Introduction to Abstract and Linear Algebra
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Lecture - 06
Equivalence Relation

So, we are talking about Equivalence Relation, so we will discuss more on that; so just to recap.

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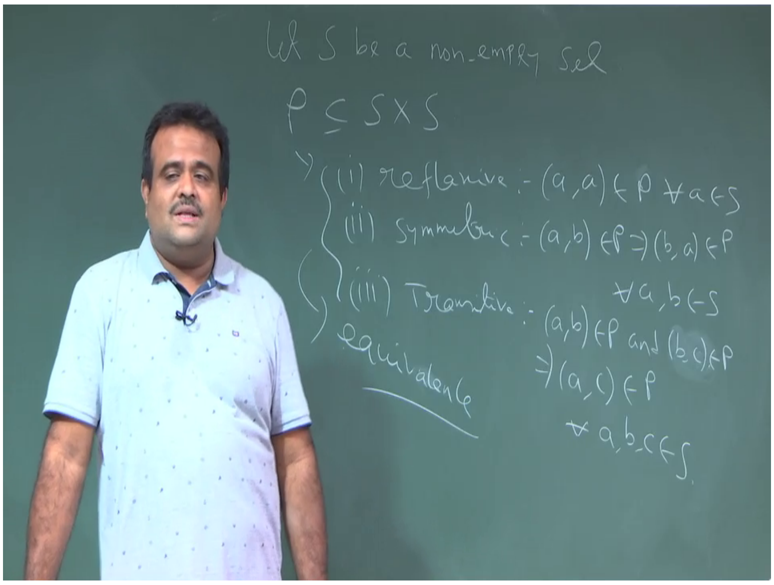

So, what is the equivalence relation? Suppose let S be a be a non empty set, then we know that relation binary relation is basically a subset of the Cartesian product any subset of Cartesian product is a relation.

Now, we call this relation to be equivalence relation, if it is reflexive; that means, if a is belongs to rho for all a symmetric; that means, if a b is belongs to rho, then this must be imply b a is belongs to rho and this must be true for all a b. Whenever a b is belongs to rho then b a must be belongs to rho and then transitive. So, if we have a b belongs to rho, and b c belongs to rho b c belongs to rho then this must imply a c belongs to rho, and this must true for all such a b c.

Now, if all this satisfy 3 satisfied, then the relation rho is called equivalence relation rho is called an equivalence relation ok. So, we have discussed some example of the equivalence relation in the last class. So, now, we will talk about the equivalence classes, we defined the equivalence classes on this non empty set S.

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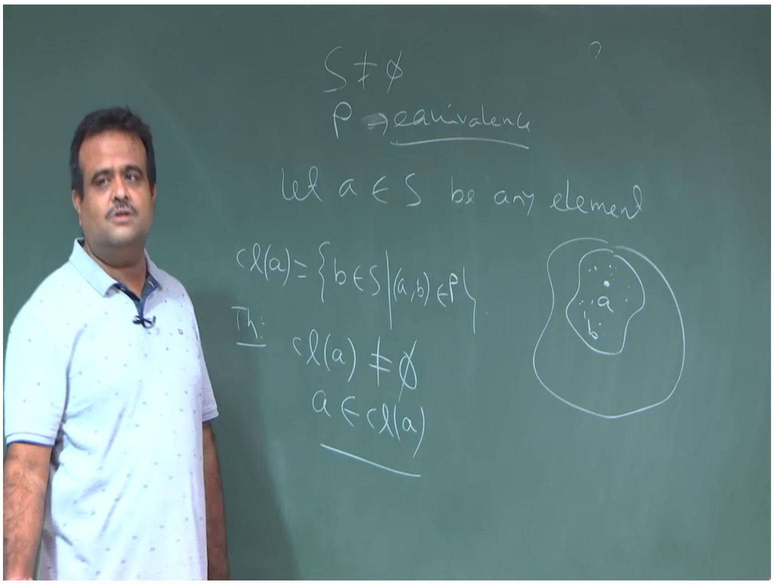

So, suppose we have a set S which is non empty and we have a equivalence relation on this, equivalence relation on this ok. So, now we define equivalence classes let a be any element we take an element from S. So, this is our S, we take an element from S a ok. Now we define the class this is this way or we can write this way, anyone of this way class of a this is nothing but this is the set of all b from a such that a is related with b; that means, a b belongs to rho and this is equivalence.

