Statistical Inference
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Lecture - 35
Application of NP- Lemma – I

In the last lecture, I have introduced the concept of most powerful test of a statistical hypothesis and we were developing the Neyman Pearson theory. In that, the first result was the so called Neyman Pearson fundamental Lemma. And this test gives the most powerful test for testing a simple null hypothesis against a simple alternative hypothesis.

As an example, I had given the normal distribution testing for the mean. Today, I will discuss various other applications of this Neyman Pearson lemma and how then it can be extended to cover the cases when we will may have composite hypothesis, in the null hypothesis or in the alternative hypothesis. So, we will consider these application today.

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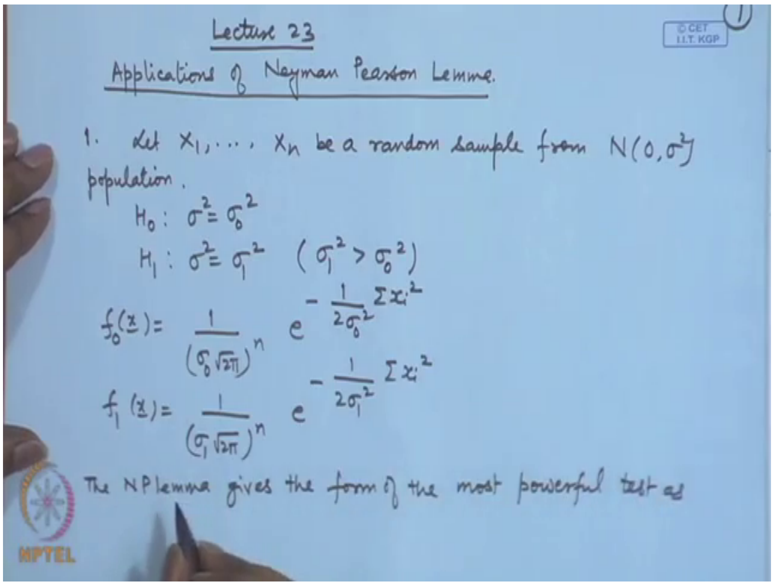

Let me start with suppose we have a say X 1, X 2, X n, let X 1, X 2, X n be a random sample from say normal 0 sigma square population. So, we were interested in testing the say null hypothesis sigma square is equal to say sigma naught square against say sigma square is equal to sigma 1 square. Now, sigma 1 square is not equal to sigma naught square. So, let us consider say the case sigma 1 square is greater than sigma naught square. So, in order to consider the application of the Neyman Pearson fundamental lemma, we should write down the distribution which is the joint density of X 1, X 2, X n under the null hypothesis and the alternative hypothesis. We call it f naught and f 1.

So, f naught x that is equal to 1 by sigma naught root 2 pi to the power n e to the power minus 1 by 2 sigma naught square sigma x i square. So, f 1 x will then be equal to 1 by sigma 1 root 2 pi to the power n e to the power minus 1 by 2 sigma 1 square sigma x i square.

Now, the Neyman Pearson lemma gives the form of the most powerful test as. So, we will consider the rejection region this is a continuous distribution. If you remember the form of the Neyman Pearson lemma, the form of the test function I will recollect here. It is given in this particular fashion.

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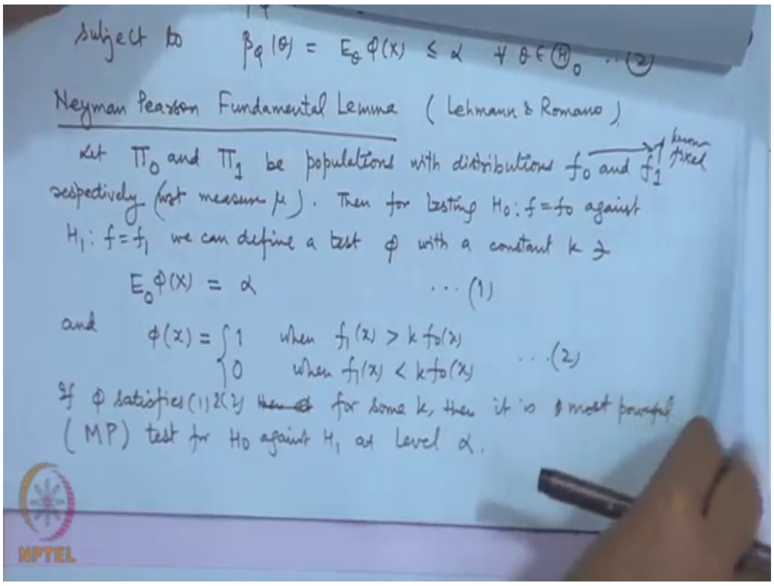

The form of the test in the Neyman Pearson lemma is given by reject H naught when f 1 x is greater than k times f naught x and accept when f 1 is less than k f naught and we are considering the rejection with probability gamma.

