Introduction to Abstract and Linear Algebra

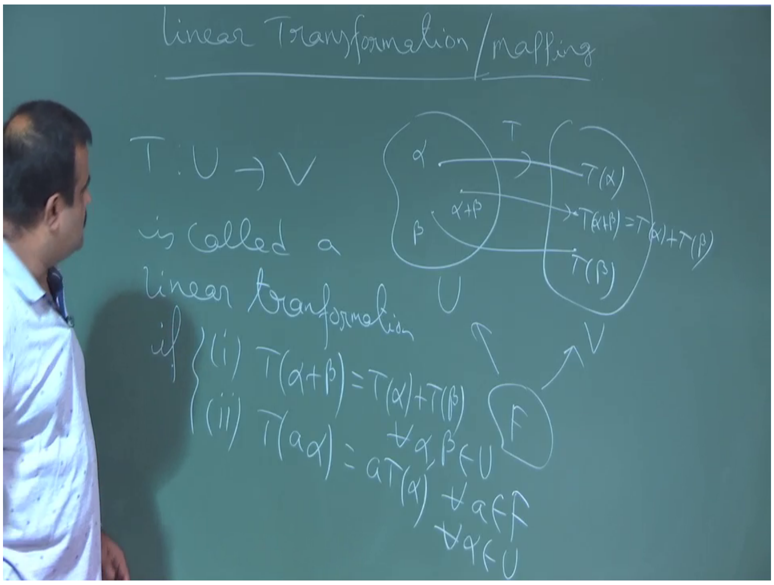
Prof. Sourav Mukhopadhyay
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture – 27

Linear Transformation

So, we will start with the Linear Transformation or it is also called linear mapping between two vector space.

(Refer Slide Time: 00:20)

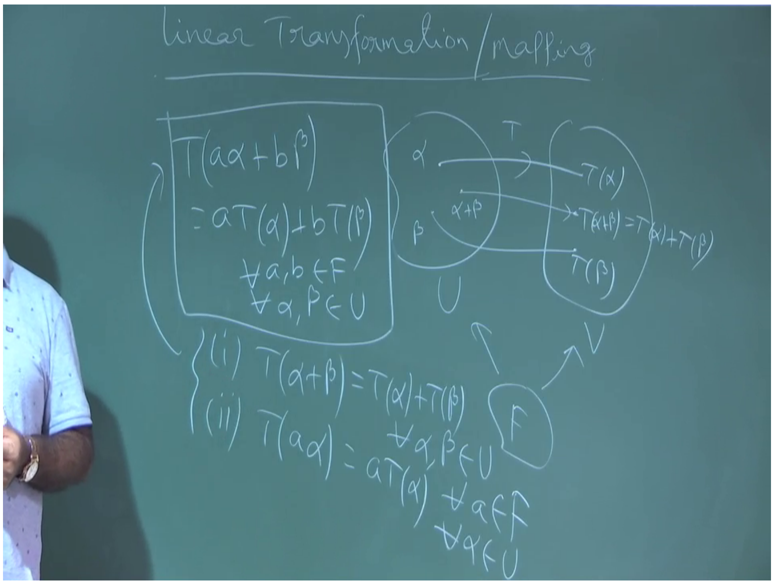


So, basically we have two vector space U and V say U and V over the same field F so that is which works. So, U, V are two vector space over the same field and a mapping from U to V is called a linear mapping or a linear transformation, if the following condition satisfied. There are two conditions, if T of alpha beta is equal to T of alpha plus T of beta.

So, we take 2 alpha beta from here alpha beta. So, and this must true for and T of alpha say this one. Now, T this is T alpha, this is say T beta it could be same also and this is basically T of alpha beta. And, this should be equal to T of alpha plus T beta and this was true for all such alpha beta, this is one condition. So, for all alpha beta belongs to U so, this is a linear property.

Now, we have with scalar T of a alpha is equal to a T alpha and this must true for all a scalar and for all vector from U ok. So, that is the reason we should have a same field because, T of alpha is a element in V and if you are not having same field then we cannot just multiply a with this is the scalar multiplication in V. So, this must be satisfied. So, combining these two so, if these two condition is satisfied for a mapping from U to V then the mapping is called linear mapping or linear transformation, in short it is called LT. Now, to combine these two property we can just write in a single property.