So, it is same as b a belongs to rho, a b belongs to rho. So, this class; that means, this is the class of all the elements b, which are related with which are related with a. So, we follow our notation this way. So, box. Box is another notation in the integer, which is the integral part. So, anyway this two are different notation I mean if there is any confusion, you can use cl a class of a ok.

So, this is the class of a, this is the set of all element b such that a is related to b ok. Now this class is non empty. So, this is we can claim cl of a is non empty, there must be at least one element in this class why this is non empty why there has to be on element who is the element? A is the element because this is a reflexive, this relation is reflexive. So, a must be belongs to rho so; that means, a is always in this class. So, there is at least one element a so; that means, it is non empty, this classes are always non empty ok.

So, now we will have some result on this class. So, this is the first result, this class is non empty now another result is.

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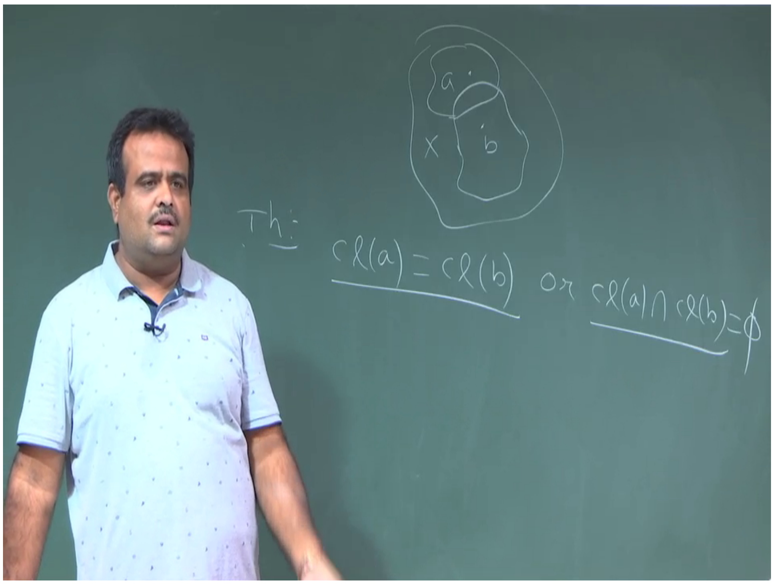

So, if we take two element a and another element b now if you consider the class of a class of b ok. Now the theorem is telling either class of a is same as class of b or they are disjoint. So that means, this type of situation will not occur, this is a class of a this is class of b this is not possible. So, either one of this either this will happen either these two class are same or these two class are different. I mean sorry this is empty set either they are these two sets are same or they are and they these two set are empty ok.

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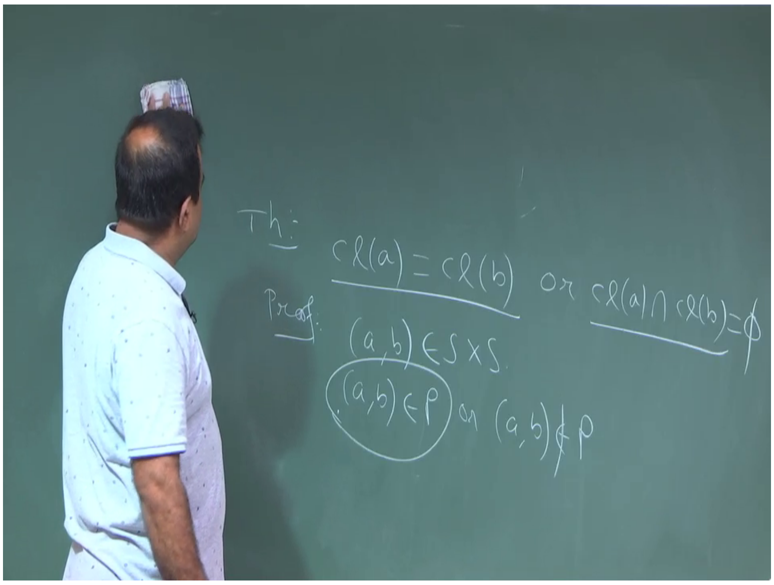

So, how to prove that? So, this is true for any a b. So, this is true for any a b. So, how to prove that? You have to prove this. So, a b belongs to S cross S. Now if a b belongs to S cross S that means, there are two possibilities a b either belongs to rho or a b does not belongs to rho.