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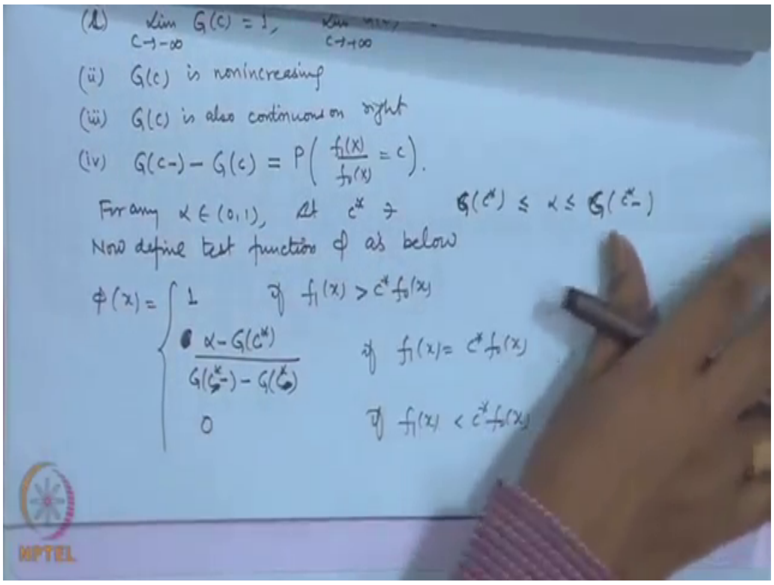

There is a constant here when f 1 is equal to constant times f naught x. Now, in the case of continuous distribution, this probability will be 0, the probability of this occurrence. Therefore, we do not have to write this thing. Rather, we can include the equality at one of the places either at the rejection or in the acceptance.

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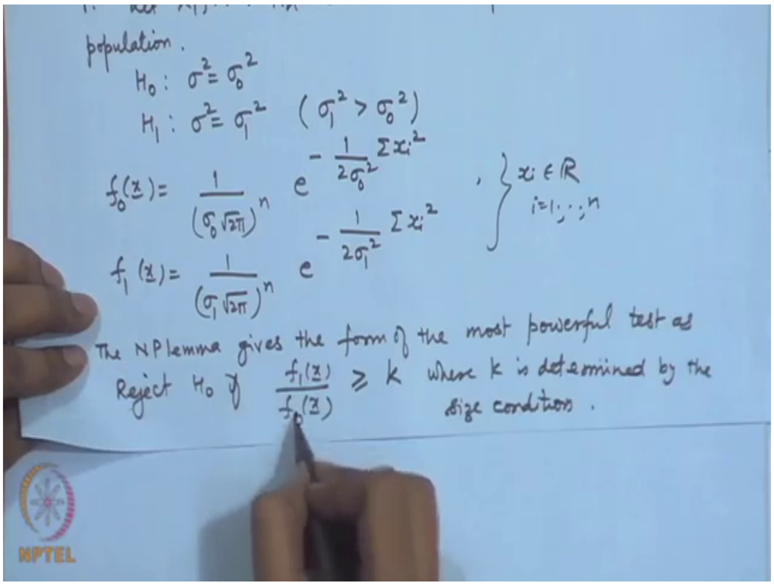

So, for convenience, I will include it in the rejection region. So, the test is reject H naught if f 1 x by f naught x is greater than or equal to k, where k is determined by the size condition.

So, let us write down this f 1 by f naught x greater than or equal to k. Since, this densities are valid for whole real line that is this x is belong to R for is equal to 1 to n. So, this ratio is defined for all values of x 1, x 2, x n on the real line.

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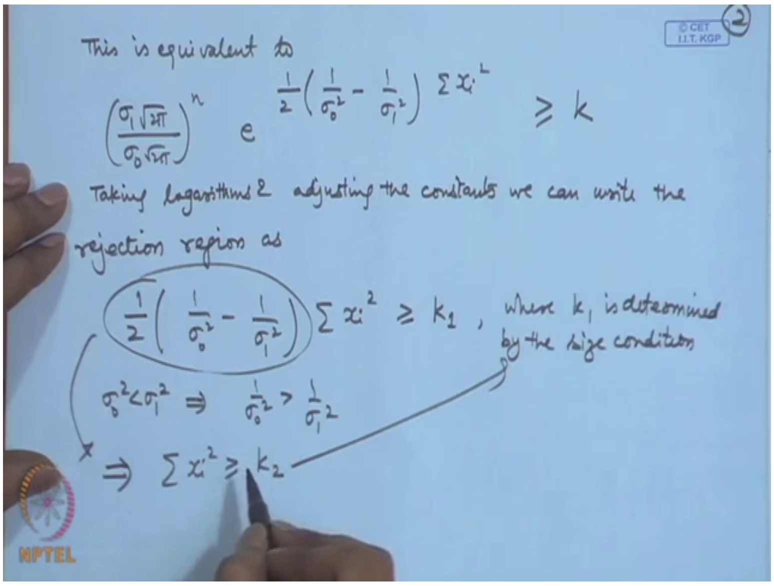

So, we write the region as, this is equivalent to, so you will have sigma 1 by root 2 pi divided by sigma naught root 2 pi to the power n e to the power 1 by 2, 1 by sigma naught square minus 1 by sigma 1 square sigma x i square greater than or equal to k.

