Statistical Inference
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Lecture – 30
UMVU Estimation, Ancillarity – II

Before we discuss other examples, let me also give some further relationship between the completeness and independence etcetera. Now there is a famous result called Basu's theorem where we consider certain statistics whose distribution does not depend upon the parameter. So, I define what is known as Ancillary statistic.

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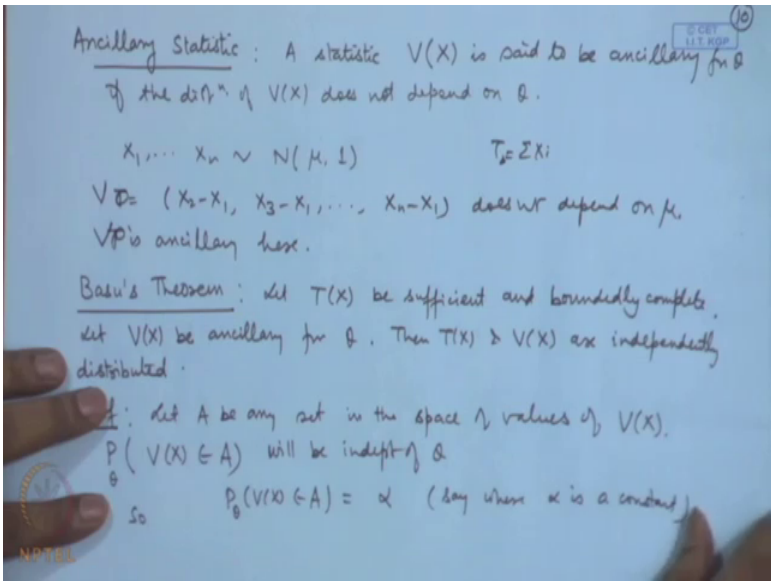

So, a statistic let me call it V of X is said to be ancillary if the distribution of ancillary for say parameter theta, if the distribution of V X does not depend on theta. For example, if I consider say X 1 X 2 X n follows normal mu 1 and I consider T as say X 2 minus X 1 X 3 minus X 1 and so on X n minus X 1, then the distribution of this does not depend on mu.

So, T is ancillary here. Let me call it V here because T is for the sigma X i here or X bar. Then we have the following theorem called Basu's theorem named after D Basu. Let T be sufficient and boundedly complete. So, if it is complete automatically bounded completeness will be true let V X be ancillary for theta. Then T X and V X are independently distributed.

Let us look at the proof of this. So, let A be any set in the space of values of V ok. So, if I consider probability of V X belonging to A, then this will be independent of theta. Because, the distribution of V X does not depend upon theta, so this is going to be independent of theta.

So, if we want to write a statement like this P theta V X belonging to A, this is some constant say alpha, alpha is a constant. Now let us consider a function say W of T that is equal to probability of V X belonging to A given T. Now this is a probability, so W is a bounded function, W is a bounded function.

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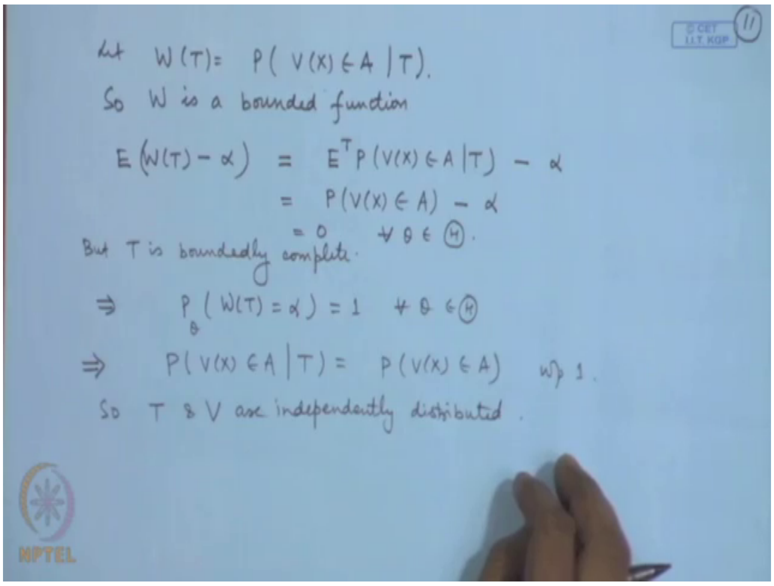