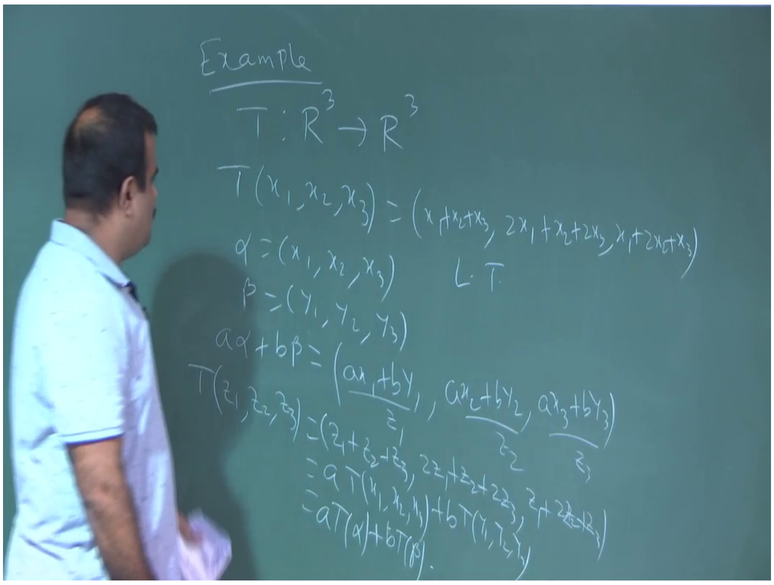
(Refer Slide Time: 02:57)



So, instead of these two we can just write T of a alpha plus b beta, if this is same as a T alpha plus b T beta. And, if it is true for all a b scalar and for all alpha beta vector, then we call this mapping is a linear mapping or linear transformation. Just we are combining, we take for first condition we need to take a and b to be the identity element from this field and second condition we take b to be the 0 and a is this.

So, these two either we take these two condition or we take only in single condition and this must true for all scalar a b and all vector alpha beta from U. So, if this is satisfied then we called a linear transformation to be linear, then we called this mapping T to be a linear transformation or linear mapping ok. But, this U V has to be from the vector space over the same field U and V are the vector space over the same field ok. Now, we will take some example of linear transformation. So, let us take some example of linear transformation ok.

(Refer Slide Time: 04:25)



Suppose we consider the vector space R 3. So, we define a mapping from R 3 to R 3 like this. So, T of x 1 x 2 x 3 so, R 3 is the R is the set of real number. So, if we define like this x 1 plus x 2 plus x 3 is the first, then 2 x 1 plus x 2 plus 2 x 3 then x 1 plus 2 x 2 plus x 3 ok.

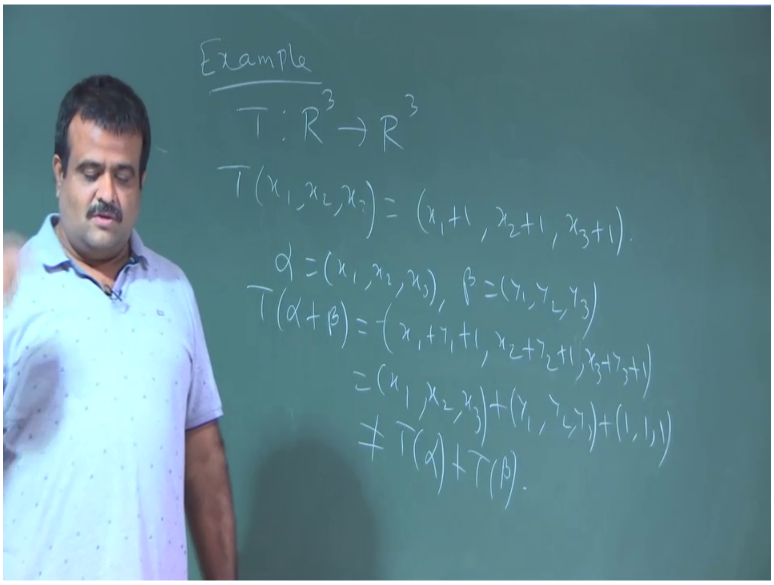
So, if we have this transformation then we are going to show that this is a linear transformation from R 3 to R 3. So, to show this we have to take 2 alpha beta from here. So, we take alpha say x 1 x 2 x 3 and beta is say y 1 y 2 y 3. So, basically what we need to show? We need to show T of a alpha plus b beta where a b are 2 scalar, scalar means this is over the real field a real field. So, scalar means here real numbers.

