Statistical Inference
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Lecture - 39
UMP Tests- III

In the last class I have also considered the cases for two sided composite hypothesis and there is one particular case, when we are having the null hypothesis as a two sided. Like theta less that or equal to theta 1 or theta greater than or equal to theta 2 against theta lying in the interval theta 1 2 theta 2. In these cases also the uniformly most powerful test exists provided the distributions are in the 1 parameter exponential family. And the Q theta function which is there in the 1 parameter exponential family should be strictly monotone.

So, basically we have given 2 results; one is that if the families of distributions have monotone likelihood ratio then for the one sided testing problems like theta less than or equal to theta naught against theta greater than theta naught or the dual of it for these problems UMP test can be derived. So, now I will discuss various applications of these in both the results; let me start with the normal distribution ok.

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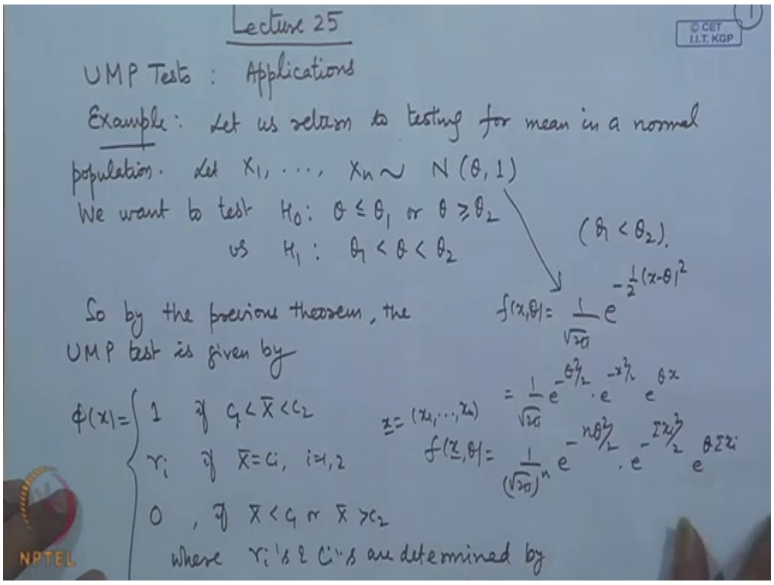

So, let us return to testing for mean in a normal population; that means, we are having a set of that X 1 X 2 X n follows a normal; so, its a random sample from a normal theta 1 distribution. And we want to test H naught theta less than or equal to say theta 1 or theta greater than or equal to theta 2 against say H 1 theta 1 less than theta; less than theta 2. So, here of course, we have assumed theta 1 is less than theta 2.

So, in the previous class I have given the theorem. So, here we have 1 parameter exponential family; see if we write down the distribution it is 1 by root 2 pi e to the power minus half x minus theta square that is equal to 1 by root 2 pi e to the power minus theta square by 2 e to the power minus x square by 2 e to the power theta x. And if we write for fx theta where x is x 1 x 2 x n then this is 1 by root 2 pi the power n e to the power minus n theta square by 2 e to the power sigma x i square by 2 e to the power theta sigma x i.

So, this is in the form of a 1 parameter exponential family the Q theta function is theta it is strictly increasing function. So, strictly monotone therefore, the theorem which I gave in the last lecture let me just show it again; let us look at the statement of the result.