Now, in this problem, sigma naught sigma 1 are known constants. So, I can adjust this here, I can also take log. Taking logarithms and adjusting the constants, we can write the rejection region as 1 by 2 1 by sigma naught square by minus 1 by sigma 1 square sigma x i square greater than or equal to say k 1. I have changed the name of the constant here because I will be adjusting this here and then I have taken the log here. So, the some other constant is coming. Now, earlier we mentioned that k is determined by the size condition. So, we will say that k 1 is determined by the size condition. Now, note here, we had sigma naught square less than sigma 1 square.

So, this means that 1 by sigma naught square is greater than 1 by sigma 1 square. Now, again this is a constant. So, I adjust this here. So, this is equivalent to saying sigma x i square is greater than or equal to some constant say k 2. Now, let us look at the determination of k 2. So, if k 1 is determined by the size condition, then k 2 is also determined by the size condition. Now, in order to determine this k 2, we need the probability of rejecting H naught when it is true and we will put it is equal to alpha.

So, let us look at this. So, basically what we need here is the distribution of sigma x i square because when we consider the probability statement here, this will involve the distribution of sigma x i.

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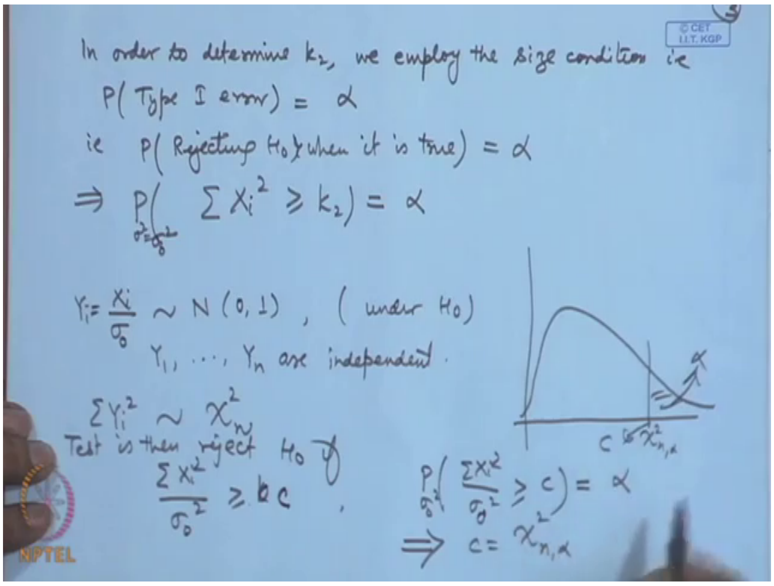

So, we write it like this. In order to determine k 2, we employ the size condition that is probability of type I error is equal to alpha that is probability of rejecting H naught when H naught is true, so when it is true that is equal to alpha. Now, let us look at this here we are saying sigma x i square greater than or equal to k 2 when the distribution is sigma naught square. That is sigma square is equal to sigma naught square, this should be equal to alpha.

Consider here, the original random variables X i's we had considered a random sample from normal 0 sigma square. So, if you consider X i by sigma that follows normal 0 1. And they are independent, let me call it Y i. So, if we consider sigma naught here, then under H naught X i by sigma naught that is Y i, this will follow normal 0, 1 and Y 1, Y 2, Y n are independent.

So, if we consider sigma Y i square, that will follow chi square distribution on n degrees of freedom. So, this test then we can consider as sigma X i square by sigma naught square greater than or equal to sum c. For example, test is then reject H naught if sigma X i square by sigma naught square is greater than or equal to k 2 by sigma naught square which I write as c here.

Now, if we want probability of this sigma X i square by sigma naught square greater than or equal to c, when sigma naught square is the two parameter value, if we want this probability to be alpha. Then, this implies that c should be chi square n alpha. That means, if we consider the curve of chi square distribution, then chi square n alpha that is this probability should be equal to alpha. So, c is here, this point will be c.

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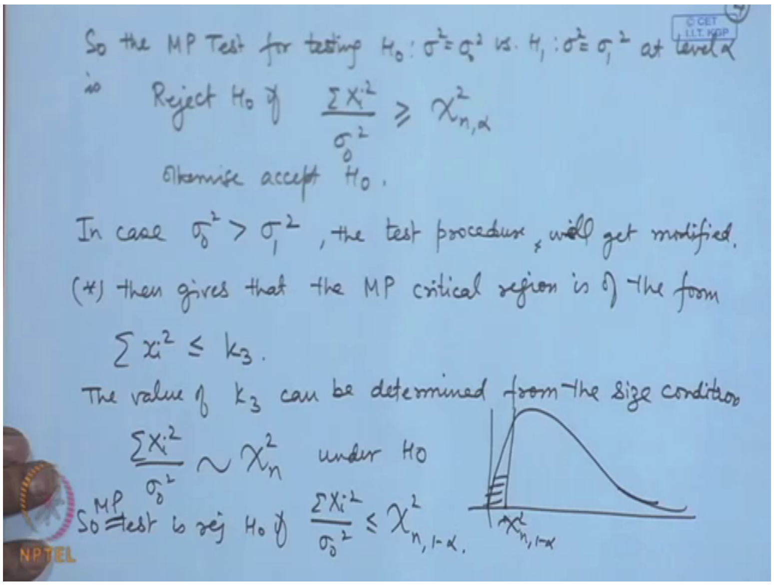