Now if a b belongs to rho so; that means, this case then we have this one we will prove that and if a b does not belongs to rho, then they have two disjoint class they will not have any common element in the class. So, that we have to prove let us prove the first one.

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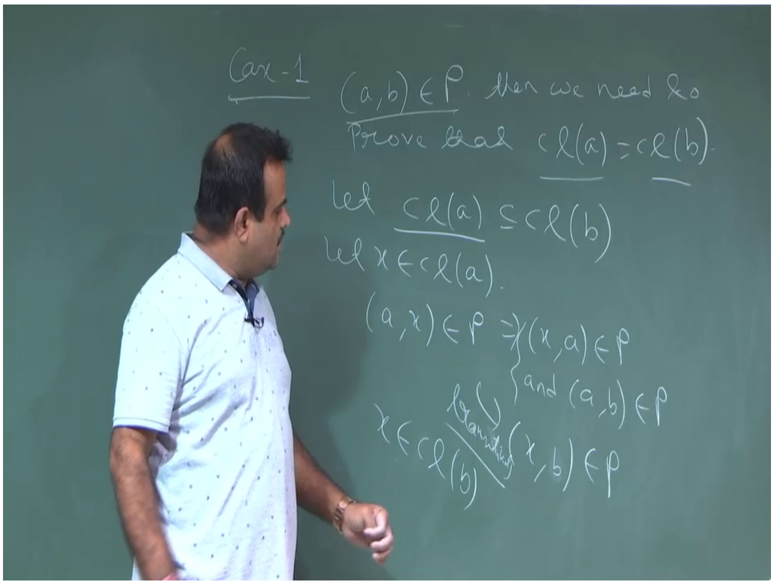

So, the case 1 if a b belongs to rho, then our claim is their class is same then class of a is basically same as class of b. Then we need to prove that class of a is basically class of b and so this how to prove this ok. So, how to prove these two class are same. So, for that we need to these are basically sets. So, for that this must be a subset of this and that must be subset of this.

So, let. So, let us try to prove this one, class of a is a subset of class of b. So, further we need to take an element from here, let x belongs to class of a then we have to prove that x belongs to class of b. So, x belongs to class of may a means what? The x belongs to class of a means, a is related with x that is the definition of class of a class of a is a consistent.

Now if a this is this is so; that means, a x belongs to rho. So, now, this means x a belongs to rho because symmetric property of rho and we know this a b belongs to rho we know. So, and also we know a b belongs to rho ok. Now these two both give us x a belongs to rho sorry x b, because of transitivity property because rho is symmet equivalence. So, it is the transitivity property transitivity property so; that means, x belongs to x x x belongs to class of b.

So, if you take a element from class of a, it is an element of class of b and this is happening because a b are related. Similarly we can prove this is the subset of this, we have to take y from here and then you have to show this y is belongs to this so; that means,. So, this is the other way let us try to prove that very similar way.

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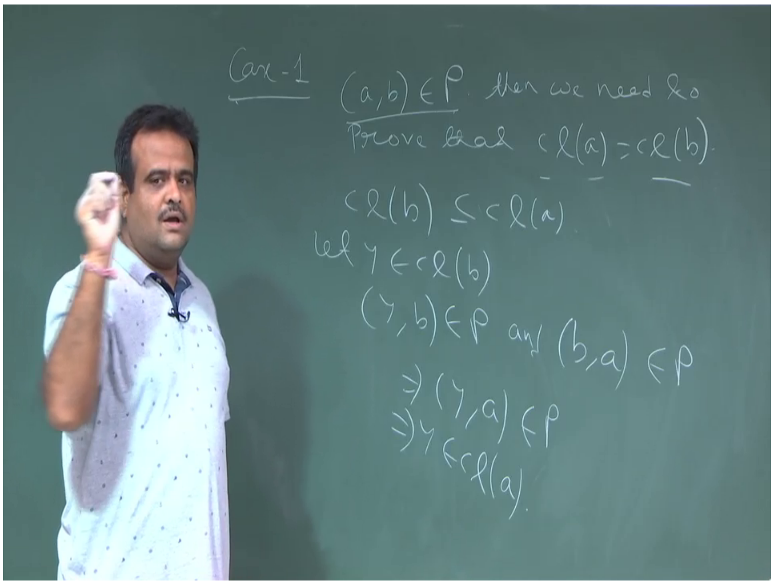

