Matrix Solvers
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Lecture – 08
LU Decomposition

In last class, we have seen how Gauss elimination process can be utilized to decompose a matrix A by into a lower triangular and a upper triangular matrices products or what is called A is equal to LU or the typical LU decomposition of a matrix.

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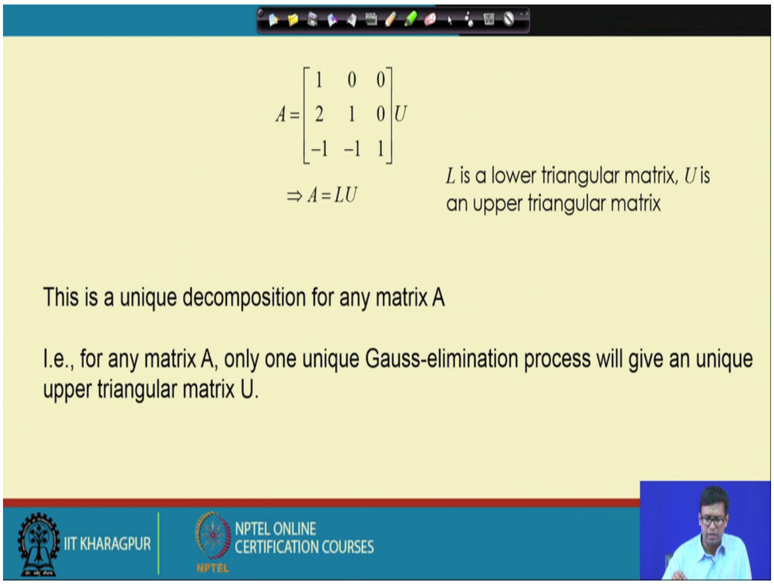

In today's class we will look into some of the applications. What we have seen in last class a matrix A through gauss elimination step can be expressed as A is equal to LU or product of a lower triangular matrix and then upper triangular matrix and this is a unique for any matrix A; there is only one unique gauss elimination process which will give unique L and also an unique upper triangular matrix U.

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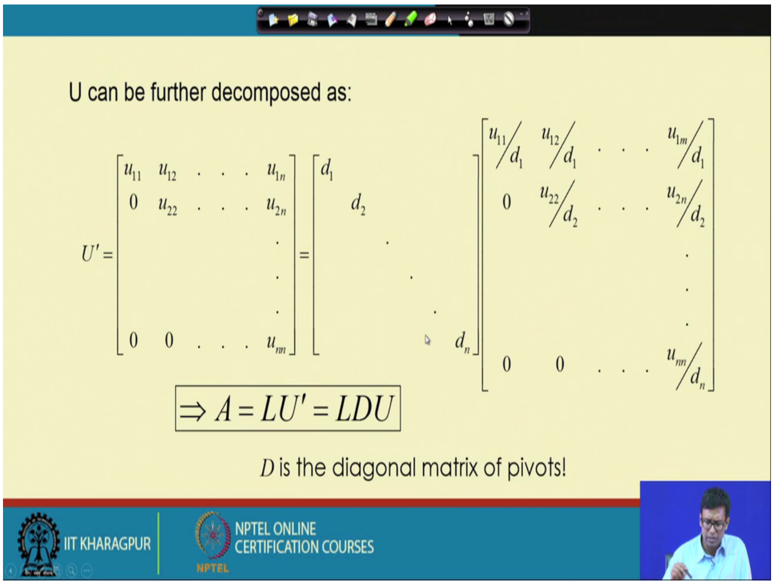

And you have also seen that you can be further decomposed because the pivots are non-zero into a diagonal matrix of pivots and another upper triangular matrix. So, you get A is equal to LDU, these decompositions are possible, there is a D is a diagonal matrix of pivots.