So, the test is then becoming, so the most power full test for testing H naught sigma square is equal to sigma naught square against H 1 sigma square is equal to sigma 1 square at level alpha is reject H naught if sigma X i square by sigma naught square is greater than or equal to chi square n alpha. Otherwise, accept H naught, that is we do not reject H naught here. Now, I will consider one variation in this problem here. Here, I have considered sigma 1 square greater than sigma naught square. Accordingly, our test is rejecting for larger values of sigma X i square. On the other hand, suppose I change here in place of sigma 1 square, I takes in sigma 1 square less than sigma naught square.

If I do that, then you look at the derivation of the test procedure. This quantity will become negative. If sigma naught square is greater than sigma 1 square, then 1 by sigma naught square will become less than 1 by sigma 1 square. That means, this quantity will be become negative. Then, the test procedure will get reversed. We will get sigma x i square less than or equal to some constant say k 3.

And therefore, in case sigma naught square is greater than sigma 1 square, the test procedure will get modified. So, for example you may consider let me call this condition as a star. A star then gives that the most powerful critical region is of the form sigma x I square less than or equal to say k 3. And as before, the way we have derived the probability of type I error is equal to alpha that will give me the value of k 3. So, in that case what will happen? The value of k 3 can be determined from the size condition.

Now, once again we will have sigma x i square by sigma naught square that will follow chi square n under H naught. So, now what is happening is that we need this less than or equal to quantity. So, this will become chi square n 1 minus alpha. So, test is reject H naught if sigma X i square by sigma naught square less than or equal to chi square n 1 minus alpha.

So, this is the most powerful test, N P test. So, here you have seen that how the application of Neyman Pearson lemma is helpful in driving the most powerful tests for a fixed size. That means, when we are fixing the probability of type 1 error, the most powerful test is giving me the exact method of deciding whether to accept or reject a null hypothesis. In this particular example you see, exactly we are getting the observations are X 1, X 2, X n.

So, given the observations, you calculate sigma X i square by sigma naught square and compare it with the tabulated value of chi square n alpha. Suppose alpha is equal to say 0.05 and n is a 10, then you consider the corresponding value of chi square 10 variable on 0.05. This value will be given in the tables of chi square distribution and we are in a position to take an exact decision. On the other hand, we may also consider the P value; that means, what is the value of alpha for which we will be rejecting, what is the minimum value of alpha?

So, in case if alpha is not specified beforehand, then we can consider the minimum value that and we can base our scientific decision on that fact. That this kind of situation occurs for example, in many medical problems or clinical trials where we may have to take a decision based on the given circumstances. So, we need not fix alpha in advance. This point about p value had mentioned earlier when I was giving the basic concepts here.

So, that can be done for almost all the test of this nature, that we can consider actually the P values. Now, apart from the normal distribution, let me also give applications to other distribution such as exponential distribution, double exponential distribution or we may not even be able to write down the form in a closed fashion. We may have f naught as one density, f 1 is another density.

So, I will consider few examples and exhibit that this Neyman Pearson lemma in each of these cases gives a solution; that means, we are in a position to take a decision whether to accept or reject a null hypothesis when the cases are simple versus simple. Let us consider say exponential distribution.

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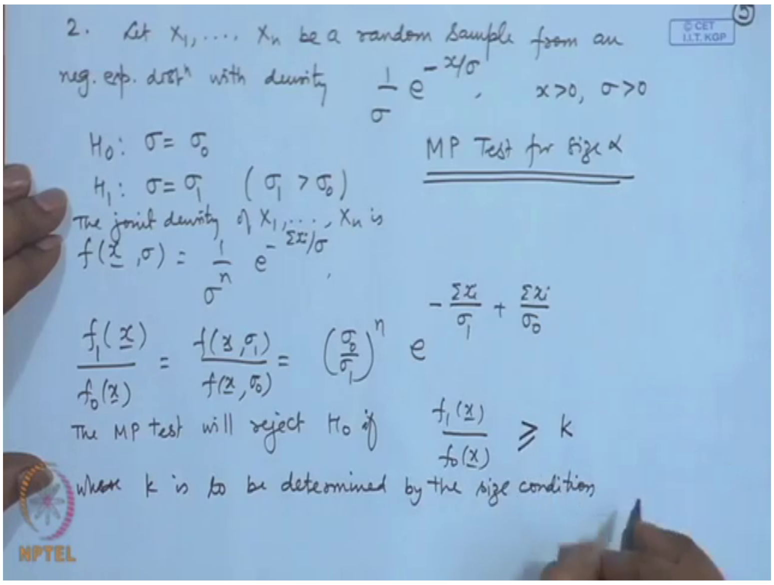

So, let X 1, X 2, X n be a random sample from an negative exponential distribution say with density function 1 by sigma e to the power minus x by sigma; x is positive, sigma is positive. Let us consider say hypothesis testing problem say sigma is equal to sigma naught against sigma is equal to sigma 1. And once again, for convenience let us consider in the beginning say sigma 1 is greater than sigma naught. We want the most powerful test for given size alpha. We will use the Neyman Pearson lemma for determination of this.

