Statistical Inference
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Lecture - 40
UMP Tests - IV

In the next lecture, I will be showing that in this case we have to take some restriction on the class of the assets which we call unbiased tests and within that class actually UMP Tests can be derived; so, that we will be taking up in the next lecture. Now let me continue the applications of this monotone likelihood ratio property and the derivation of the UMP test for various distributions.

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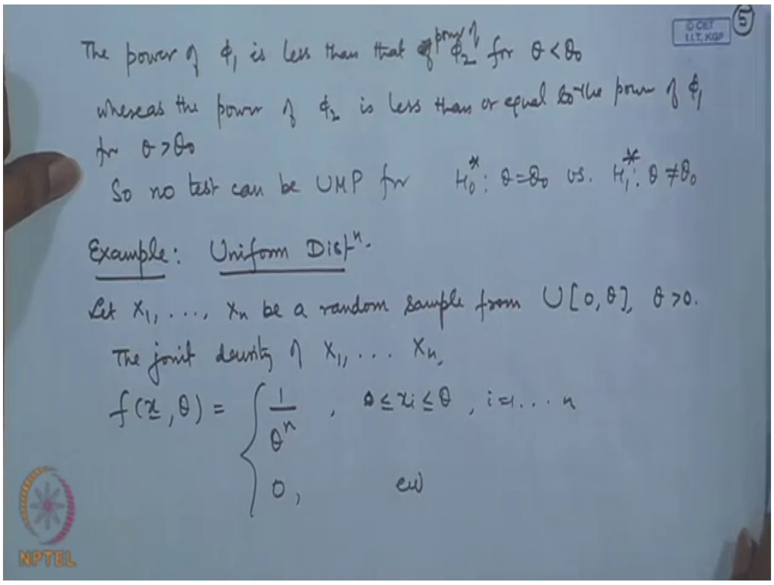

So, let me consider uniform distribution. So, we have X 1, X 2, X n. This is a random sample from uniform 0 theta distribution. And if we want to derive the UMP test for one sided testing problems etcetera, we should firstly, have the monotone likelihood ratio property. Let us check whether that is true or not. So, let us write down the joint density joint density of X 1, X 2, X n. I write it as f x theta is equal to 1 by theta to the power n and each of the x i is between 0 to theta and it is 0 elsewhere.

Now, we can write in a compact form as we have done when we were considering the discussion of the sufficiency or the maximum likelihood estimator.

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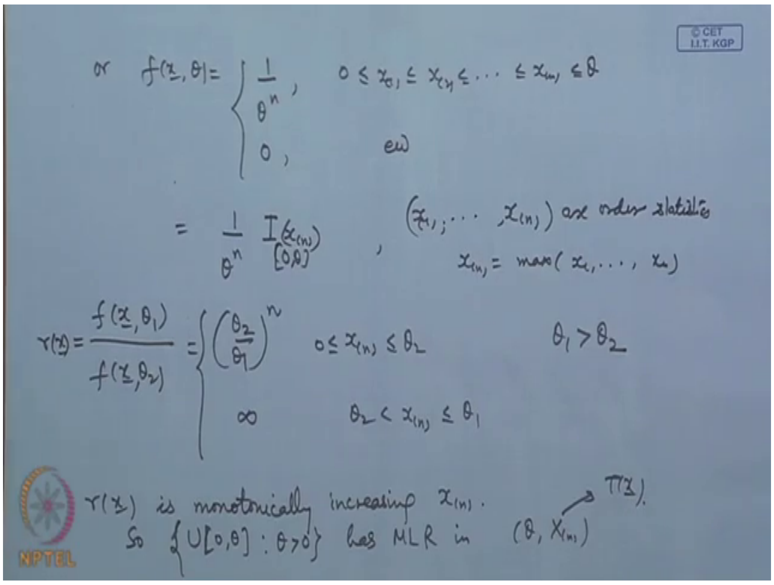

We can write this as 1 by theta to the power n 0 less than or equal to x 1 less than or equal to x 2 less than or equal to x n less than or equal to theta. It is 0 elsewhere which we further write as 1 by theta to the power n indicator function of the largest order statistics here x n is the. So, this x 1 x 2 x n, these are the order statistics. So, x n is actually the largest. So, we can write in terms of this. So, if I consider say f of x theta 1 divided by f x theta 2, let me take say theta 1 greater than theta 2 then this ratio will become theta 2 by theta 1 to the power n and ratio of the indicator functions.

So, if I choose x n to be less than or equal to theta 2, then both the densities are valid and we will get this indicator function value as 1. If I take theta to less than x n less than or equal to theta 1 in that case f x theta 1 is a positive density where as this density becomes 0. So, this becomes infinite. Now if x n is greater than theta 1, then of course, both the densities are 0 and we do not have to write that region. So, what we are observing is this likelihood ratio r x is monotonically increasing in x n. So, we can say this family of uniform distributions. This family has monotone likelihood ratio in theta and X n this is our T x.