So, this we have to show this a T alpha plus b T beta ok. So, if we can prove that and if we can prove this is for all a b and for all alpha beta then we are done alpha beta from R 3 for all a b from R. This we can try I mean just we have to take this what is. So, what is basically a alpha plus b beta this is nothing, but a x 1 plus b y 1 comma a x 2 plus b y 2 comma a x 3 plus b y 3.

So, if we take the transform so, we can take this as a z 1 z 2 z 3 by taking this, then if we take the transformation on this T of this thing. So, this is z 1 comma z 2 plus z 3. We know by definition this is z 1 plus z 2 plus z 3 comma 2 z 1 plus z 2 plus 2 z 3 comma z 1 plus 2 z 2 plus z 3. Now, if you put the value of z 1 z 2 z 3 we can easily show that this is nothing, but a T of x 1 x 2 x 3 plus b T of y 1 y 2 y 3.

So, this is nothing, but a T alpha plus b of T beta, this one has to verify this; just is a simple calculation we give us this. So, this we have to only this we have to little bit workout on the exercise book so, you can achieve this. So, this is basically a linear transformation, this is a linear transformation from R 3 to R 3. So, now we will take an example where the transformation or mapping is not a linear.

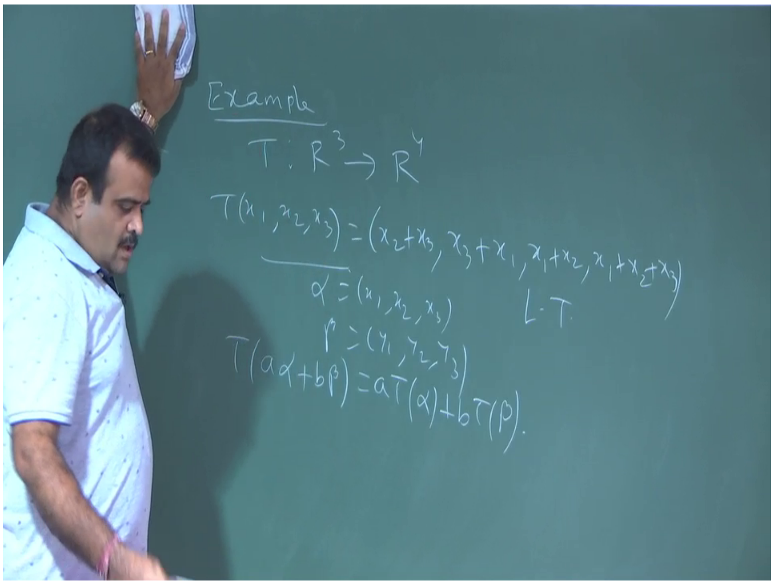
(Refer Slide Time: 08:17)



So, say for R 3 to R 3 so, we take this to be say x 1 plus 1 x 2 plus 1 x 3 plus 1. This is also a mapping from R 3 to R 3. Now, we will see this is not a linear transformation. Because, if we take alpha to be x 1 x 2 x 3 and beta to be y 1 y 2 y 3, then T of alpha plus beta is basically x 1 x 2 x 3 plus y 1 y 2 y 3. So, T of this so, T of alpha plus beta is nothing, but 1. So, T of x 1 plus y 1 so, this is basically x 1 plus y 1 plus 1 x 2 plus y 2 plus 1 comma x 3 plus y 3 plus 1.

So, this is this is equal to x 1 x 2 x 3 plus y 1 y 2 y 3 plus 1 1 1. So, this is not same as T of alpha plus T of beta. So, to be a linear transformation this has to be T of alpha plus T of beta. So, this is an example where this mapping is not a linear mapping ok. So, we can take a another example also from say R 3 to R 4.

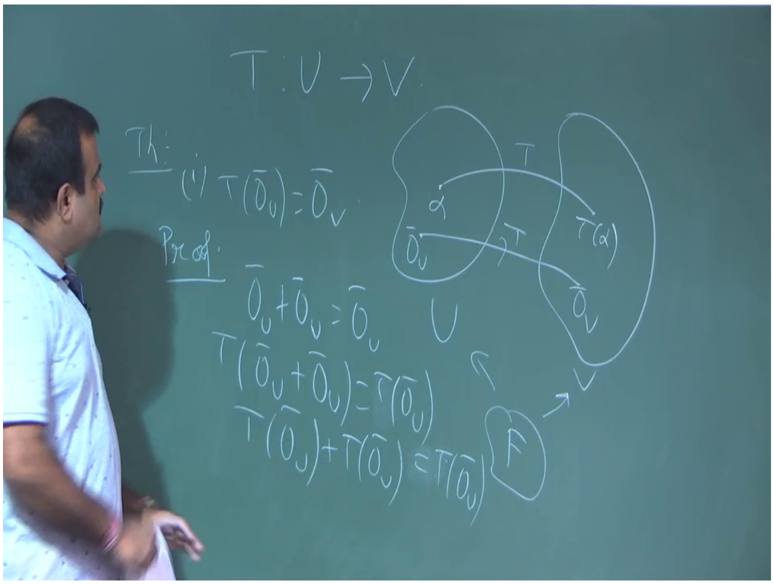
(Refer Slide Time: 10:11)