So, let us consider the form of the joint distribution of X 1, X 2, X n; joint density of X 1, X 2, X n is given by f x sigma. So, 1 by sigma to the power n e to the power minus sigma x i by sigma; note here that for all x i positive, this density is positive. Therefore, we can consider the ratio that is f 1 x by f naught x, that is the densities corresponding to sigma 1 and sigma naught value of the parameter.

So, when you write down the ratio, you will get a constant here, sigma naught by sigma 1 to the power n and then e to the power minus sigma x i by sigma 1 plus sigma x i by sigma naught. So, the most powerful test will reject H naught, if f 1 by f naught is greater than k where k is we determine from the size condition. Once again, a point to be noted here is that we are dealing with the continuous distributions. So, the probability of equality is 0, that is this is equal to k. Therefore, we may include rejection region, this equality point here. We may put it in the acceptance region also. It does not make any difference in the nature of the test because the probability of equality will be 0. So, where k is to be determined by the size condition.

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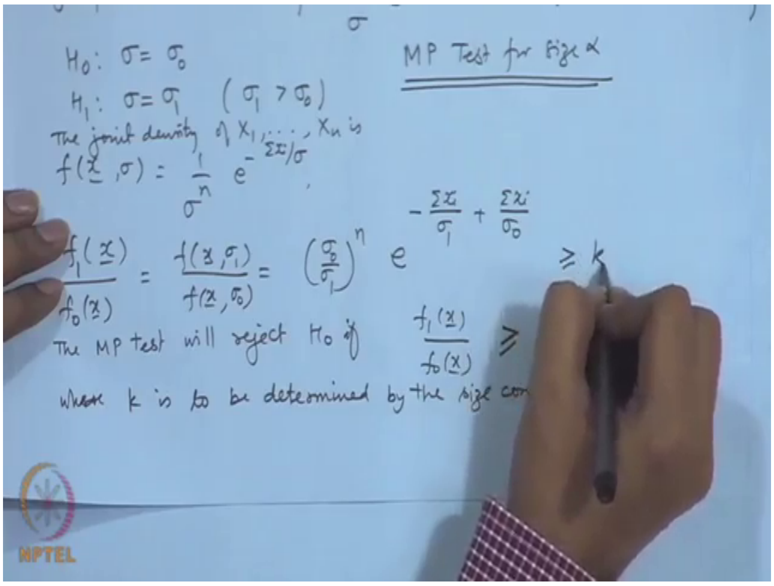

So, if you consider this ratio here, I am saying this greater than or equal to k. Now, this is a constant sigma naught and sigma 1 are known. So, I can adjust this with this coefficient on the right hand side and I can also take logarithm here. If I take the logarithm here, I will get sigma x i into 1 by sigma naught minus 1 by sigma 1.

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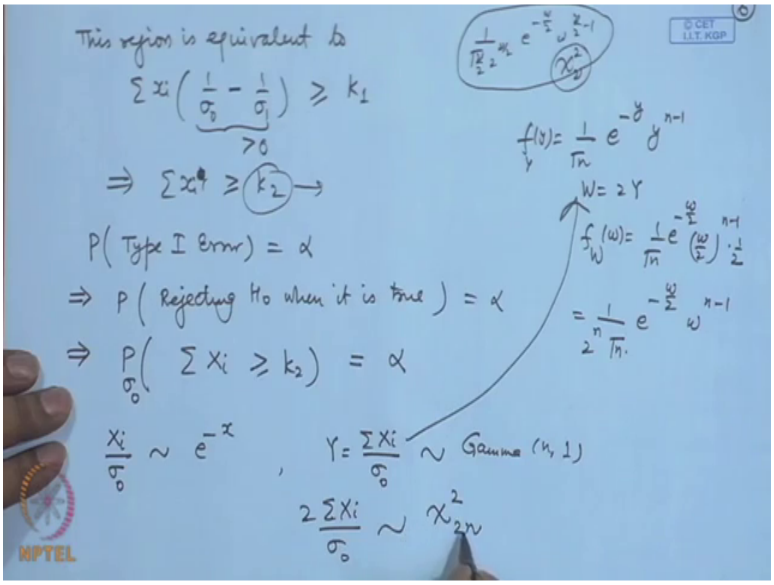

So, this region is equivalent to sigma x i 1 by sigma naught minus 1 by sigma 1 greater than or equal to some constant k 1. Now, as before in the normal distribution case, this constant 1 by sigma naught minus 1 by sigma 1 the sign of this will be positive because I am taking sigma naught to be less than sigma 1. So, this is positive.