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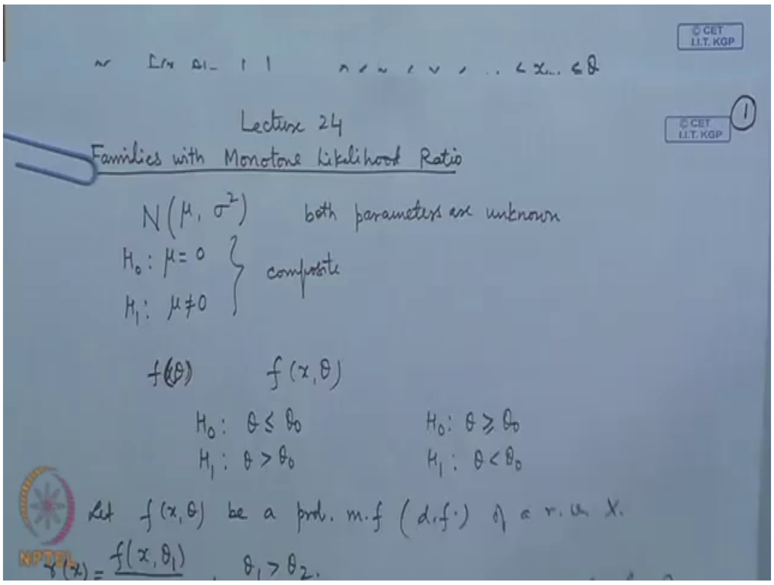

If you want to apply the monotone likelihood ratio property and the corresponding UMP tests this theory, then this is what we were requiring.

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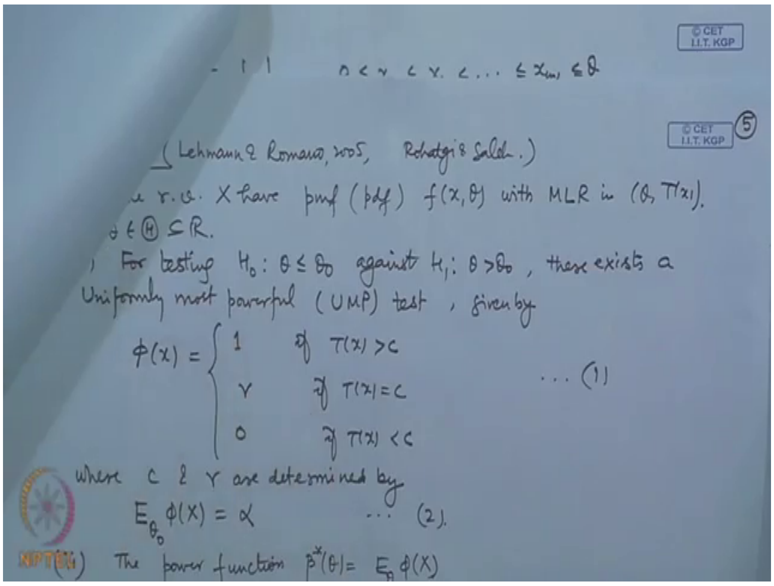

That if we are looking at the families with MLR in theta T x, then for one sided testing problems, we have the UMP test here. So, we will apply this now. So, let us derive the test in this particular case.

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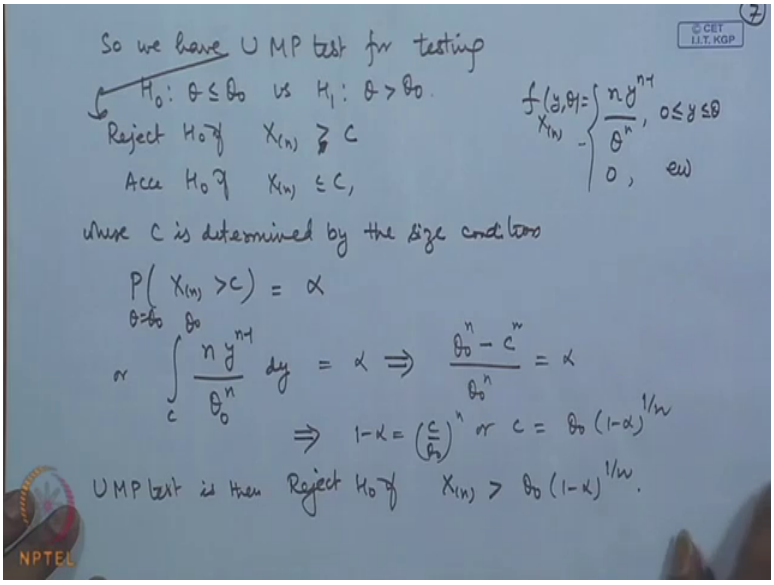

So, we have UMP test for testing say H naught theta less than or equal to theta naught against say H 1 theta greater than theta naught. So, test is this UMP test will be reject H naught if X n is greater than or equal to some constant C. This X n is having a continuous distribution. The density function of this will be n y to the power n minus 1 by theta to the power n. So therefore, we do not have to consider the randomization we can consider rejection for greater than or equal to or greater than and acceptance; if X n is less than or equal to C where, C is determined by the size condition that is probability of X n greater than C when theta is equal to theta naught is equal to alpha.