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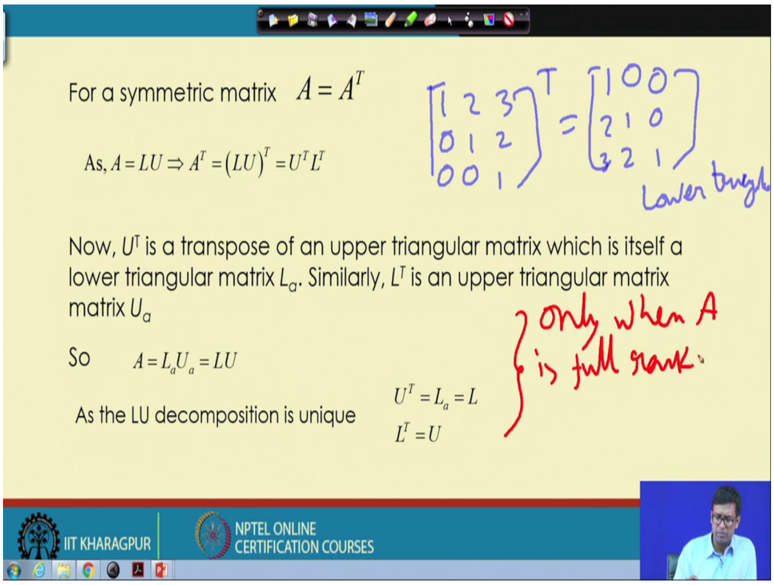

There is a special case when A is symmetric matrix say A is equal to A transpose. And we can see as A is equal to A transpose A transpose can be written as LU transpose A is LU also which is U transpose L transpose. Now, this is A in possible when A is a full rank matrix. LU decomposition is possible when gauss elimination forward substitution works on A or A is a full rank matrix. So, we get A is equal to U transpose L transpose. Now, U transpose is transpose of an upper triangular matrix.

For example, if I have upper triangular matrix 1 2 3 0 1 2 0 0 1, its transpose will be 1 2 3 0 1 2 0 0 1, which is a lower triangular matrix. So, U transpose is transpose of an upper triangular matrix which is itself a lower triangular matrix say L a. Similarly, L transpose is also an upper triangular matrix say U a. So, we can write A is equal to which is U transpose L transpose, now L a L a U a and this is A is also equal to L L U. Therefore, this LU decomposition we have seen that it is an it is a unique decomposition, there is only one paired of L and U which is possible for a particular matrix A.

So, L a must be equal to L; and U a must be equal to U. So, U transpose which is L a is L; U transpose is equal to L for a symmetric matrix; and L transpose is equal to U and that is how we should see that U transpose is L, so we can write A is equal to a L L transpose or L transpose is equal to U.

So, we can write A is equal to L transpose U transpose U, U transpose U or L transpose L L transpose. So, sorry A is equal to L L transpose or A is equal to U transpose U that is what we can write. And everything is possible when and this is a caveat here only when A is full rank or A is invertible or A x is equal to b has unique solutions. It is only possible in this case if A is a symmetric matrix we can write that this can be decomposed as A is equal to L transpose L or U transpose U.

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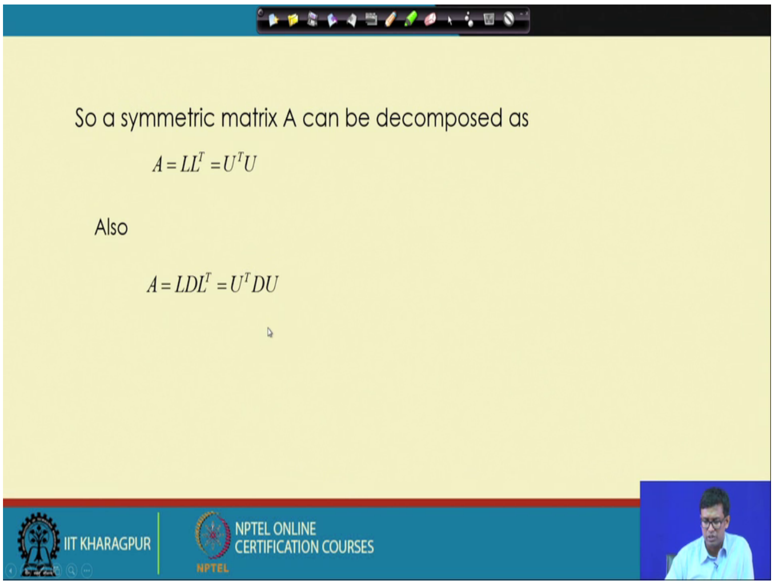

Similarly, you can write a we have seen L can be decomposed as L D U which is now it can be written as L D L transpose or U transpose D U.