So, R 3 to R 4 how we are defining this mapping so, T of x 1 x 2 x 3 to be so, this is basically x 2 plus x 3 comma x 3 plus x 1 comma x 1 plus x 2 comma x 1 plus x 2 plus x 3. So, this is four-dimensional ok. Now, I leave as a exercise to you that is to show this is a linear transformation. So, basically we need to take 2 alpha beta x 1 x 2 x 3 beta is equal to y 1 y 2 y 3.

Then we need to show that T of a alpha plus b beta is equal to a T for any 2 real number a b. So, if we can show this then we are done. So, this I leave you as a exercise this should so, this is a linear transformation from R 2 to R 3 to R 4 ok. So, now we will define we have some properties of the linear transformation. So, the first property is this.

(Refer Slide Time: 11:45)

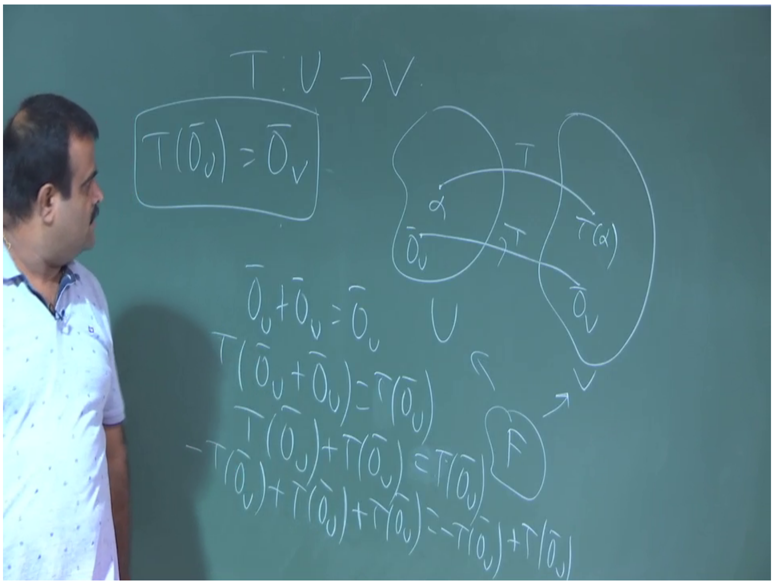


Suppose you have a linear transformation from U to V. So, this is we have two vector space U and we have another vector space V over the same field F ok. So, we have a mapping from here to here, we have alpha is mapped to T alpha. So this is a mapping from U to V and to; and suppose this mapping is a linear map, linear transformation or linear mapping.

Now, this theorem is telling the two part one is this 0 vector 0 of U 0 vector of U is mapping to 0 vector of V. So, this is telling us that T of U is basically V. So, 0 vector should map to the 0 vector ok. How to prove that? So, to prove this we take 0 plus 0 is 0 because, this is the identity element. We know the vector space means that it is abelian group under plus.