Now, let us consider expectation of W T minus alpha. Now what this is going to be? This is expectation of probability V X belonging to A given T. Now this expectation is over what? This conditional probability is a function of T. So, this is expectation over T minus alpha. Now this will become nothing, but probability of V X belonging to A minus alpha, which is actually equal to 0 for all theta. But T is boundedly complete, T is boundedly complete.

So, this implies that probability that W T is equal to alpha must be 1, but what is this statement? This statement is equivalent to saying probability of V X belonging to A given T is equal to alpha. What was alpha? Alpha was probability V X belonging to A; that means, the conditional probability of V given T is same as unconditional probability of V, this is with probability 1.

So, T and V are independently distributed. Let us look at 1 or 2 applications of this here. So, if we consider this problem here, X 1 X 2 X n follows normal mu 1 and here T is equal to sigma X i, this is complete and sufficient. So, this is complete and sufficient and X 2 minus X 1 X 3 minus X 1 X n minus X 1 has a distribution which does not depend upon mu. Then T and V will be independently distributed.

And of course, this is all also a well known result in the normal distribution theory that sigma X i and S square X bar and S square are independently distributed. So, that is a the proof is actually through this only that, we firstly show that X bar and X 2 minus X 1 X 3 minus X 1 etcetera are independent. And therefore, since S square is directly a function of this therefore, X bar and S square are also independent. So, that is confirmed here.

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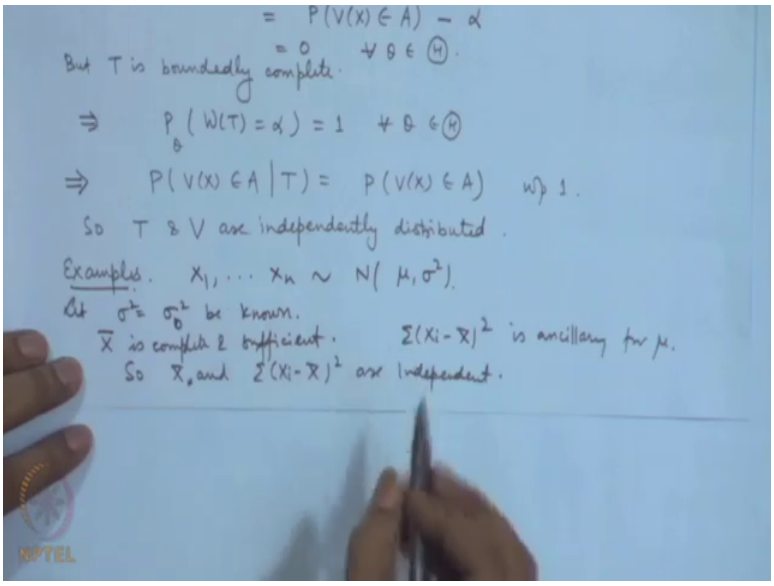

Let us generalize this example to normal mu sigma square. So, let us consider say X 1 X 2 X n follows normal mu sigma square. So, let us take say sigma square is equal to sigma naught square be known. If that is so then X bar is complete and sufficient. And at the same time, if we consider sigma X i minus X bar whole square this is ancillary for mu. Therefore, X bar and sigma X i minus X bar whole square are independent.

Now, if we are writing this statement here this sigma naught square does not play a role here, because this was arbitrarily fixed, so here if we say it for all sigma naught square; that means, X bar and sigma X i minus X bar whole square are independent in general here.

