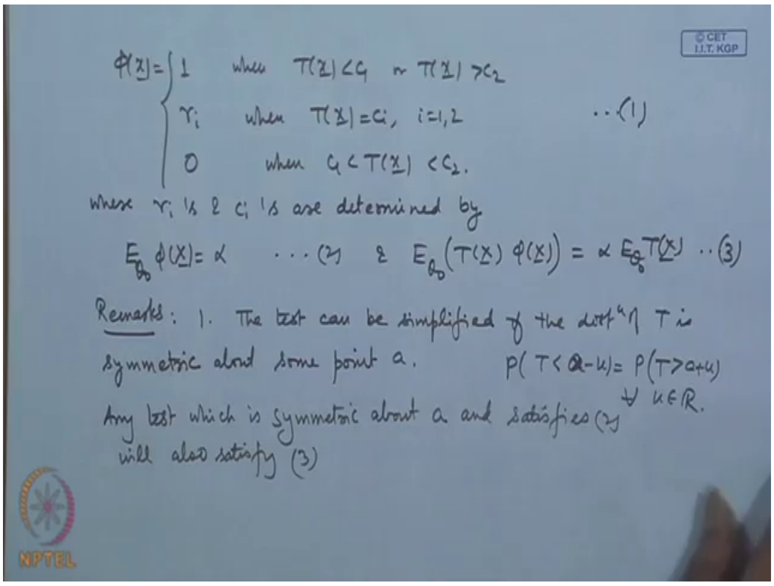


It is phi X is equal to 1 when T x is less than C 1 or T x is greater than C 2. It is equal to gamma i when T x is equal to C i i is equal to 1 2, it is equal to 0. When C 1 is less than T x is less than C 2 where gamma i s and Ci s are determined by expectation of theta naught phi X is equal to alpha and expectation of theta naught T x phi X is equal to alpha times expectation of theta naught T x. Note here that here we have a condition in which the statistic T is also involved. So, this was not there in any of the previous results here. So, for the proof, I refer to the book of Laman and also the book of Rohatgi, I am not going to discuss the proof in detail here.

However let me give certain comments here. The test can be simplified if the distribution of T is symmetrical about some point a. For example, so if I said a symmetric about the point a, then actually will have probability of T less than some x minus u is equal to probability of sorry a minus u is equal to probability of T greater than a plus u for all u on the real line. So, any test which is symmetric about a and satisfies 2 will also satisfy 3.

So, automatically this condition will be satisfied and therefore, we do not have to consider two conditions in these cases. So, let me just give it here. For example, you may consider here.

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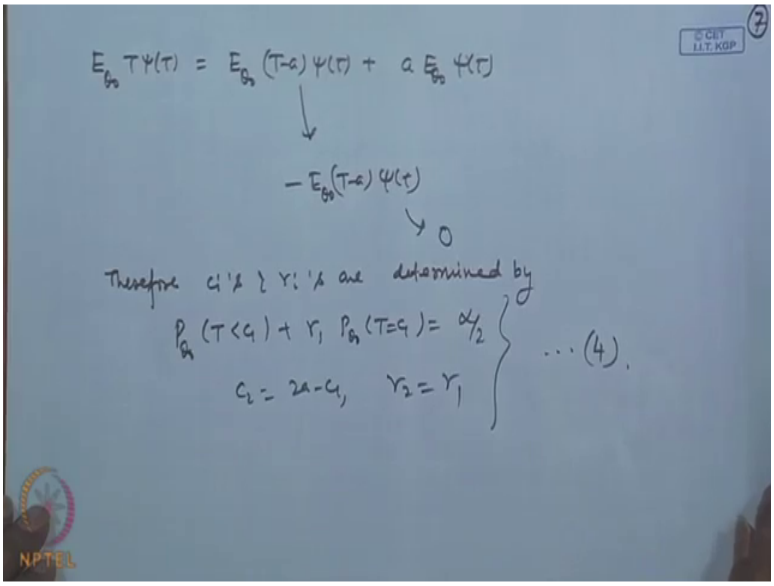


So, we can actually show that this value is equal to minus expectation theta naught T minus a psi T; that means, this is actually equal to 0. So, if this is actually equal to 0 then so, this will become equal to 0. That means, this condition becomes this condition. So, it is all automatically becoming true. Therefore Ci's and gamma i's are determined by alpha by 2 C 2 is equal to 2 a minus C 1 and gamma 2 is equal to gamma 1. So, the conditions get actually modified in place of writing down these two conditions we can actually reduce to these two conditions here.

Another important point which I would like to mention here is that the tests that we have stated in the two previous theorems for the two sided alternative hypothesis testing problems, these are UMP unbiased they are actually is strictly unbiased. What is the meaning of a strictly unbiased that as soon as we move away from the point theta naught, then the if we are going to the alternative hypothesis set. It is becoming a strictly greater than alpha; if we are going to the null hypothesis set, it is becoming is strictly less than alpha.

I will continue the concept of unbiasness in the testing problems in the following lecture.