Now, if I have this distribution this probability can be easily evaluated. This is turning out to be simply integral n y to the power n minus 1 by theta to the power n d y from C 2 theta naught. This is equal to alpha which is equivalent to theta naught to the power n minus C to the power n divided by theta naught to the power n is equal to alpha. Now this can be further simplified we get 1 minus alpha is equal to C by theta naught to the power n or C is equal to theta naught into 1 minus alpha to the power 1 by n. So, UMP test is then reject H naught if X n is greater than theta naught into 1 minus alpha to the power 1 by n. Let me also demonstrate the power function etcetera for this.

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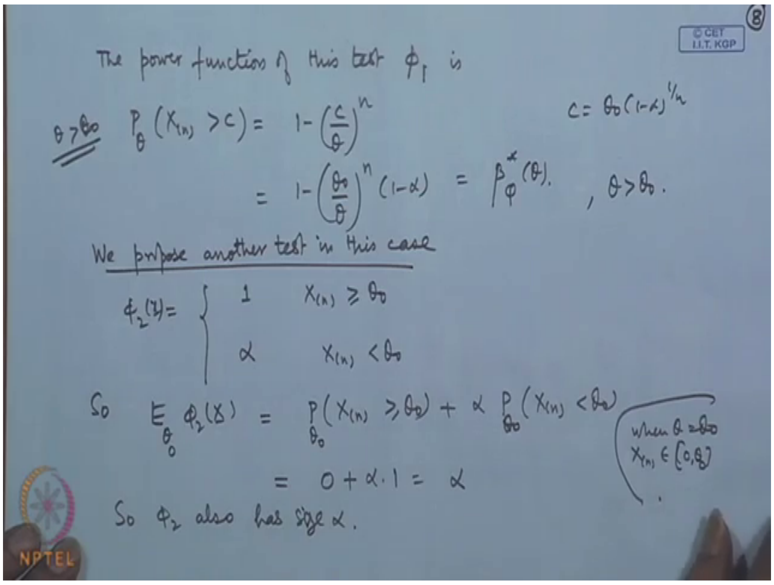

The power function of this test, let me call this test as phi 1 say. So, that is probability of X n greater than C where C is actually theta naught into 1 minus alpha to the power 1 by n and here theta will be greater than theta naught. So, this is equal to 1 minus C by theta to the power n that is equal to 1 minus theta naught by theta to the power n into 1 minus alpha. Let us call it say beta phi star theta. Here theta is greater than theta naught.

Here I will also like to give one example see we have derived using the theorem which why given the last class that is for the families with the monotone likelihood ratio. UMP test can be has a particular form for the one sided testing problems. Now using that, we are able to exactly derive the form of the UMP test as this X n greater than theta naught 1 minus alpha to the power 1 by n. Let me call it phi 1 ok.

Now, we propose another test in this case, let me call it phi 2 and the test is 1 if X n is greater than or equal to say theta naught and it is equal to alpha if X n is less than theta naught. Notice the difference here in the previous case, I was only rejecting or accepting; that means, it was a non randomized test, but this particular test is a randomized test because I am rejecting if X n is greater than or equal to theta naught.

But I am also rejecting with probability alpha if X n is less than theta naught. So, if I consider the expectation of phi 2 under theta naught then it is equal to probability of X n greater than or equal to theta naught when theta is equal to theta naught plus alpha times probability of X n less than theta naught when theta is equal to theta naught. Now when theta is equal to theta naught then X n has the range 0 to theta naught when theta is equal to theta naught because the distribution of X n is n y to the power n minus 1 by theta to the power 1 from 0 to theta.

So, if I have assumed here that theta is equal to theta naught is the two parameter value which is actually required to calculate the size of the test. So, this probability will be 1 where as this property will be 0. So, you will get alpha. So, phi 2 also has size alpha ok.

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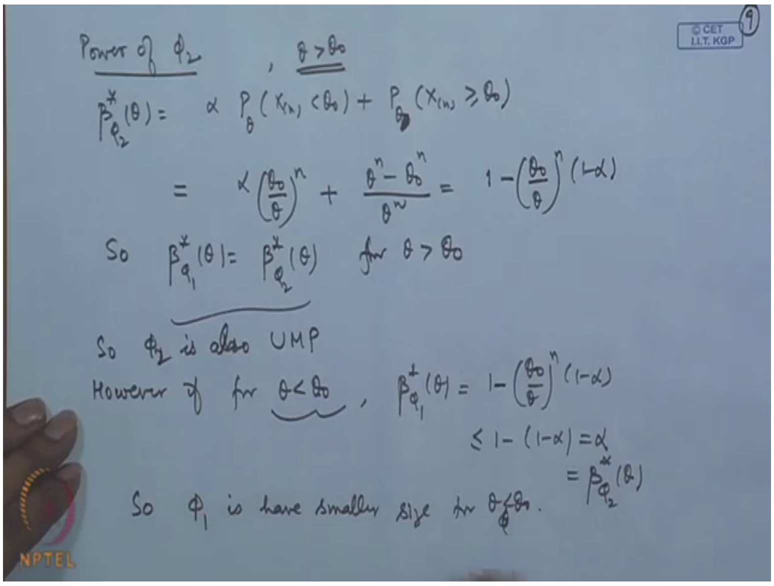