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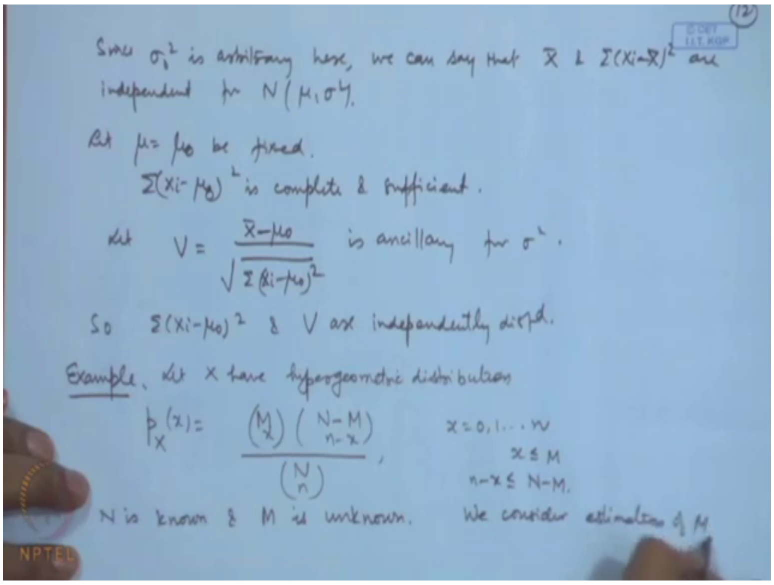

So, we can say here, since sigma naught square is arbitrary we can say that X bar and sigma X i minus X bar whole square are independent for normal mu sigma square case here. Let me take another application here. Suppose I fix mu is equal to mu naught, if we take this then sigma X i minus mu naught square is complete and sufficient. Let V be of the form say X bar minus mu naught divided by square root sigma X i minus mu naught square. You can see here if I divide by sigma here in the numerator and the denominator then the distribution will become free from the parameters here, this is ancillary here.

So, sigma X i minus mu naught square and V they are independent here. Let me consider some further applications of the minimum variance unbiased estimation. Let X have hypergeometric distribution that is the probability mass function is given by M c x N minus M c n minus x divided by N c n. Here x is from 0 1 to n and of course, subject to the restrictions that x is also less than or equal to M and n minus x is less than or equal to N minus M. Here N is assumed to be known and M is unknown ok. So, we consider estimation of M. So, if we write down the distribution it is already in the factorizable form. So, X is certainly sufficient ok.

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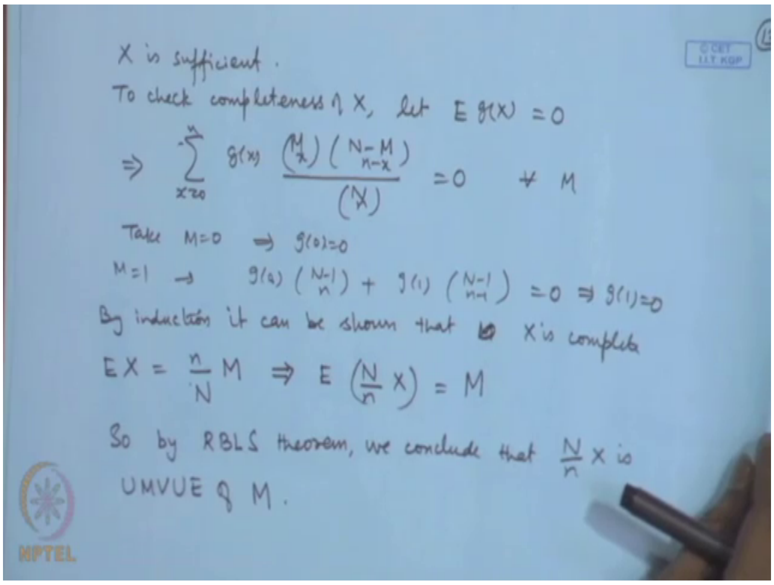

So, X is sufficient here. Let us look at the completeness. To check completeness of X, let us take expectation of a function of x is equal to 0. Then that is equivalent to saying g x M c x N minus M c n minus x divided by N c n is equal to 0. For x equal to 0 2 n subject to those conditions here for all M. If I take M is equal to 0 here, then this will give me g 0 is equal to 0. If I take M is equal to 1 and that will give me g 0 n minus 1 c n plus g 1 N minus 1 c n minus 1.