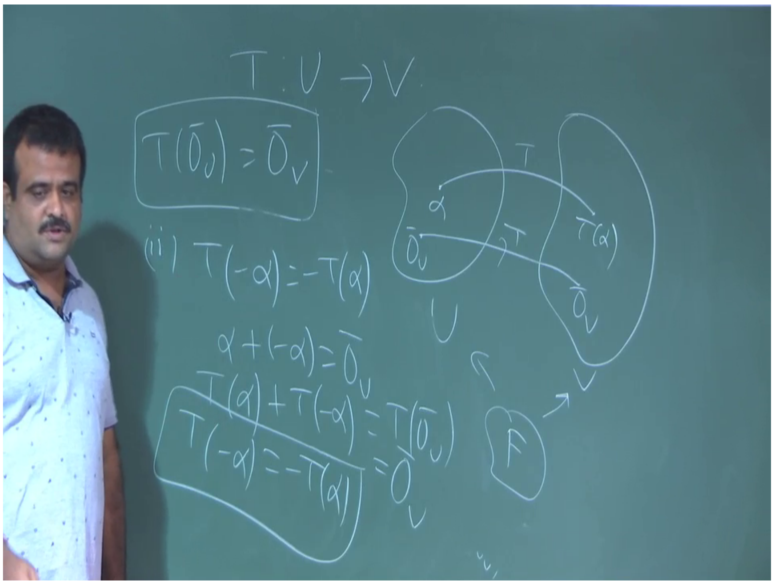
So, this is the identity element. So, 0 plus 0 we can write this 0. Now, we take linear transformation on this. So, this is T of now since the T is linear so, this is T of 0 U plus T of 0 U plus equal to T of 0 U ok. Now, if we subtract T of 0 U from both side and then we use the associatively law of this plus then we get, that we take T of 0 U minus T of 0 U both side plus T of 0 U is equal to minus T of 0 U plus T of 0 U.

(Refer Slide Time: 13:51)



So, this by association law, this will cancel out and then this will give yourself. So, T of 0 U nothing, but 0 V so, that 0 vector is going to the 0 vector of the other vector space ok. This is one property, another property is that alpha.

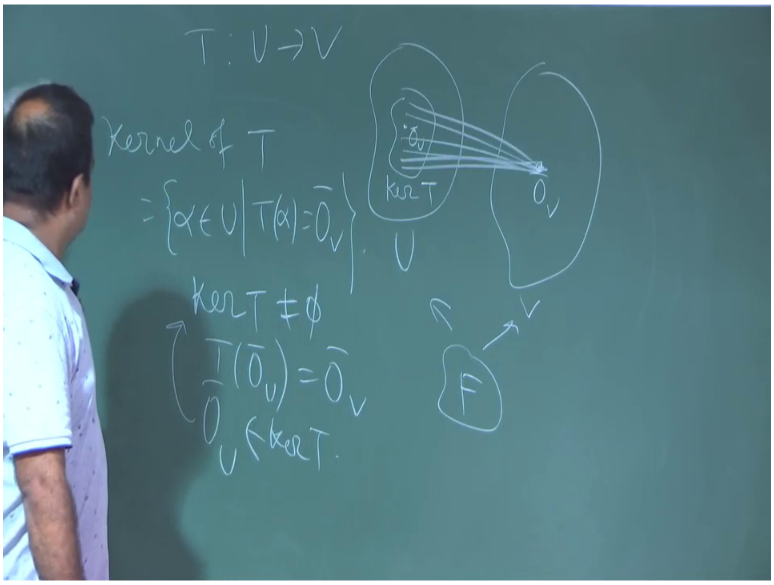
(Refer Slide Time: 14:35)



So, this is the second property T of minus alpha is equal to minus of T alpha. Again, we can prove this by this way. So, we take a alpha so, alpha plus of minus alpha we know this is the 0 vector of U. Now, we take the transformation on this so, this is linear so, T of this plus T of minus alpha is equal to T of this. Now, just now we have seen this is T of V.

So, this implies T of minus alpha is equal to minus of T alpha this is from the result ok. So, these are very the this we can say these are the result ok these are the properties of a linear transformation from a one vector space to another vector space ok. So, now we will define a kernel of a vector space.

(Refer Slide Time: 15:47)