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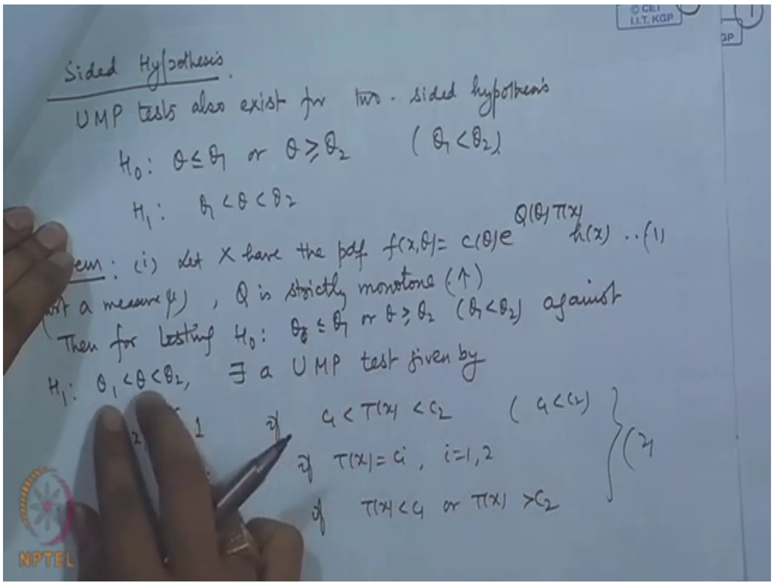

If we have f x theta is equal to c theta into e the power Q theta T x to h x, where Q is we have taken to be a strictly increasing function. Then for testing two sided null hypothesis against a interval for the alternative hypothesis UMP test exist. And the tests as this form which is having this it is based on T x; T x function which is available here. Therefore, we can straightforwardly write down here the test based on sigma x i or x bar. So, by the previous theorem the UMP test is given by phi x is equal to 1 if c 1 is less than X bar less than c 2.

It is gamma i if X bar is equal to c i i is equal to 1 2, it is equal to 0 if X bar is less than c 1 or X bar is greater than c 2. So, the presentation of the test is that we reject the null hypothesis H naught if X bar lies between c 1 and c 2. And we reject with probability gamma i if X bar is equal to c i for i is equal to 1 2. And we accept H naught if X bar is less than c 1 or X bar is greater than c 2 where, this gamma i's and c i's are determined by the size conditions.

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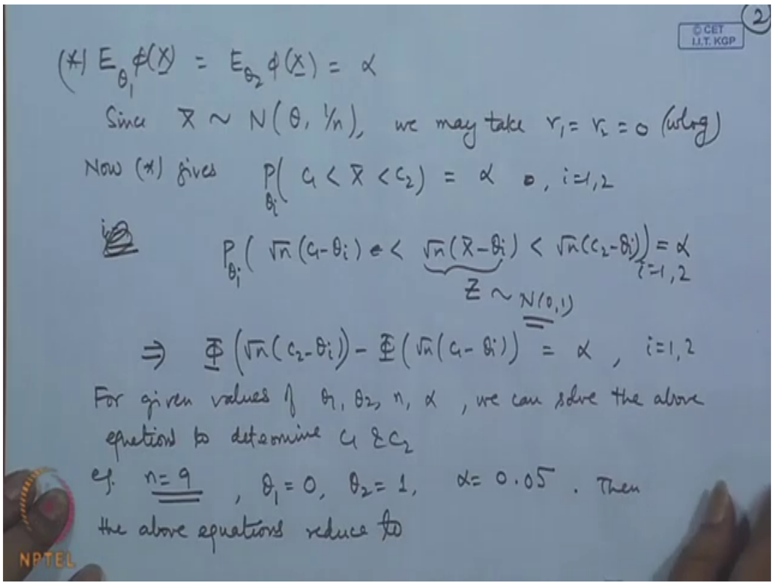

Expectation of phi X under theta 1 and under theta 2 to be equal to alpha. Now, note here I am considering X bar is equal to c i that distribution of X bar will be; the distribution of X bar is normal theta 1 by n.