So, now, we need to show that class of b is a subset of class of a now how to prove that? Let y belongs to class of b so; that means, y and b are related. So, this means b y is related this is by the property of symmetricity symmetric, and we know that a b is related sorry y b is related, y b is related we do not need this symmetricity property to use y b is related and we know that a b is related; that means, b is also related because rho is symmetric. Now from these two we can say y a is related because of transitivity property. If y is related means y belongs to class of a.

So; that means, this two are same. So, class of a and class of b are same basically if we take two element which are related. So, this is one observation another observation is if they are not related, if you take two point a b two element and they are not related then their class is disjoint. So that means, if we take a and if we take a. So this is a class of a, and if you take another element b which is related with this a then class of b is same as class of a.

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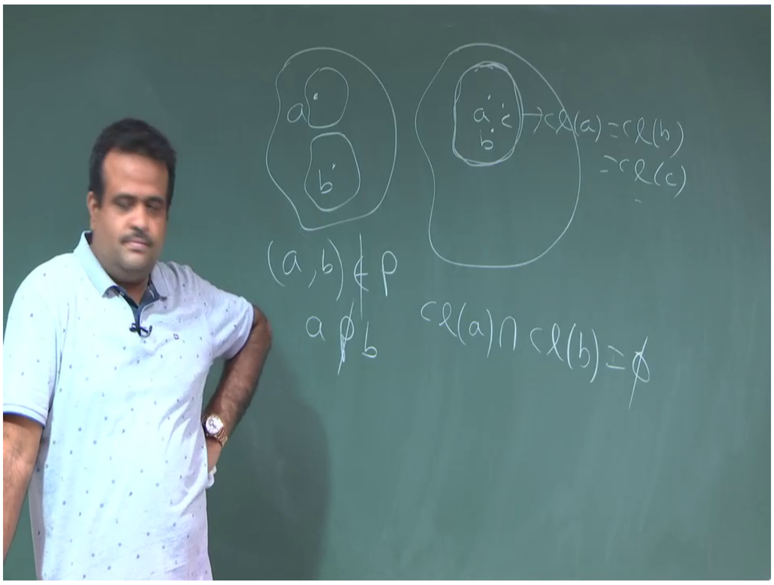

So, this is class of a, which is same as class of b and if you take another element c which is same as class of c like these. So, they are basically give us the same subset ok.

So, now another part; so if you take two element a b, we cannot related. If a b is not related; that means, a is not related to b, then we have to show then had they cannot have a common element then we have to show this class of a and class of b is disjoint, they have no common element. So, this is this is the second part of the theorem.

So, how to prove this? So, let us try that by using the contradiction method suppose they are suppose they are nonempty they are not empty.