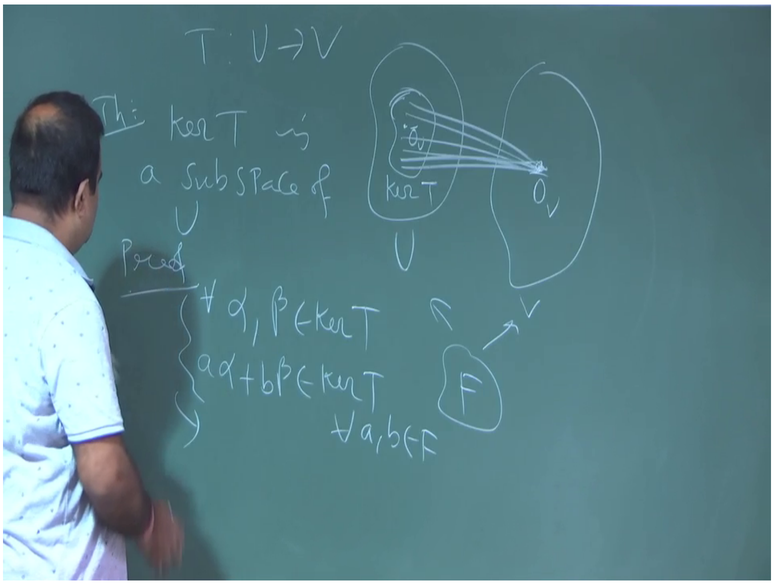


So, now we define a kernel of a vector space sorry so kernel of a linear transformation sorry so, suppose we have a trans linear transformation from U to V, same thing you have a two vector space U and V over the same field ok. So, we defined kernel of T and suppose this is a linear transformation. Kernel of T is basically set of all vector from U such that they are mapping to 0 vector of V. So, which is basically the set which are all mapping to the 0 vector of so, this is the 0 vector of V. So, these are all this side is in short it is written as ker of T kernel of T ok.

So, now this kernel of T is not empty. Why? This ker T is not empty because, we know theta 0 vector of U is there, because just now we have seen T of this is basically going to this. So, at least this belongs to kernel of T so, this means this is not empty. So, at least this 0 vector is a member of this 0 vector of U ok. So, now we will we will have we will see some properties of this kernel. First of all we will see the kernel is a subset of U.

Now, we will see this kernel is a subspace of this vector space. So, how to prove that? So, this is again make a theorem we will prove.

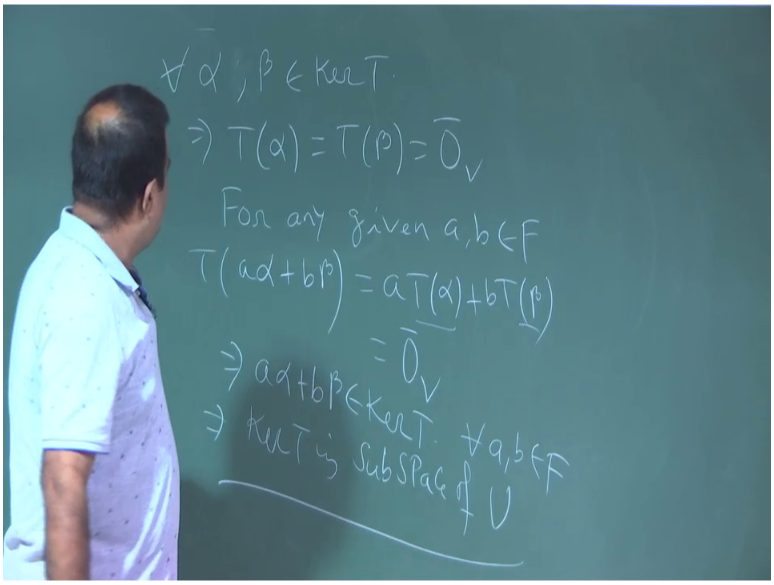
(Refer Slide Time: 17:39)



Kernel of T is a subspace of U ok. So, this is the subset is again form a vector space over the same field. So, how to prove that subspace? So, for moving the subspace we know we know that we have to take two element alpha, beta from that subset. And, if we can show that a alpha plus b beta is belongs to that subset.

And, if that is true for all our a b and if this is true for all alpha beta, then they we have proved a theorem from which we can say this is a subspace of U. So, now we have to show this. So, let us try that so, we take alpha beta from let us remove this ok.

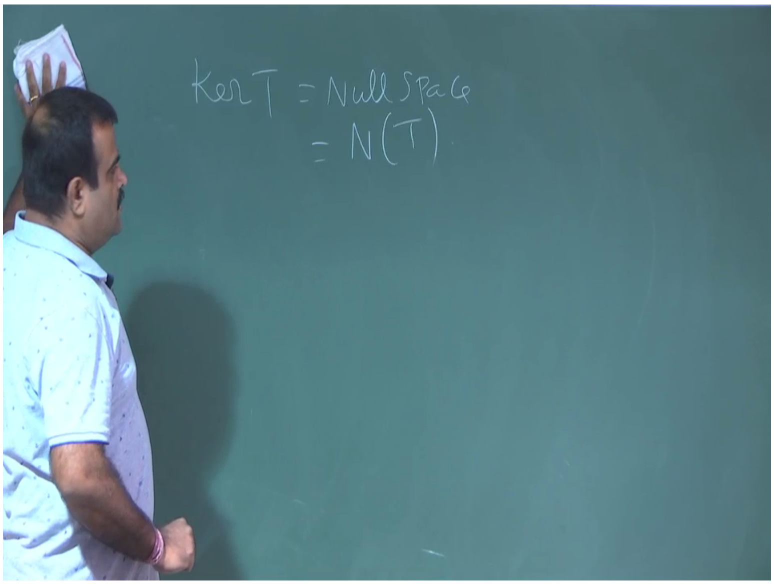
(Refer Slide Time: 18:53)