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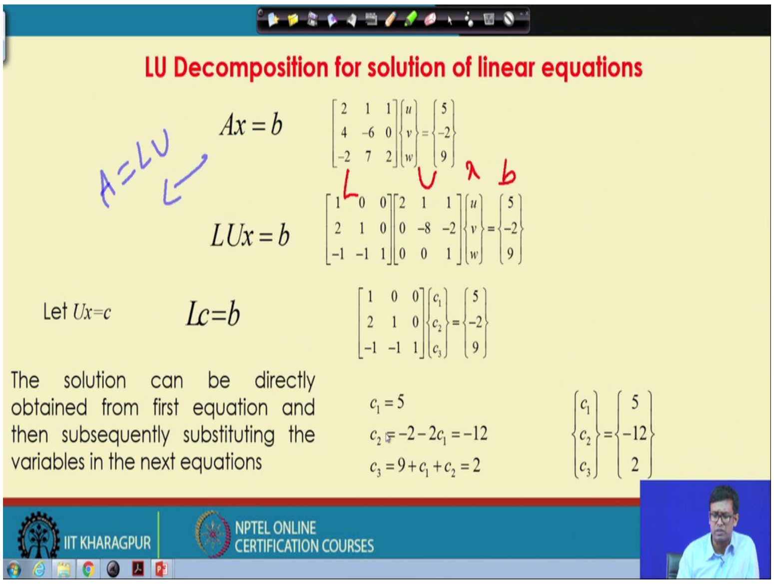

Now, we will see how what are the applications of LU decomposition. And LU decomposition can be utilized for system of linear solution of linear equations also. For example, I have A x is equal to b which is this equation presented here. And I A is equal to now A can be decomposed into A is equal to LU. So, if I write L U x is equal to b or we have early found out the terms LU for this A matrix in last class which is this is the lower triangular matrix and this is the upper triangular matrix. This is L; and this is U; and this is x; and this is b.

Now, we will say that U x is a matrix into a vector, the product is also a vector. And we will see that this vector is will I denote it by a vector c. So, if U x is equal to c, we will write L c is equal to b, which is this into c is equal to b.

Now, L c is equal to b is a triangular equation system. So, directly I can find out c 1 is equal to 5, I will substitute into the next equation, I can write c 2 is equal to minus 2 minus 2 c 1 something like that and so the solution can be directly obtained from first equation and then that can be substituted into second equation, and then it can be further substituted into the third equation and so on for large systems.

So, you can directly see c 1 is equal to 5, c 2 is equal to minus 12, c 3 is equal to 2. So, L c is equal to b can be solved in one step if L U decomposition is already available to me. Now, once we know L c is equal to b, you will solve U x is equal to c. So, this is much this is a very easy solution just one step solution we will get L c is equal to b.

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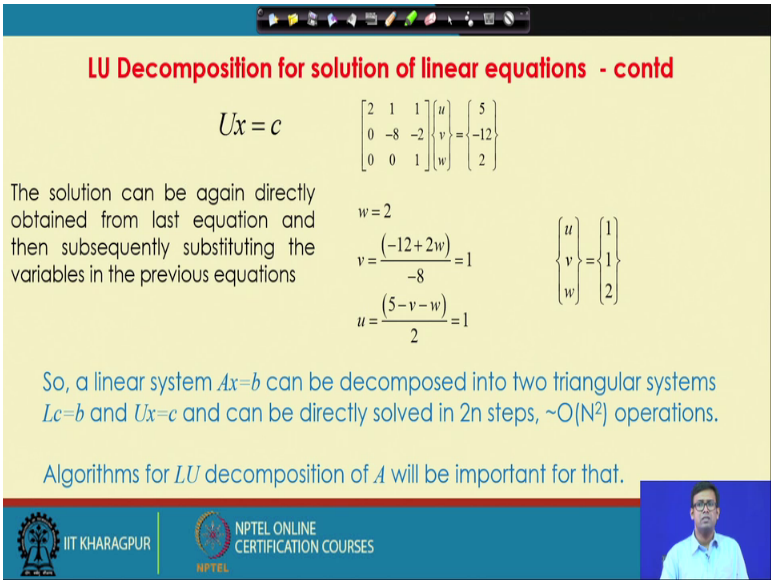

And similar thing will happen for U x is equal to c, the equation is this particular equation. Interestingly if you check back last class the lecture nodes, you will see this is the equation what we got at the end of forward substitution of the Gauss elimination process on the given A x is equal to b. And it is also very easy to solve it. It is like back substitutions, so you find what is w. Substitute w in the second equation find what is v; and substitute v in the second equation and find what is u.