Let us now look at the power of phi 2 power of phi 2. So, power function of phi 2 is equal to alpha times probability theta X n less than theta naught plus p theta naught X n greater than or equal to theta naught. Now we have already considered the distribution of X n which is of this form.

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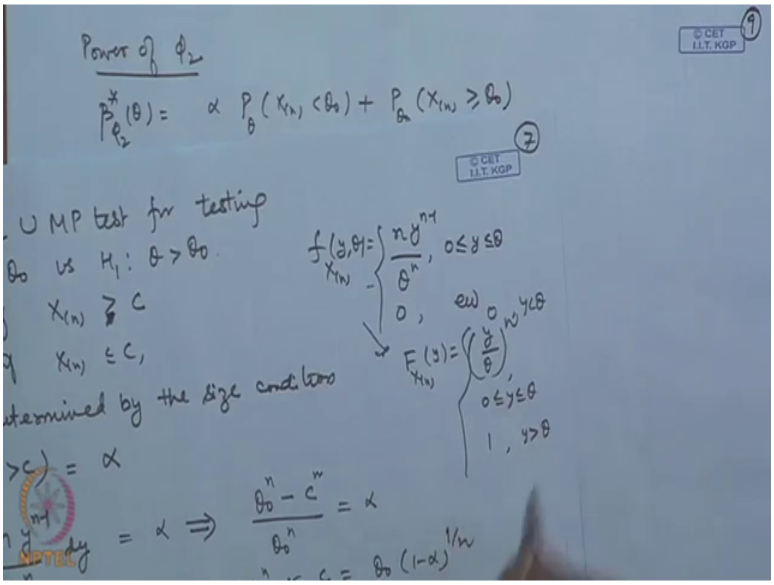

So, what is CDS here? That is y by theta to the power n for 0 less than or equal to y less than or equal to theta. It is 0 for y less than theta is equal to one for y greater than theta. Therefore, I can consider this thing here. I am considering the alternative set theta is greater than theta naught. So, we are going only up to theta naught. So, this probability will be theta naught by theta to the power n because this is y by theta to the power n that is the probability up to y.

So, this is theta naught by theta to the power n plus this is the probability from theta naught to theta because in this particular case sorry this is only up to theta here. So, this will be theta to the power n minus theta naught to the power n by theta to the power n. This will become 1 and this term I can combine. So, I can write it in a slightly modified fashion as theta naught by theta to the power n into 1 minus alpha. Now let us consider the power function of the phi 1. The power function of phi 1 was 1 minus theta naught by theta to the power n 1 minus alpha the power function of phi 2 is also same.

So, what we have proved that power functions of the two for theta greater than theta naught are the same. So, phi 2 is also UMP. However, if we consider the power function that is for theta less then theta naught beta phi 1 star theta that is 1 minus theta naught by theta to the power n into 1 minus alpha that is going to be less than or equal to. See theta naught minus theta is theta naught minus theta to the power n is greater than 1; if theta is less than theta naught this is greater than 1. So, if I take minus here this will be less.

So, this is less than or equal to one minus 1 minus alpha that is a equal to alpha that is beta phi 2 star theta for theta less than theta naught. So, phi 1 is having smaller size for theta less than theta naught. So, I will consider here see, I have proposed I have derived one test phi 1 as the UMP test by the usual name and PSM theory. I proposed another test phi 2 I showed that the power function is the same. So, that is also UMP test. However, now what I am doing I am showing that the second test as uniformly the size is equal to alpha whereas the first test has it less than or equal to alpha.