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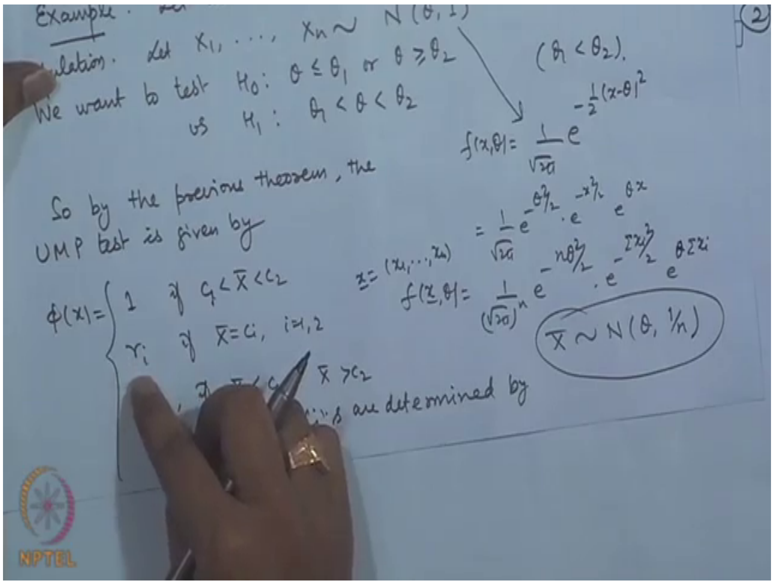

This is a continuous a distribution therefore, without loss of generality we can take gamma i is to be 0. If it is to be 0 then this point is included here. Since, X bar follows normal theta 1 by n we may take gamma 1 is equal to gamma 2 equal to 0 without loss of generality. So, now we want this condition size condition star probability of c 1 less than X bar less than c 2 is equal to alpha, for i is equal to 1 2. Now, when i is equal to 1 then X bar follows normal theta 1 1 by n.

So, for i is equal to 1 this condition then can be written as we can transfer to the standard normal variable. We will get root n c 1 minus let me write for i minus less than root n X bar minus theta i less than root n c 2 minus theta i is equal to alpha, for i is equal to 1 2. Now, when theta is equal to theta i this is normal 0 1. So, this is reducing to phi of root n c to minus theta i minus phi of root n c 1 minus theta i is equal to alpha i is equal to 1 2.

So, for given values of theta 1, theta 2, n, alpha we can solve the above equation to determine c 1 and c 2 and of course, this will be numerical solutions. As an example let us take say suppose, I take say n is equal to 9 let me take say theta 1 is equal to say 0, theta 2 is equal to say 1 and say alpha is equal to 0.05. Then what will be these equations?

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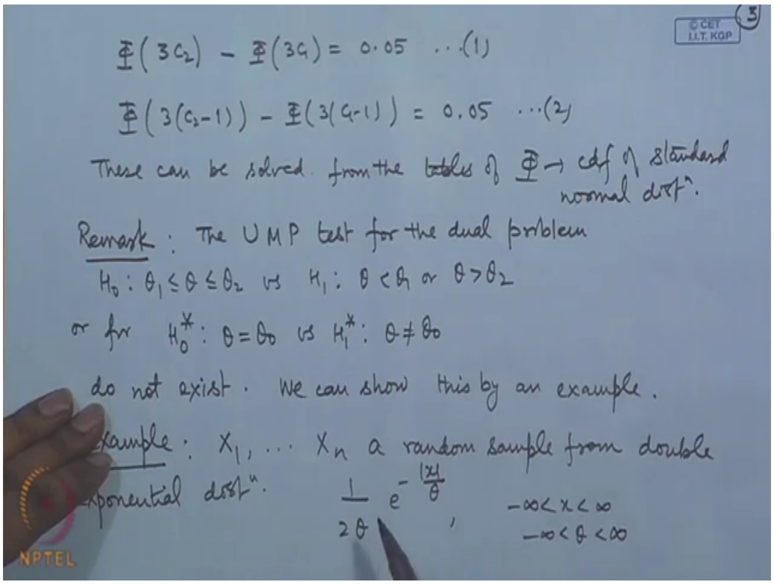

Then the above equations reduce to the first equation will become; now root n is 3 c 2 if I am writing theta is equal to 0 then it will be phi of 3 c 2 minus phi of 3 c 1 is equal to 0.05 that will be 1 equation. And the other equation will become 3 of c 2 minus 1 minus phi of 3 into c 1 minus 1 is equal to 0.05. These can be solved from tables of capital phi function, this is the cdf of a standard normal distribution that we have been using. So, once again I have demonstrated here that under the given conditions UMP test for a testing problem can be provided.