So, the solution can be again directly obtained from last equation, and then subsequently substituting the variable into the previous equations. So, we will go in the backward direction and find out what is w; what is v; and what is u; and u, v, w is equal to 1 1 2. So, I have a and this can be done for a large system of equations also, it is first solving a lower triangular matrix into c is equal to the right hand side vector, and then solving upper triangular matrix into the solution vector x is equal to c.

So, a linear system A x is equal to b can be decomposed into two triangular solved systems; L c is equal to b, and U x is equal to c. And then they can be directly solved in 2 n steps; first n steps are needed to solve the first n equation, and then n steps are needed to solve the U x is equal to sin x 10 equations. And the order number of operations come down to N square, why N square, because in the say if I see this n step, in the first this step where I am doing operation into two variables and doing another deficiency. So, each step is maximum doing n number of operation. So, if there are n step coming up, and n step going down, the maximum number of operation will be sum multiplied into N square.

We have seen in gauss elimination, the total number of operations where n cube. And if we do a LU decomposition, the operations are N square provided I need to have the LU matrices already. If I try to get LU matrices to gauss elimination, I will again need n cube operations to get LU matrices, because just to do the forward substitution I need n cube operations. So, I need some better algorithms for LU decomposition. And other than Gauss elimination, we should look for some other algorithms for LU decomposition also, which at least for certain cases which can give us the, this decomposition in steps less than the Gauss elimination steps.

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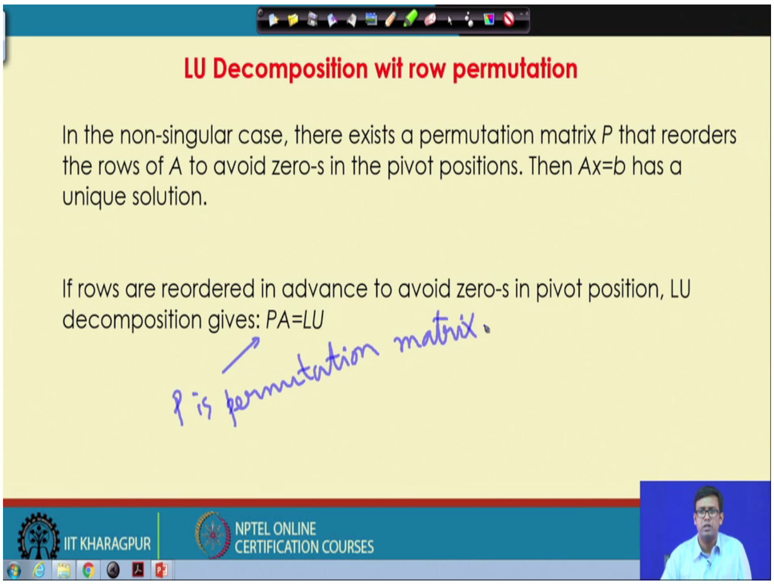

So, then there can be cases and we have looked in the we have not looked into the cases, where row permutation is needed, where we get a zero pivot and we have to do a row permutation. In the non-singular case, if there is a this is a singular matrix case, there is no permutation matrix; row permutation is not possible; Gauss elimination will not work; I cannot write A is equal to LU.

But if a non-singular cases there, I can at certain stage, I can multiply A with a permutation matrix P and still can go for LU decomposition. So, there exists a permutation matrix P that will reorder the rows of A to avoid zero in the pivot position, and then A x is equal to b has an LU decomposition.

And then the rows, if we are already reorder the rows in advance, so that we do not get zero pivot, and we can perform the LU decomposition. We can write P A is equal to LU, where P is the permutation matrix; P is permutation matrix. So, instead of A is equal to LU, we get P A is equal to LU or permutated form of A can be decomposed into LU, but all the process of LU decomposition, and uniqueness of LU decomposition should also hold.