So, this region is equivalent to sigma x i greater than or equal to sum k 2. And once again, this k 2 is to be determined from the size condition. So, if I consider probability of type I error equal to alpha; that means, probability of rejecting H naught when it is true that is equal to alpha, then this is implying probability of sigma X i greater than or equal to k 2; when sigma naught is the true parameter value, then it is equal to alpha.

That means, I need to look at the distribution of sigma X i when sigma is equal to sigma naught. Now, we know that the sum of independent exponentials of this nature is actually a gamma. So, we can consider the derivation of the constant k 2 based on this. So, let us look at this. If I consider, say X i by sigma naught, then that will follow exponential with parameter simply 1. If I consider, say sigma X i by sigma naught, then that will follow gamma n 1. If I consider twice sigma X i by sigma naught, then that will follow chi square distribution on 2 n degree of freedom. See we can write down the density here, suppose I am considering this as say y.

So, what is the distribution of y? Fy is equal to 1 by gamma n e to the power minus y, y to the power n minus 1. Say if I consider say w is equal to 2 y, then what will be the distribution of w ? 1 by gamma n e to the power minus w by 2, w by 2 to the power n minus 1 into half that is equal to 1 by 2 to the power n gamma n e to the power minus w by 2 w to the power n minus 1.

So, if we consider the form of a chi square distribution, the chi square distribution on new degrees of freedom is given by 1 by gamma nu by 2, 2 to the power nu by 2 e to the power minus w 2 by w to the power nu by 2 minus 1. This is the form of a chi square distribution on new degrees of freedom. So, if you compare this with this, actually we are getting 2 n degrees of freedom. So, chi square twice sigma X i by sigma naught, this will follow chi square distribution on 2 n degrees of freedom when H naught is true.

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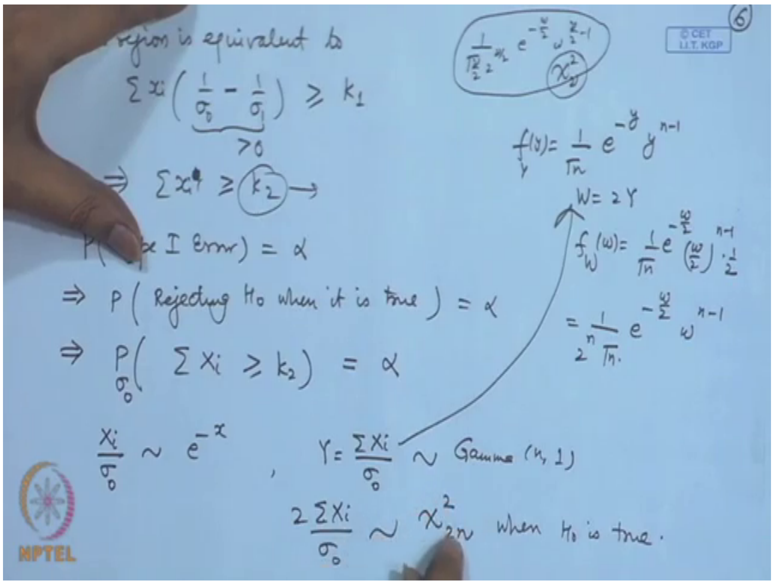

Therefore, the rejection region can be written in the terms of chi square value on 2 n degrees of freedom.

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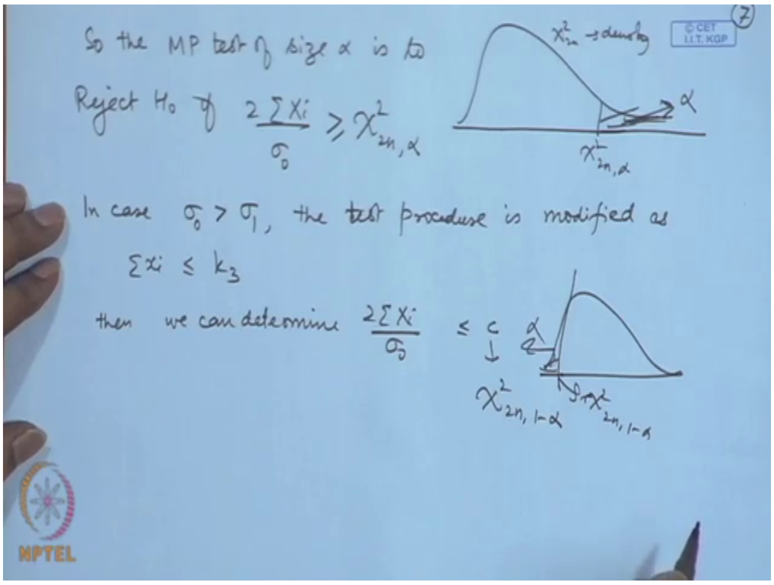

So, if we consider say chi square 2 n density, so this point is chi square 2 and alpha, so this probability is alpha say. So, the most powerful test of size alpha is to reject H naught if twice sigma X i by sigma naught is greater than or equal to chi square 2 n alpha, accept otherwise. So, you can easily see here that we are able to give exact decision making procedure given the level of significance.