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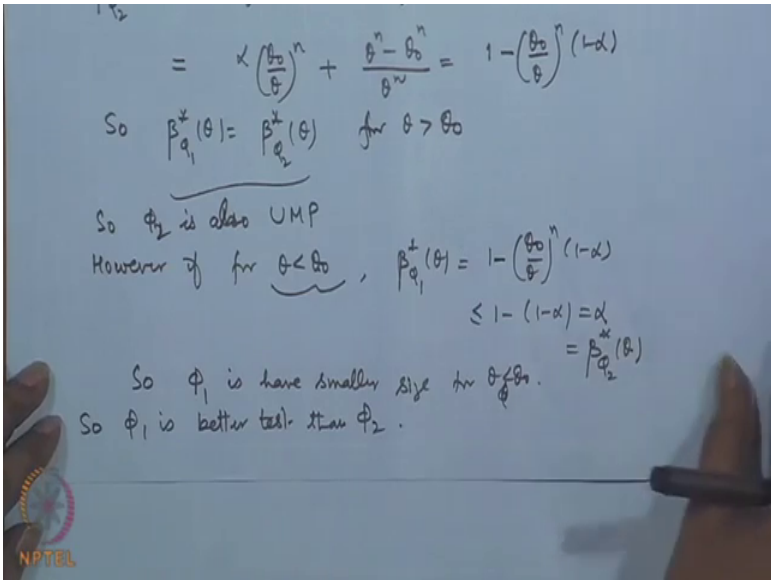

So, I will consider phi 1 is better test than phi 2. In this particular case let me also demonstrate the reverse hypothesis that is the dual.

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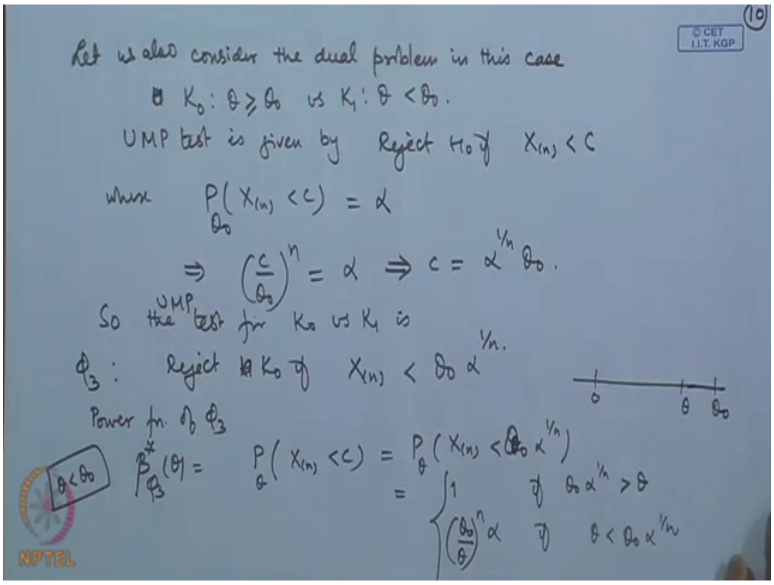

Let us also consider the dual problem that is a H I called its K naught theta greater than or equal to theta naught against K 1 theta less than theta naught. So, UMP test is given by reject H naught if X n is less than C where probability of X n less than c under theta naught should be equal to alpha. Now, once again this value is simply equal to when theta is equal to theta naught, this value will be equal to C by theta naught to the power n that is equal to alpha. That means, C is equal to alpha to the power one by n theta naught.

So, the test case test for K naught versus K 1 is this is the UMP test reject; let me call it some name phi 3 reject K naught if X n is less than theta naught alpha to the power 1 by n. Compare it with the test that we derived for the dual problem that is H naught versus H 1. Here it was X n is greater than theta naught into 1 minus alpha to the power 1 by n and here it is reject K naught if X n is less than theta naught into alpha to the power 1 by n. Notice here that in both the cases, we have shifted little bit from theta naught. So, note here this alpha is less than 1.

So, alpha to the power of 1 by n is also less than 1. So, the cutoff point theta naught, but slightly less than that whereas for this one if you see the cutoff point is again a little less than theta naught not exactly greater than or equal to theta naught ok. We may also consider power for this part power function of phi 3 that is beta phi 3 theta that is equal to probability of X n less than C when theta is the parameter value, but theta is less than theta naught here that is equal to P theta X n less than theta naught alpha to the power 1 by n.

Now, the range of X n is from 0 to theta and this theta is less than theta naught. So, there can be two cases here. This alpha to the power 1 by n because theta naught into alpha to the power 1 by n. This value is actually less than theta naught. So, this could be here or this could be here. So, this is equal to 1 if theta naught alpha to the power 1 by n is greater than theta. Otherwise this value is equal to theta naught by theta to the power n into alpha if theta is less than theta naught alpha to the power 1 by n.

You can compare the power functions in the two cases here I have derived the power function for the other part also. The power function for phi 1 was given by 1 minus theta naught by theta to the power n 1 minus alpha here and here you can see this value is theta naught to the power n by theta alpha here. Now, these two tests; these two tests can be combined also. If I can combine I can write in this particular case for if I am considering theta is equal to theta naught against theta naught equal to theta naught, then for theta greater than or equal to theta naught the rejection region is given by X n greater than something and that something we have determined actually.

So, if we distribute that probability we can slightly modify this statement here and similarly for theta less than theta naught, the rejection region is X n less than something. So, X n greater than something X n less than something I can combine these two statements to get a UMP test here for the two sided problem also. Let me continue with some further applications here for the UMP test here.