Now, we will try to look into some of the algorithms for LU decomposition, because faster algorithms are needed for having the solution first, and without going into that complexity of n queue order n cube, which is needed for Gauss elimination.

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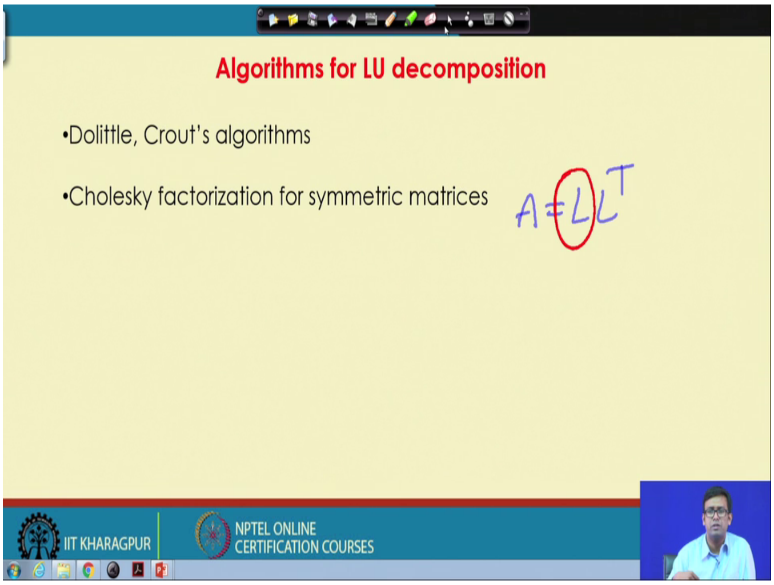

And the there are few algorithms like Dolittle and Crout's algorithm, these are few of the few of the popular algorithms for LU decomposition. And for symmetric matrix for there is a Cholesky algorithm factorization. Interesting thing is that for symmetric matrix is that you do not do LU, rather you find A is equal to A L L transpose.

So, if you can find out only the L matrix, your work is done; half of the work has to be done in a for a symmetric matrix. And these algorithms for certain cases, they are number of steps needed, and performance is comparable with Gauss elimination, but for many physical systems we get banded matrices or sparse matrices, where this algorithms can give us faster LU decomposition.

And these are popular algorithms also all the algorithms will discuss in this course has a unique feature that these algorithms can be very easily transformed into a computer program. And that is way that is the source of their popularity; these are popular algorithms, because it is easy to write computer programs using these algorithms.

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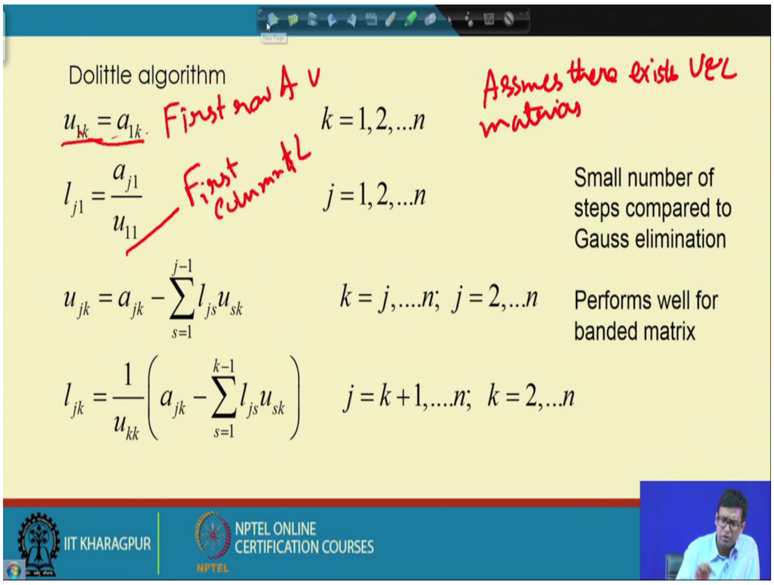

So, you if we will see Dolittle's algorithm, and this algorithm starts from assuming that for an a matrix and all these are for a matrix, which are full rank matrix; where LU decomposition exist. It starts assuming that a matrix has an LU decomposition, and it forms the first row using the so assumed, so it starts assumes there exist U and L matrices.