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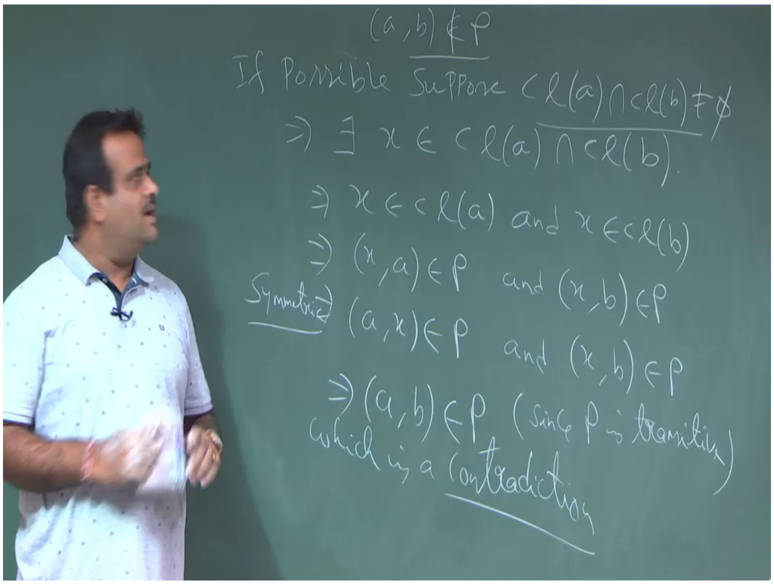

Suppose if possible if possible suppose class of a and class of b is not an empty set so; that means, this implies there exists an element x, this is the symbol for there exist. There exist an element x, which is the which is in the intersection and we need to arrive to a contradiction and here we know that a b is not related, a b does not belongs to rho that is that that information we have ok.

So, now this is implies what, x belongs to class of a and also x belongs to class of b. Now x belongs to class of a means what? X and a are related and x belongs to class of b means what? X and b are related now this imply by the symmetric property, we can say a x belongs to rho this is by symmetricity because rho is equivalence relation. So, rho is symmetric also, and x b belongs to rho. Now this x a belongs to rho x b belongs to rho and rho is equivalence; that means, rho is transitivit transitive also. So, if rho is transitive then these two will give us what?

A b belongs rho since rho is transitive, but that is not possible we already assumed a b is not belongs to rho which is a contradiction which is a contradiction against our assumption that, they are not empty contradiction, which is a contradiction. Hence our assumption is false. Hence, this if this is if they are not related, then the class of a and class of b is having no common element.

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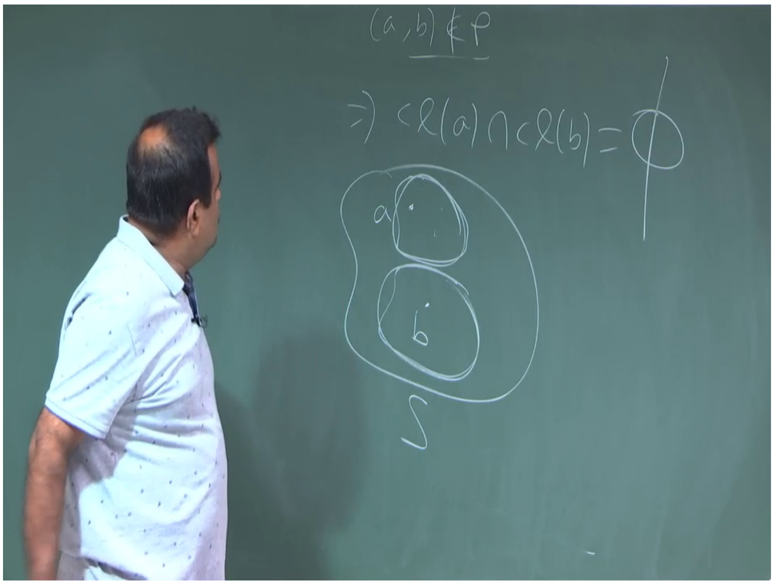

So, this implies class of a class of b is empty set ok. So, in the other one if we this is our S, if we take two elements a b which are not related then it is if this is the class of a this is this is the class of b, they will have no common element. Because once they have a common element then, there has to be in the same class and there. So, all the elements over here which are related, they have a same class with this we have a same class with this.

So, this will give you give us the idea of the partition ok. So, we just rewrite this theorem which we have proved just now.