So, note here either we have the distributions in the exponential family. Usually here we are considered one parameter exponential family because so far whatever testing problems we have discussed, we have considered only one parameter. When we have more than one parameter for example, in the normal distribution we may consider normal mu sigma square or in exponential distribution we may consider location and scale both. In those cases, we will see that in place of the uniformly most powerful test you have to restrict attention to only unbiased test and then we will be able to get the UMP unbiased test.

So, those things we will be considered in the following lecture here my intention is to show either we considered the distributions in the exponential family or we consider the distributions with the monotone likelihood ratio.So, therefore, we can be able to derive the UMP tests.

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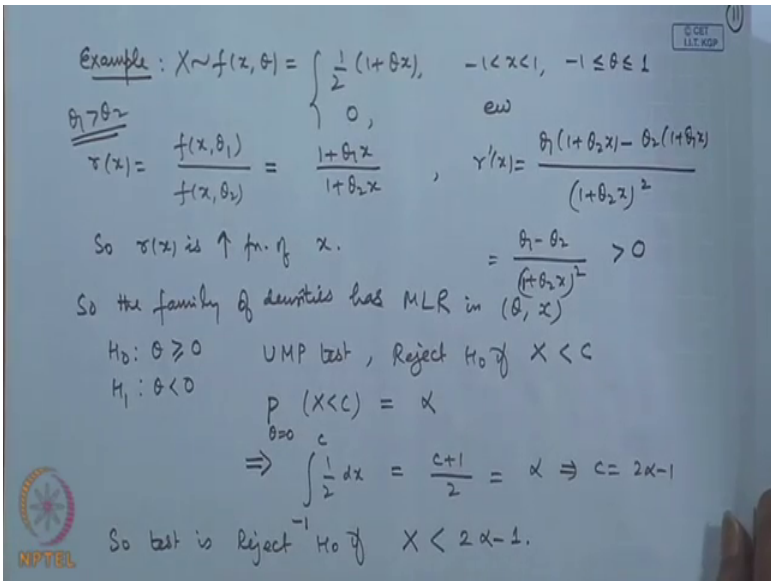

Let me take slightly different example which is naught a very conventional one half 1 plus theta x minus 1 less than x less than 1 minus 1 less than or equal to theta less than or equal to 1 and it is of course, 0 elsewhere. Now if you look at this certainly, it is not in the form of an exponential family. So, what we can do? We can consider whether the monotone likelihood ratio is there or not. So, let us look at that. If I consider the ratio of the let me consider say one observation here ok, in place of n I am considering for the convenience one observation f x theta 1 divided by f x theta 2. Let us take say theta 1 is greater than or greater than theta 2.

So, this ratio will become 1 plus theta 1 x divided by 1 plus theta 2 x. So, whether this is increasing or decreasing, it will depend upon theta 1 greater than theta 2. Let us look at for example, what is derivative of this. So, derivative will give you theta 1 into 1 plus theta 2 x minus theta 2 into 1 plus theta 1 x divided by 1 plus theta 2 x whole square. So, this is equal to theta 1 minus theta 2 divided by 1 plus theta 2 x square and since theta 1 is greater than theta 2; this is positive. So, what we are concluding is that r x is increasing function of x. So, this distributions so, the family of densities that we have considered here has monotone likelihood ratio in theta and x. So, now, it is nice that we can actually derive the UMP test. Suppose I consider one sided testing problem theta greater than or equal to 0 against say theta less than 0.

So, UMP test will be reject H naught if X is less than C and C is determined from the size condition; so, probability of X less than C when theta is equal to 0, this should be equal to alpha. Now when theta is equal to 0, the density will become half that is simply the uniform distribution. So, this is actually half from minus 1 to C that is equal to C plus 1 by 2 that is equal to alpha; that means, C will be equal to twice alpha minus 1. So, test is reject H naught if X is less than twice alpha minus 1. So, we are able to get a exact form of the testing procedure here; this is the UMP test in this particular problem.

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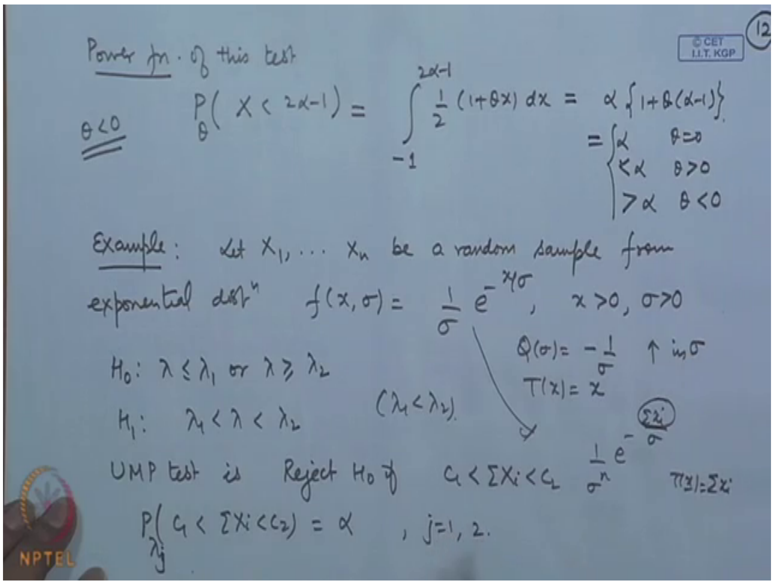

Let us also consider power function that is probability of X less than 2 alpha minus 1 for theta when theta is less than 0. So, in this case, the density is half 1 plus theta x. So, we integrate from minus 1 to twice alpha minus 1.