Now, if the level of significance is not specified in the beginning, then you can look at what is the probability of this that minimum level at which this test will be rejected, this null hypothesis will be rejected. So, that will be the P value. So, I have been, I am considering both of this P value thing and level of fix level of significance in all these situations. Once again, note here that if I have a modification in my original null hypothesis. In place of sigma one being greater than sigma naught, if sigma 1 is less than sigma naught, then there will be a modification here because this coefficient will become negative.

If this coefficient becomes negative, then the region will turn out to be sigma i less than or equal to something here and therefore, the rejection region will then become left handed. In case sigma naught is greater than sigma 1, the test procedure is modified as sigma x i less than or equal to say k 3. Then, we can determine sigma X i by sigma naught twice less than or equal to say c. So, c will become than equal to chi square 2 n 1 minus alpha because now this is the left handed point here. This probability is alpha, so, chi square 2 n 1 minus alpha.

Now, you can see here that in many of these problems, we are able to work out the exact distribution here and one interesting thing here is that the range of the random variables is a same. Therefore, this writing down the ratio f 1 by f naught etcetera is quite convenient and when we write down the final test function here, then we are able to derive the distribution of that.

Now, in many cases this will be dependent upon the situation. We may not have state forwardly the full region divided by full region. We may have partial regions; sometimes the range of the variable will be dependent upon the parameter. Therefore, the range of the two densities may not be exactly the same. I will explain this through a couple of examples.

So, let me take case for when the full region is the same, but the distribution gets the form of the density gets modified midway. That means, for partial values of x, you have a form of density function. For another part, we may have another density function. So, let me take up this case and I will also consider one case when the range of the variable is dependent upon the parameter. Therefore, the two densities are positive not on the full region, but on partial regions. So, let us consider these cases.

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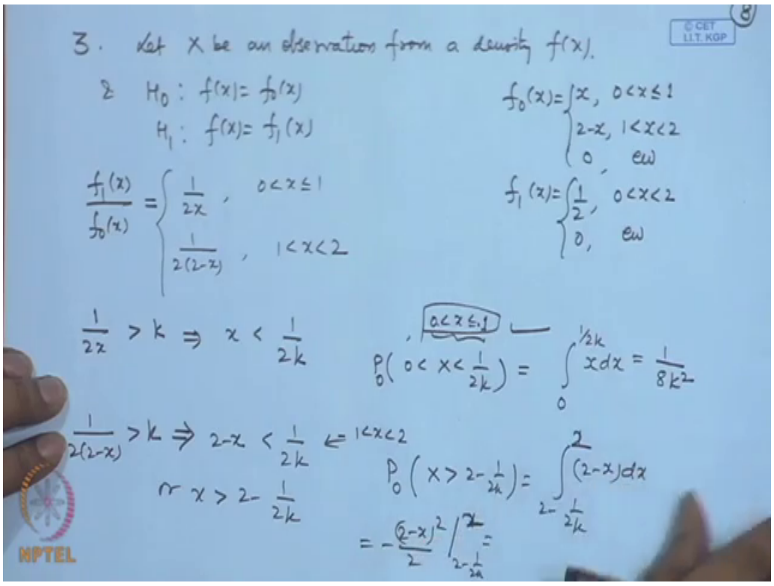

Let X be an observation from a density f x and H naught f x is equal to f naught x H 1 f x is equal to f 1 x. And f naught and f 1 are define like this, f naught is the triangular distribution it is equal to x for 0 less than x less than or equal to 1 and it is equal to 2 minus x for 1 less than x less than 2. It is actually the triangular distribution and of course, it is 0 elsewhere. And f 1 x is half for 0 less than x less than 2.

So, this is nothing but the uniform distribution on the interval 0 to 2. Now, you note here the distribution under x naught is a distribution over the range 0 to 2, but the form of the density function changes at the point 1 whereas the second density is having the same form throughout. So, when we write down the form of the most powerful critical region the Neyman Pearson lemma, we have to be careful in writing down the regions. So, for example, consider this f 1 by f naught. Here, we assume that our decision making process is based on one observation. Of course, we may consider n observations also and of course, it will increase the difficulty or we can say complication in the nature of the derivation.

So, this a value is equal to now you look at f 1 by f naught that will be 1 by 2 x if 0 is less than x less than or equal to 1 and it will be equal to 1 by twice 2 minus x for 1 less than x less than 2. Now, the question is if an x is there which is outside this region, the thing is that under H naught and H 1 that will have probability 0. So, we will not consider that situation here. So, if I consider the rejection region 1 by 2 x greater than k, then this is equivalent to saying x is less than 1 by 2 k.

Now, this is for the portion 0 less than x less than or equal to 1. So, if you consider probability of this region that is 0 less than X than 1 by 2 k, this is for under H naught and here we will consider for 0 to 1 only for 0 to 1 the density is x. So, if you integrate this, it is becoming x square by 2. So, you will get 1 by 4 k square divided by 2 that is 1 by 8 k square. If we consider 1 by twice 2 minus x greater than k, then this is equivalent to 2 minus x less than 1 by 2 k or x is greater than; x is greater than 2 minus 1 by 2 k.