So, we take a alpha for alpha beta from the kernel of T. So, this implies T alpha is equal to T beta is equal to they are mapping to 0 vector of V, because that is the definition of kernel. Kernel consists of all the vectors in U which are mapping to the 0 vector of V ok. So, now we take this a alpha plus b beta for any given alpha for any given a b. So, for any given a b from the scalar, we take T of a alpha plus b beta.

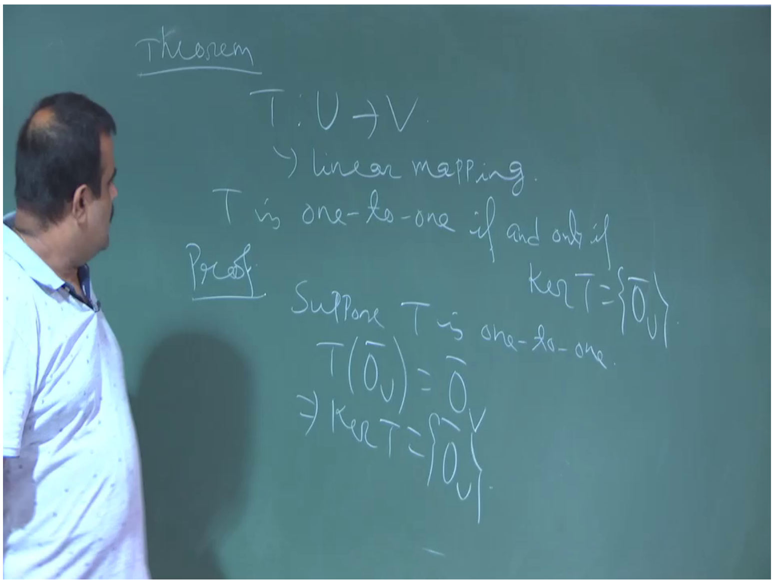
Now, since this T is a linear transformation this must be written as beta. Now, this is 0 this is 0. So, this will give us basically so, this implies a alpha plus b beta belongs to kernel of T. So, if alpha and this is true for all alpha beta and this is true for all a b from the field. So, this implies my theorem which we have proved in the vector space class, that kernel of T is a subspace of U, kernel of T is a subspace of U ok. So, this is also called null space.

(Refer Slide Time: 20:47)



So, sometimes kernel of T is also called null space and it is written as N T. So, now we will just have the some theorem on it.

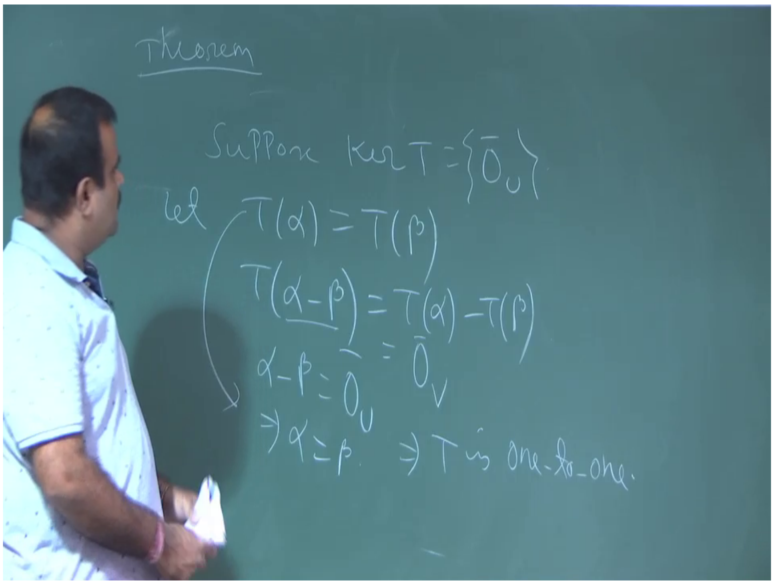
(Refer Slide Time: 21:23)



So, like suppose you have a vector space from U to V and U V are over the same field. And this is a linear transformation, suppose this is a linear transformation from U to V ok. This theorem is telling T is one to one or injective if and only if kernel of T is only the 0 vector of U ok. So, how to prove this? So, it has two part: one is the first part is it is one to one; if it is one to one so, let us try to put that.