So, after simplification this value turns out to be simply alpha into 1 plus theta into alpha minus 1. So, actually you can see at theta is equal to 0, this is exactly equal to alpha for theta greater than 0, it will be less than alpha. So, this is equal to alpha or theta is equal to 0. It is less than alpha. If I take theta to be greater than 0 if theta is greater than 0 alpha minus 1 is negative. Therefore, this value will become less than alpha and it is greater than alpha for theta less than 0. So, the result which actually I stated when we were giving the result about the UMP test is exactly shown to be satisfied here. Let me read out from the statement that we gave that day.

So, if the distribution of X has f x theta has MLR in theta T x and the most powerful test is of this form, then what we said here that for the values of theta which are bigger that is it will be greater than or equal to power and for lower side, it is actually increasing function; phi beta phi star theta is a increasing function of theta. I think I yeah this is beta star theta is strictly increasing function of theta for which is this is true.

So, this is followed here. Let us consider here exponential distribution let X 1 X 2 X n be a random sample from exponential distribution. So, we are considering simply one parameter exponential distribution with scale parameter setup. Now this is simply one parameter exponential family. If I consider Q sigma that is equal to minus 1 by sigma this is increasing in sigma and T x here is x. So, this is one parameter exponential family with the setup that we have stated in the theorem. So, even for the two sided null hypothesis, we will be able to derive a UMP test.

So, if I consider say H naught lambda less than or equal to say lambda 1 or lambda greater than or equal to lambda 2 against lambda 1 less than lambda less than lambda 2. Here lambda 1 is less than lambda 2 UMP test is reject H naught. If C 1 is less than now if you see here when I write down the distribution of n of these observations, then it will become 1 by sigma to the power n e to the power minus sigma x i by sigma. So, T x then in that case will become equal to sigma x i. So, we will get sigma x i less than C 2 where probability of C 1 less than sigma x i less than C 2 when lambda 1 or lambda 2 is true this is equal to alpha that is j is equal to 1, 2.

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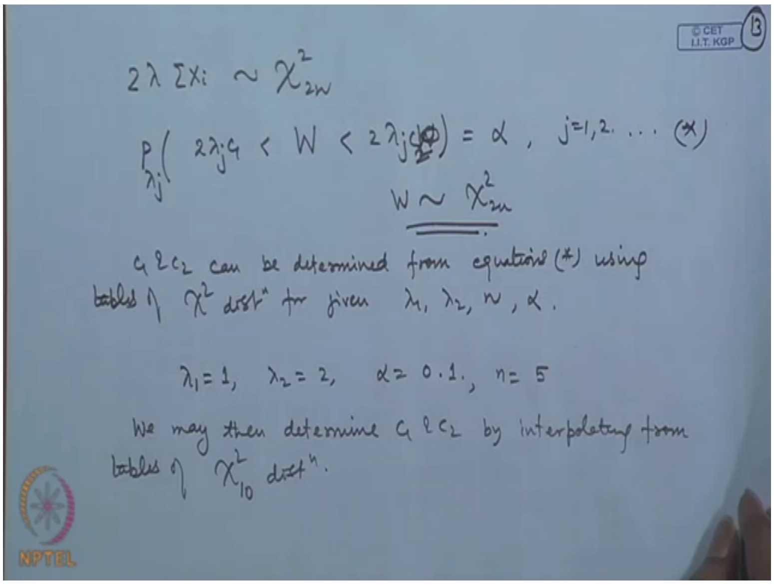

Now, we can see that lambda times sigma x i and if I take two times that it will have chi square distribution on two one degrees of freedom. So, we can write down this conditions probability of twice lambda C 1 less than so, this is W variable less than twice lambda. So, let me put j here W, this is equal to alpha for j is equal to 1, 2. This is when lambda j is true where w follows chi square 2. So, let me call this equation star. So, C 1 and C 2 can be determined from sorry this is C 2 here. C 1 and C 2 can be determined from equations star using tables of chi square distribution for given lambda 1 lambda 2 n and alpha.