Now, this part is for 1 less than x less than 2. So, the probability of X greater than 2 minus 1 by 2 k that is equal to 2 minus 1 by 2 k 2 1 2 minus x d x. So, that is equal to 2 minus x whole square by 2 with the minus sign from 2 minus 1 by 2 k 2 1. So, this is again evaluated. If you put here no sorry, this is a put 2. So, if you look at the value at to this is becoming 0 and when we put 2 minus 1 by 2 k, this is again 1 by 2 k whole square so, it is again 1 by 8 k square.

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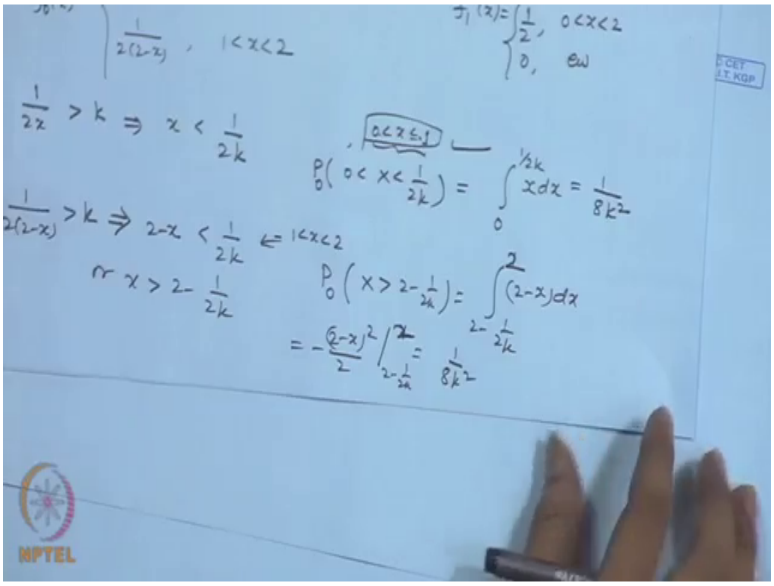

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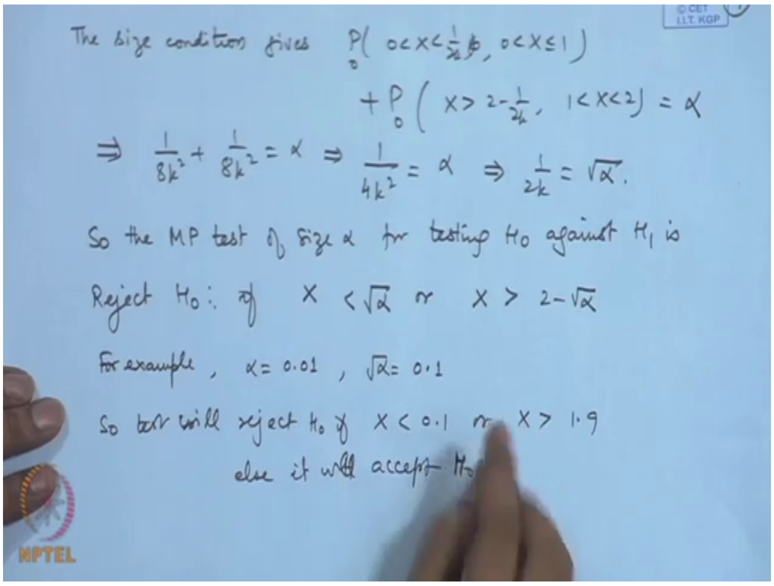

So, if we write down the size condition here that is the probability of, so the size condition gives probability of type I error that is 0 less than X less than 1 by 2 k for 0 less than X less than or equal to 1 plus X greater than 2 minus 1 by 2 k for 1 less than x less than 2 is equal to alpha. Note here, that these regions are dependent upon this condition. So, we have to considered the probability under this. We have calculated both of these probabilities. So, it is becoming 1 by 8 k square plus 1 by 8 k square is equal to alpha 1 by 4 k square is equal to alpha; that means, 1 by 2 k is equal to a square root of alpha.

So, the region of rejection is becoming x is less than root alpha or is x is greater than 2 minus root alpha. So, the most powerful test of size alpha for testing H naught against H 1 is reject H naught, if X is less than root alpha or X is greater than 2 minus root alpha. Once again you note here that we are able to provide exact test here, that is the test tells exactly what decision one has to take given a value of X.

So, for example, let us choose a alpha is equal to say 0.01, then alpha is equal to root alpha will become 0.1. So, test is done test will reject H naught if X is less than 0.1 or X is greater than 1.9; else it will accept H naught; that means, if I am having an observation between 0.1 to 1.9, then the test will accept H naught; that means, it will have no region to reject H naught.

And other hand if X is less than 0.1 or X is greater than 1.9, then this is not supporting H naught; that means, you will tend to reject H naught here. In this particular example, I have shown that even the form of the distribution maybe changing over the range of the sample space; however, the Neyman Pearson lemma is able to provide a exact test at a given so that I will be taking up in the following lecture.