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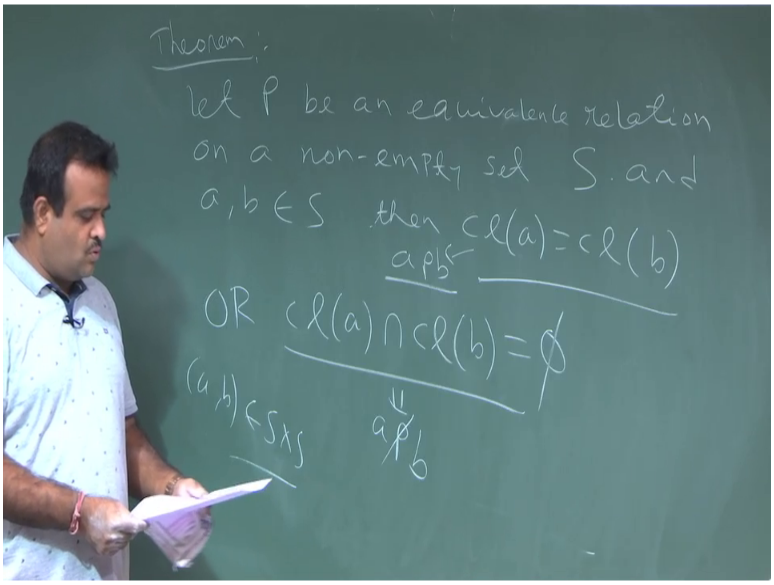

So, let us just rewrite this theorem in a proper way. this is telling that this is the statement of the theorem, let rho be an equivalence relations, rho be an equivalence relation over a non empty set S on a nonempty set S and if you take any two element from S, then either class of a is equal to class of b either one of this true or class of a intersection class of b will be empty. So that means, that depending on this will be the case when a is related to b, and this will be the case when a is not related to b.

So, if a b are any two element, then a b we will be belongs to Cartesian product now rho is a subset of the Cartesian product. So, this is a this is are there are two possibilities, either it is belongs to rho or it is it does not belongs to rho if it is belongs to rho then this will be this case, and if it is not in rho then this will be the this case.

So, just now we have proved this theorem. So, now, this is theorem telling us a partition on the set S, because this is giving us the disjoint classes right. So, these disjoint classes will give us basically a partition on this set S. Now if S is finite it will be a finite, I mean it may give us a finite partition or infinite partition depending on the relation we will take an example.

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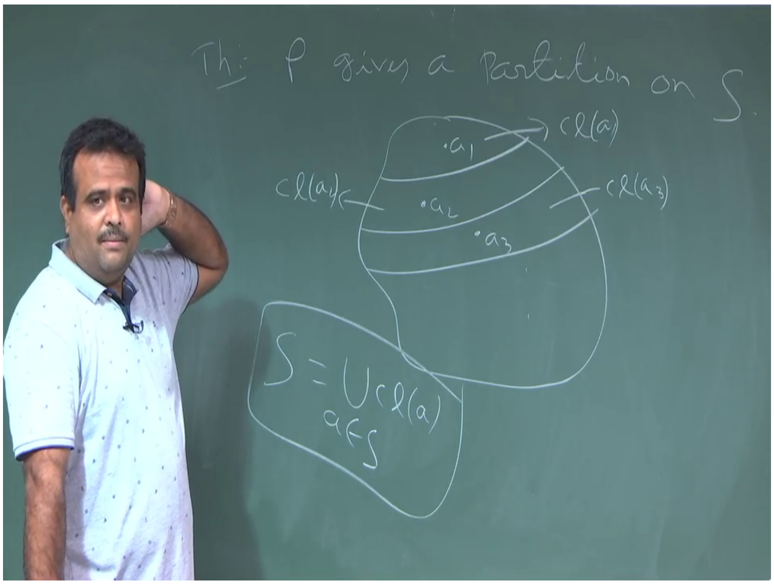

. So, basically this we will have a proper statement of this theorem, but roughly this rho which is an equivalence relation rho gives a partition on S. So, this is how on S. So, how it will give us a partition just the theorem we take an element, and we consider all the this is a. So, this is the class of a. So, this belongs to all the elements which are basically related with a.

And then we take another element, which is not related with a say this is say one we will take another element a 2 which is not related with a, if it is related with a it will be in the same class they will perform the same class. This is class of a 2 and then we will take another element a 3, which is not related to a 1 and a 3. So, this will be c 3 like this and they are disjoint.

So, eventually they will give us a partition because partition means its a collection of disjoint subsets whose union is the whole set because it is just a. So, basically S will be written as union of this class of a where a belongs to S. So, this will give us the partition.