Now, g 0 is 0 that means, g 1 is also 0. So, by induction we can prove that it can be shown that M is that x is complete. Now what is expectation of x; that is equal to n by N into M. So, that means, expectation of N by n X is equal to M. So, x is complete and sufficient and this is an unbiased estimator of M. So, we conclude that by Rao Blackwell Lehmann Scheffe Theorem, we conclude that N by n X is UM VUE of M. Let me give one more application.

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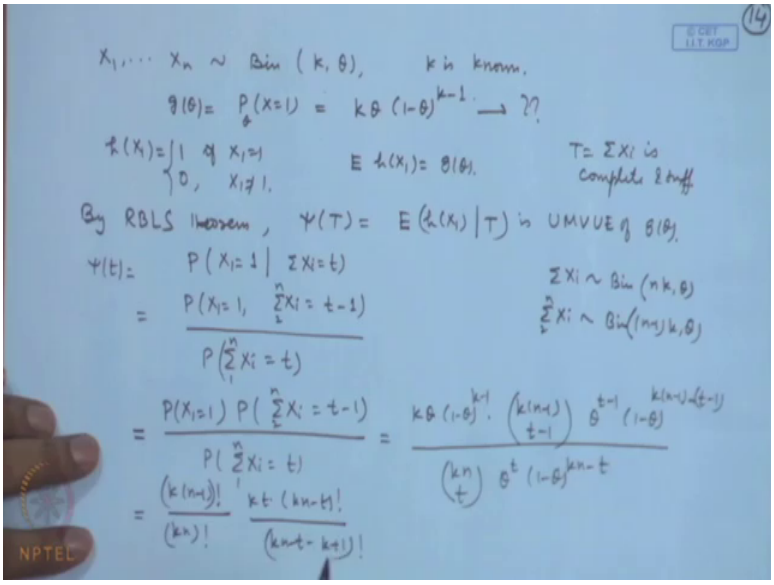

Let us consider say a random sample from a binomial distribution with parameter say k and theta, where k is known. Let us define a function say g theta is equal to probability of 1; that is k theta into 1 minus theta to the power k minus 1. We want unbiased estimator of this. Let us define a function say h X 1 is equal to 1 if X 1 is equal to 1 it is 0 if X 1 is not equal to 1. Then expectation of h X 1 is equal to g theta.

So, by Rao Blackwell Lehmann Scheffe theorem psi T that is here T is equal to sigma X i is complete and sufficient. So, that is equal to expectation of h X 1 given T, this is UMVUE of g theta. So, we can consider here evaluation of this psi T function that will be equal to probability of X 1 is equal to 1, given sigma X i is equal to T; that is equal to probability of X 1 is equal to 1 sigma X i from 2 to n is equal to T minus 1 divided by probability sigma X i 1 to n is equal to T.

Now, sigma X i will follow binomial n k theta sigma X i from 2 to n will follow binomial n minus 1 k theta. So, if we substitute these values here, probability of X 1 is equal to 1 into probability of sigma X i 2 to n is equal to T minus 1 probability sigma X i is equal to T 1 to n. Then that is equal to k theta into 1 minus theta to the power k minus 1 and then this is equal to k into n minus 1 c t minus 1 theta to the power t minus 1, 1 minus theta to the power k into n minus 1 minus t minus 1 divided by k n c t theta to the power t into 1 minus theta to the power k n minus t.