And this helps us in taking exact decisions at a given level of significance and of course, the given level of significance may depend upon the problem that is given at hand. Let me give some further applications. Now, another point which I would like to mention here, that I have considered here the region of null hypothesis as two sided. And the region for alternative hypothesis is the complementary of that is it is within an interval.

Now, one may think that if the UMP tests exists for this problem the if I interchange it, like if I write this has H naught and this is H 1; that means, the alternative is two sided. Unfortunately, in these cases it can be shown that the UMP test does not exist; I will demonstrate by it an example.

Let me give this comment here; the UMP test for the dual problem H naught theta 1 less than or equal to theta less than or equal to theta 2 versus H 1 theta less than theta 1 or theta greater than theta 2 or for let me say H naught star theta is equal to theta naught versus H 1 star theta is not equal to theta naught do not exist. So, let us take the example we can show so, let us take this example. We have considered earlier a double exponential distribution, let us consider say X 1 X 2 X n a random sample from double explanation 1 by 2 theta e to the power minus modulus x by theta.

And here of course, both theta and x have the range on the whole real line. This problem you have discussed in the previous lecture. I had demonstrated the UMP test for the two sided for one sided testing problem, that is for theta less than or equal to theta naught against theta greater than theta naught. Now, here I will show that if I consider this type of hypothesis then the UMP test does not exist.

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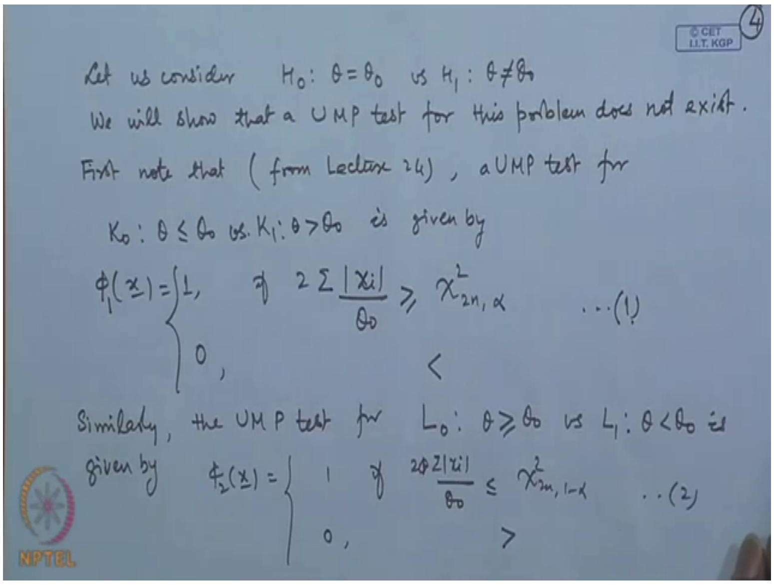

So, let us consider say H naught theta is equal to theta naught against say H 1 theta is not equal to theta naught. We will show that a UMP test for this problem does not exist. So, if you go back to the development that, I gave it in the last lecture; what we have shown here if you considered with example discussed in the last lecture. I have considered the one sided testing problem theta less than or equal to theta naught against theta greater than theta naught. And we derived the UMP test of the having form that reject H naught if sigma modulus X i is greater than or equal to c.

And we were able to determine this constant also; the final form was reject H naught if 2 by theta naught sigma modulus of X i greater than or equal to chi square 2 n alpha. So, let us write this first note that; so, I am giving the reference from lecture 24 a UMP test for; let me give some different names than H naught and H 1. We can consider say K naught theta less than or equal to theta naught against K 1 theta greater than theta naught is given by phi x is equal to 1 for twice sigma modulus xi by theta naught greater than or equal to chi square 2 n alpha and 0 if it is less.