And the converse is also true converse is also true; that means, if we have a partition, then we can have a equivalence relation we will have that, but before that let us take an example of this how to get these partitions, a real life example of the integer set ok. So, that is give us basically we are going to define that z n.

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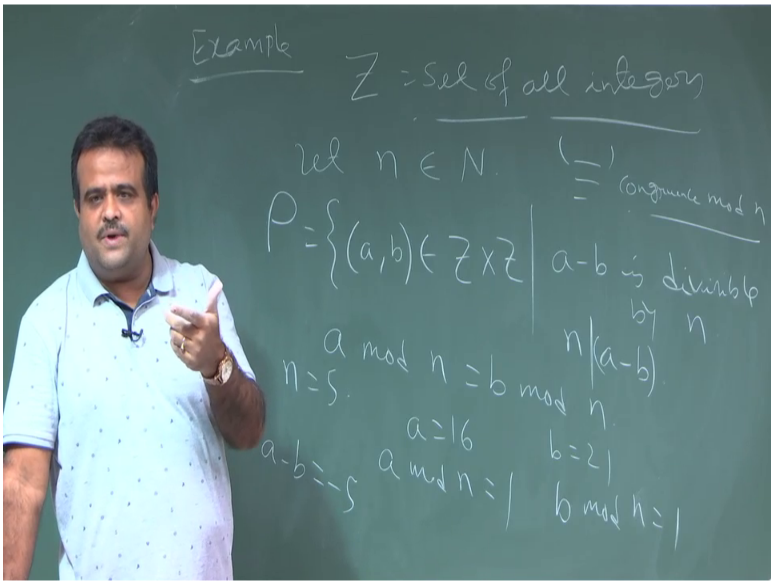

So, this is an example of such partition. So, let us take the set Z this is the set of all integer we know set of all integers ok. Now, we define we take an integer, let n it could be 5 anything I mean usually we take this to be maybe positive integer or it could be negative integer also, but let us first check the some positive integer ok. Now we define a relation rho which is basically we take two point a b 2 integer, now they will be related another rho.

If a minus b is divisible by divisible by n so; that means, n divides a minus b so; that means, what; that means, if a is an integer then a mod n a mod n means what. So, we divide a by n, then this will be the remainder basically. If this is same as b mod n suppose for example, if n is say 5, suppose n is 5 now if you take a to be 16.

So, what is a mod n? A mod n is basically 1, because 16 mod 5 its 1 remainder will be 1, and if we take say 21 b is 21 say, then also b mod 5 is b mod 5 is 1. And also we can check by this way also what is a minus b? A minus b is basically just the minus 5, which is divisible by 5.

So; that means, if we take any inte any two integer a b and if we divided by try to divided by n both the integer, if the remainder is same then they are related under this relation rho, this is called congruence relation. This relation has a symbol, this is can called congruence modular congruence mod 10 this is called congruence relation.

That means, very simple way we take two element a b, two integer a b now if we try to divide a by n we get a remainder it may be 0 also, if it is divisible a is divisible by n and we try to divide b by n. And if they belongs to if the remainder is same, then they are related under this relation this is called congruence relation ok.

So, now this is basically we have seen this is an equivalence relation or we can just prove that this is an equivalence relation. Now if this is an equivalence relation, then it wills it will give us equivalence classes. So, who are those classes basically? So, classes is basically the remainders all possible remainders. So, that is very interesting to see these classes.