The terms which contain theta they got cancelled out here and we are left with k into n minus 1, factorial divided by k n factorial into k t into k n minus t factorial divided by k n minus T minus k plus 1 factorial. So, if we consider this function here, that is the UMVUE of g theta here. Let me end with one example in the exponential distribution.

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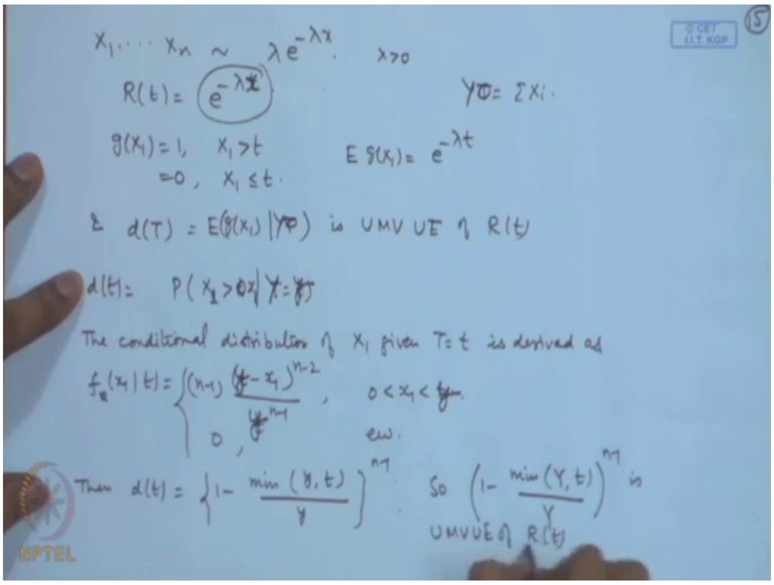

Suppose we have a random sample from exponential distribution with parameter say lambda and we are looking at the reliability function R t is equal to e to the power minus lambda t. We want the UMVUE of this. So, define the function g X 1 is equal to 1 if X 1 is greater than t, it is equal to 0 if X 1 is less than or equal to t.

So, expectation of g X 1 is equal to e to the power minus lambda t and expectation of g X 1 given T; that is equal to say d of T is UMVUE of R t, that is further reliability function. The minimum variance unbiased estimator will turn out to be the conditional expectation of g X 1 giving T. So, if I evaluate this that is nothing, but probability of X 1 greater than t, given T is equal to t where, T is equal to sigma X i here.

Now, here we need the conditional distribution of X 1 given t. In the discrete case, we were able to write down it as a joint probability divide by the probability of this term, but in the case of continuous distribution we cannot write that statement. So, what we do? We do derive the conditional distribution of X 1 given t, and this distribution can be easily derived. The conditional distribution of, the conditional distribution of X 1 given T is equal to t is derived as f of x 1 given t is equal to t minus x 1 to the power n minus 2 divided by t to the power n minus 1 into n minus 1 0 less than x 1 less than t it is equal to 0 elsewhere.

So, this probability of X 1 greater than t then turns out to be simply 1 minus minimum of X 1 greater than y. So, that is equal to the conditional probability of X 1 greater than t, given T is equal to t, turns out to be simply minimum of t and ok, so there is a confusion here, I should have used a different notation here x 1 here. So, this turns out to be there is a problem here. Let us use a different notation Y here and this is Y, this is Y is equal to say small y. So, this is y here y. So, then this will be equal to minimum of y and t divided by y to the power n minus 1. So, we conclude that 1 minus minimum of Y and t divided by Y to the power n minus 1 is UMVUE of reliability function in the case of exponential distribution.

So, we have seen here today, that the properties of sufficiency and completeness are extremely helpful in determining the problem of or solving the problem of minimum variance unbiased estimation. Essentially it reduces the problem to find out the unique unbiased estimator which can be done easily determine.

In the next class we consider the different approaches to the estimation. There is a approach of invariance and then Bayesian and minimax estimation, I will be introducing in the next classes.