Now, if you considered the dual problem here, the dual problem is to consider H naught theta greater than or equal to theta naught against theta less than theta naught. Then in that particular case the rejection region will become less here, the reason is that we are having the monotonic equation ratio in theta and sigma X i sigma modulus of X i. So, the rejection region will become less than or equal to here. And when we proceed in this fashion the constant this will become chi square 2 n 1 minus alpha. Therefore, we can write that form here; let me call this is as say phi 1.

Similarly, the UMP test for technical this L naught theta greater than or equal to theta naught against L 1 theta plus tan theta naught; this is given by phi 2 x is equal to 1 if twice sigma modulus x i by theta naught is less than or equal to chi square 2 n 1 minus alpha, it is 0 if it is greater. Now, by the property of the UMP test if I look at the power functions then the power function of this will be having the values in the theta greater than theta naught region. Now, theta greater than theta naught region is the region of the null hypothesis for the L naught. So, naturally this value will be larger than this value for theta greater than or equal to theta naught. So, let me write this comment here.

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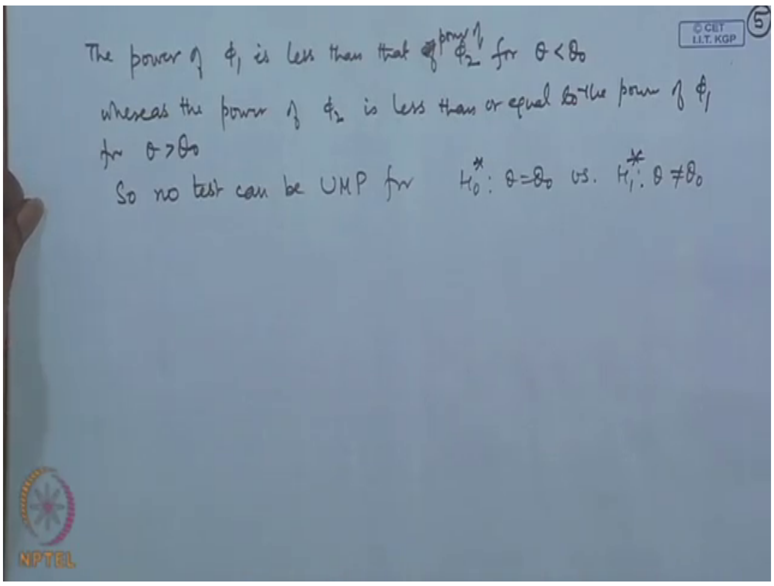

The power of phi 1 is less than that of phi 2 for theta less than theta naught. Why? Because, for theta less than naught phi 1 is having the size that is the level of significance or the probability of type 1 error that is less than or equal to alpha. The maximum value is attained at theta is equal to theta naught. So, the power function of phi 1 for theta less than theta naught is actually the probability of type 1 error which is less than or equal to alpha; whereas, for the phi 2 it is the probability of rejecting when H 1 is true. That is actually it is a 1 minus the probability of type 2 error, that value is greater than or equal to alpha because, the minimum value that is attend at theta naught.

So, what we are getting here is this is less than or equal to that power of phi 2 for this. And if I consider the power of phi 2 that is less than or equal to the power of phi 1 for theta greater than theta naught. Note here what we are claiming is that for one sided testing problems phi 1 and phi 2 both are UMP, but in the other region they have the power higher than the other one; that means, like phi 1 is UMP for theta less than or equal to theta naught. So, for theta greater than or equal to theta naught the phi 2 is UMP.

So, this one is having power less than that and similarly the other way round. So, naturally no test is UMP. So, no test can be UMP for H naught theta is equal to theta naught against H 1 theta naught equal to theta naught; let me call it star here. So, what we have concluded here is that although for one sided testing problems and for some of the two sided testing problems UMP test exist. There are certain two sided testing problems where, the UMP test does not exists and that I will be developing in the next lecture.