So, first part is suppose T is one to one and we know that T of 0 of U is going to 0 of V. Now, if it is one to one there should not be any other vector which are mapping to U of V. So, this implies kernel of T must be only this vector 0 vector because, it is one to one; no other vector should map to the 0 vector of V. Now, the other way around, suppose kernel is 0 vector then we have to prove that this is one to one. So, this is the conversely suppose just put that.

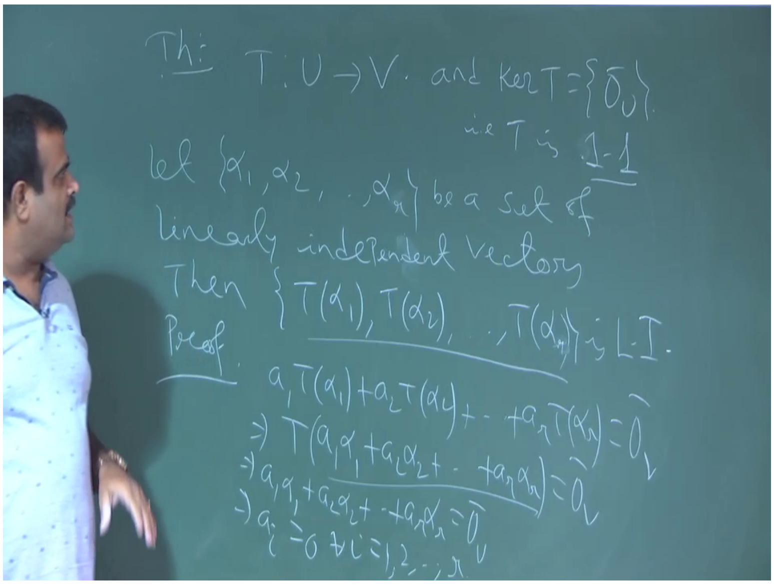
(Refer Slide Time: 23:13)



Suppose kernel of T is consists of only 0 vector of this of U, then you have to show this T is one to one. How to show that? So, we take 2 alpha beta and suppose T alpha is equal to T beta, let. Then from here we can say T of alpha minus beta is basically T of alpha minus T of beta. So, this is basically nothing but we know this alpha beta so, T of alpha beta is basically so, now these are same.

So, this will give us a 0 vector of V. Now, we know the kernel is consists of this. So, this must be alpha beta must be equal to so, this implies alpha is equal to beta. So, this implies so, T of alpha is equal to T of beta implies alpha is equal to beta and this is true for all alpha beta like T of alpha equal to T of beta. So, this implies T is one to one or injective ok. So, if the kernel of T is 0 vector then the mapping is T is one to one, now we have another theorem.

(Refer Slide Time: 24:53)



So, this theorem is telling suppose we have a linear transformation from U to V and over the same field and kernel of T is 0 vector. So, this means that is T is one to one that is T is one to one ok. So, now that this theorem is telling, if we take a let alpha 1 alpha 2 alpha n or alpha r be a set of linearly independent vector. Then this set T of alpha 1 T of alpha 2 T of alpha n, these are also linearly independent set of vector.

So, if you take a linearly independent set of vector in U and if T is one to one then it will give us a linearly set of vector in V. So, how to prove that? So, to prove this is to be linearly independent. So, this is r, to prove this is to be linearly independent, what we need to show? We need to take a 1 T of alpha 1 plus a 2 T of alpha 2 plus a r T of alpha r to be 0 vector in V. And, now we need to show that all these a is must be 0. So, how to show this? This implies from the linear transformation properties of T, we have a 1 alpha 1 plus a 2 alpha 2 a r alpha r is equal to this.

So, now this must belongs to kernel of T now the, but the kernel of T only consists of 0 vector. So, this implies a alpha 1 plus a 2 alpha 2, this is the 0 vector of U. So, this implies all a is are 0 because, these are the linearly independent set of vector in V, all a is are 0. So, this is the proof; that means, if you take a linearly independent set of vector from U and it will be again a linearly independent set of vector in V ok. So, we will continue this in the next class.

Thank you.