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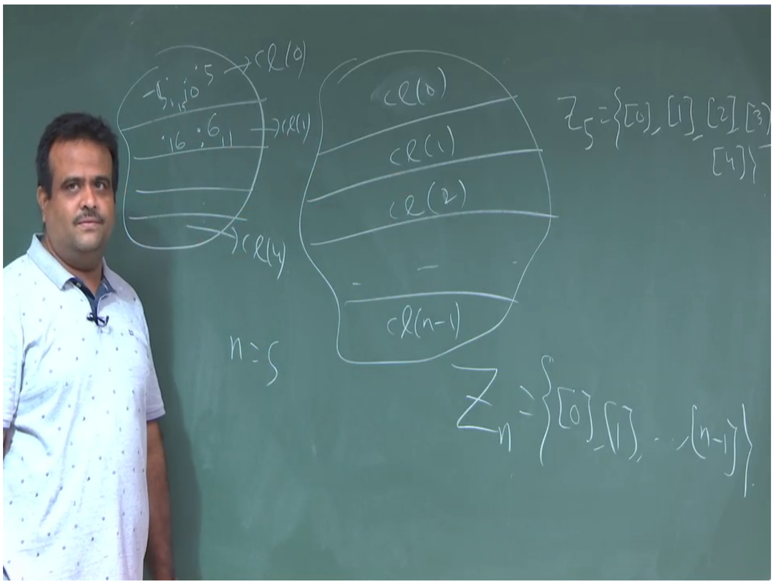

So, this classes are basically we have this set of integer Z, now we have a number n, now we divide that integer there will be remainder on like this. So, 0 class, 1 class, 2 class like this. So, n minus 1 class, because if we divide any integer by n there are these possibilities are there, but if n is equal to 5 say n is equal to 5 means. So, there are possibilities is 0 class means all the elements like 5 10 minus 5 15.

So, all these are in 0 class; that means, they are all related under this relation, and then we have one class dot dot we have up to 4 class. Because if we divided by 5 there are many remainder 0, 1, 2, 4.

So, this collection this set is called Z n. So, 0 class, 1 class and n minus 1 this is called Z n these classes are called Z n. So, what is Z 5? Z 5 is nothing but there are 4 classes class 0, class 1, class 2, class 3 and class 4. So, which are in class 1? Class 1 means these are the elements say 6, 16 11 because all they are related under this relation that 5 a rho and they have if their remainder is 1. So, this is these classes are basically this z 5 now in general if it is n. So, there are 0 to n minus 1. So, n class this is the way how we define the Z n.

Now, the theorem is another part, this is an example of the equivalence classes this is how we define the Z n integer modular n.

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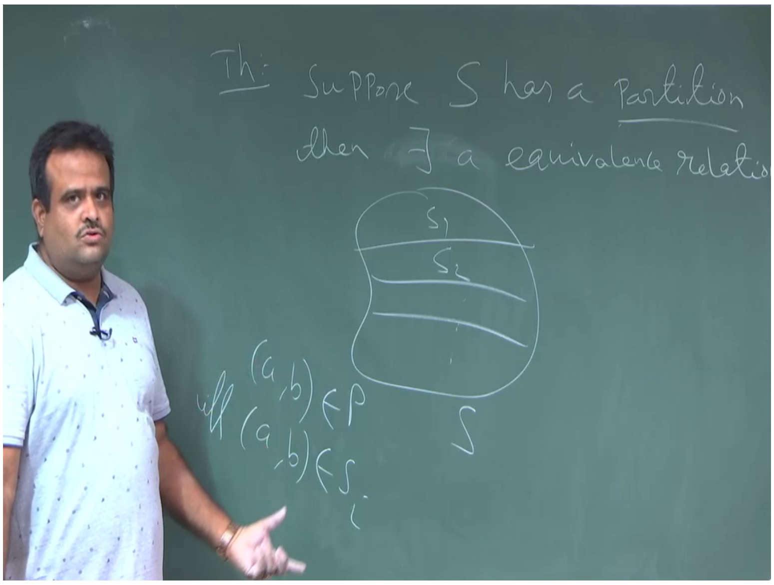

Now, the theorem has another part it is telling, suppose S has a partition, disjoint partition, then there exist n then we can we can have a, then there exists a equivalence relation rho equivalence relation rho. So, how to show this? Suppose we have a partition P is a partition this is S, P is a partition say S 1, S 2 like this. So, we defined a b belongs to rho, if a b belongs to the same class same subset partition is a subset disjoint subsets. A b belongs to rho if and only if they are in the same subset. Then we can show this is a this is a equivalence relation, because a belongs to rho if a b belongs to rho, then b a belongs to rho, if a b belongs to rho b c belongs to rho then a c belongs to rho. So, this is the other way other part of this theorem.

Thank you.