So, for example, in a given problem you may have lambda 1 is equal to say one lambda 2 is equal to say 2 alpha is equal to say 0.1, then you and say n is equal to phi, then you need to look at that tables of chi square ten distribution; chi square on ten degrees of freedom and you can determine this conditions here. We may then determine C 1 and C 2 by interpolating from tables of chi square distribution on ten degrees of freedom. Let me make mention about the location scale distributions. Under certain conditions, this location is skill distributions also have the monotone likelihood ratio property. So, let me give that thing.

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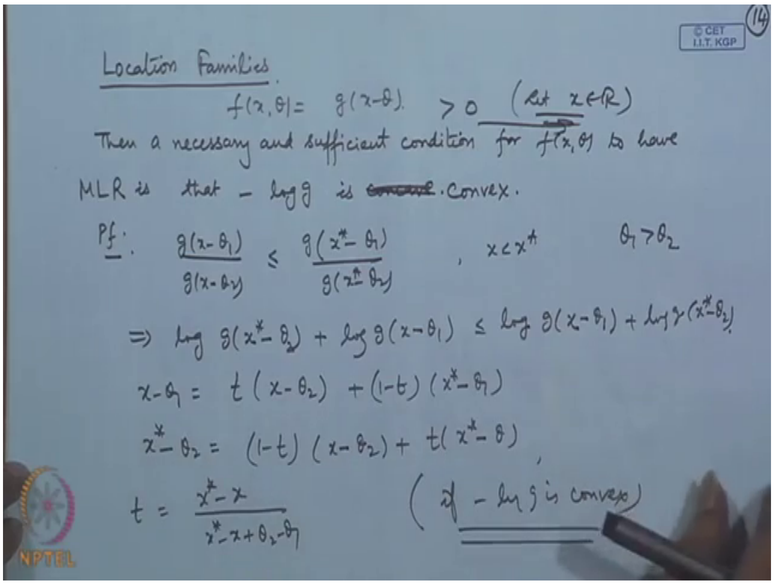

So, let us consider say location families; that means, my f x theta is of the form g of x minus theta and of course, let us takes a x belonging to r that is this is positive for all x ok. Then a necessary and sufficient condition for f x theta to have monotone likelihood ratio is that minus log of g is concave is convex sorry. This can be actually proved here; if I consider g x minus theta 1 divided by g of x minus theta 2 where theta 1 is greater than theta 2, then we have to show that this is increasing function of x.

That means, if I consider x a star for x less than x star. This is what we should show for monotone likelihood ratio that is g x minus theta 1 by g x minus theta 2 to be an increasing function. So, if I take logarithms this is reducing to log of g of x star minus theta 2 plus log of g x minus theta 1 less than or equal to log of g x minus theta 1 plus log of g of x star minus theta 2.

Now, we can actually interpret something like this x minus theta 1, we can write as some t times x minus theta 1 minus theta 2 plus 1 minus 3 times x star minus theta 1 and we may also write x star minus theta 2 is equal to 1 minus t times x minus theta 2 plus t times x star minus theta where t I am choosing to be x star minus x divided by x star minus x plus theta 2 minus theta 1. Then if we do that then if minus log g is convex then this will be true and converse is also true that is it is necessary and sufficient.

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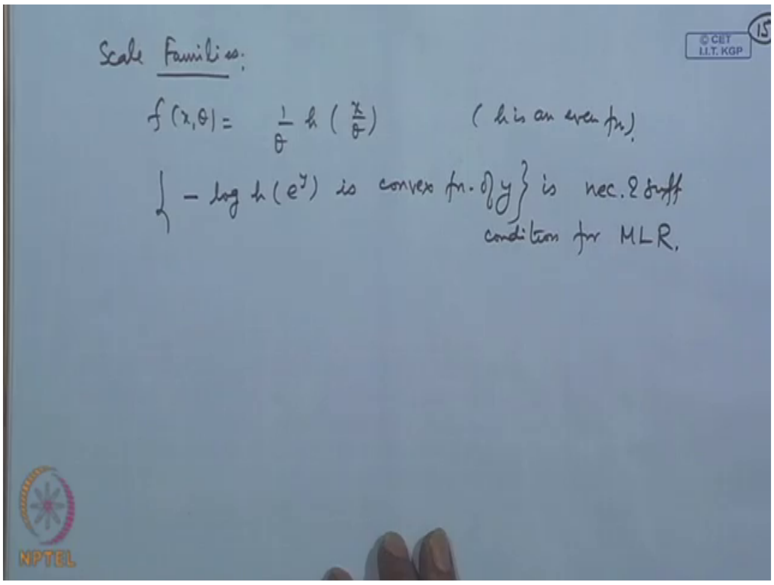

I will just mention about the scale families also. For a scale families, we can consider the shifting two like if I consider f x theta is equal to 1 by theta h of x by theta h is an even function. In that case a necessary and sufficient condition will be that minus log of h e to the power y is convex function of y. This is a necessary and sufficient condition for monotone likelihood ratio. Now we will consider the application of this Neymann Pearson theory to the cases when the UMP tests do not exist. So, we will consider some further criteria and that I will be developing in the next lecture.