Now, we can see that the first row, first row of U; and first column of first row of LU is nothing I think I made a mistake here, the first row of U is nothing but the first row of A; and first column of U comes by dividing the first column of A by the first element of U, the diagonal element of U. And similarly, it goes on in getting the next rows of a U using thus the first column of U, and first column of L, and first row of U, and forms the next second row of U, and then it forms the second column of L and so on.

So, it assumes that they are already exist U and L matrices are already existing their forms somehow starts forming the first row of U; and first column of L. And uses that to get the next rows and next columns of U, so that the product of the rows and columns gives exactly the elements of a. And this is a very quite popular algorithm, and has been utilized in number of cases.

And the advantage of this algorithm is that for in a very small number of steps compared to gauss elimination, it can solve the matrices, if this is a banded matrix. If the matrices, well banded if there are few of diagonal terms, and remaining terms are 0, these operations are very small and compare. If for a large set large matrix system, and the number of steps are less than the gauss elimination steps, and it can give us a first solution. So, this is Dolittle's algorithm for LU decomposition, and this U and L are the member of upper triangular matrix, and L is the member of the lower triangular matrix capital L.

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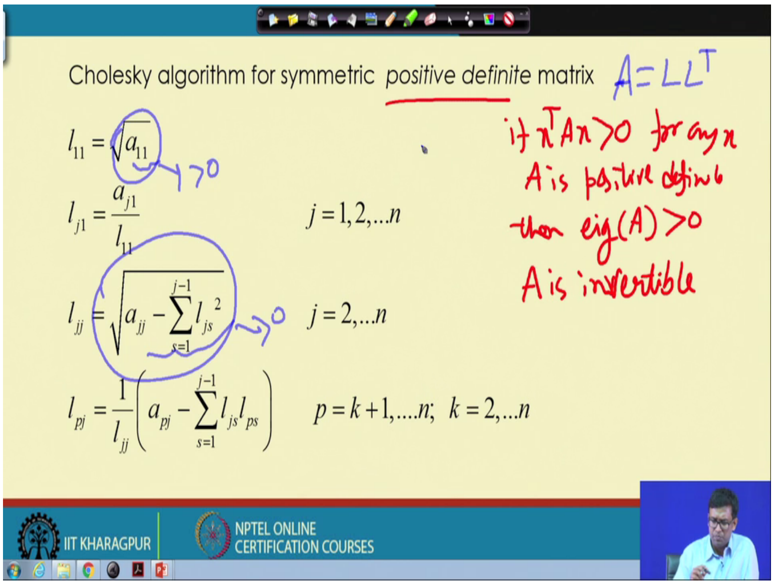

Now, you see Cholesky algorithm, and it is stated for symmetric not only for any symmetric matrix for symmetric positive definite matrix. What is positive definite matrix; we will discuss on that later, but right now, what basically the definition of positive definite matrix is, a matrix is called positive definite if x transpose A x is greater than 0 for any x, A is positive definite.

The other idea is then eigen values of A or greater than 0, all eigen values of a, A or greater than 0. However, here we can just think of and we will discuss on positive definiteness of a matrix in detail in one of the later classes. One important thing is that if it is positive definite A is invertible. So, for symmetric positive definite matrix, we get will have A is equal to L L transpose, and we will have an algorithm to find out the elements of A, which is the Cholesky factorization.

And this algorithm tells us l is root of a 1 1, so we take the first norm of the matrix a, and its square root gives us l. And we form the remaining terms of the first column of L, then we go to the next column, it is again a minus whatever is in the first column sum summation of then, and a square of that and then its square root, because the multiplication will finally give me A. And similarly, we this the diagonal term, we form the of diagonal terms of each column.

Importance is symmetric definiteness is that, that this terms must be real, that is why; this particular term must be greater than 0, this particular term must be greater than 0, and that is ensured by the positive definiteness of this matrix. So, nevertheless, this gives the final algorithm for Cholesky decomposition of a symmetric positive definite matrix to get the L matrix with A is equal to L L transpose.

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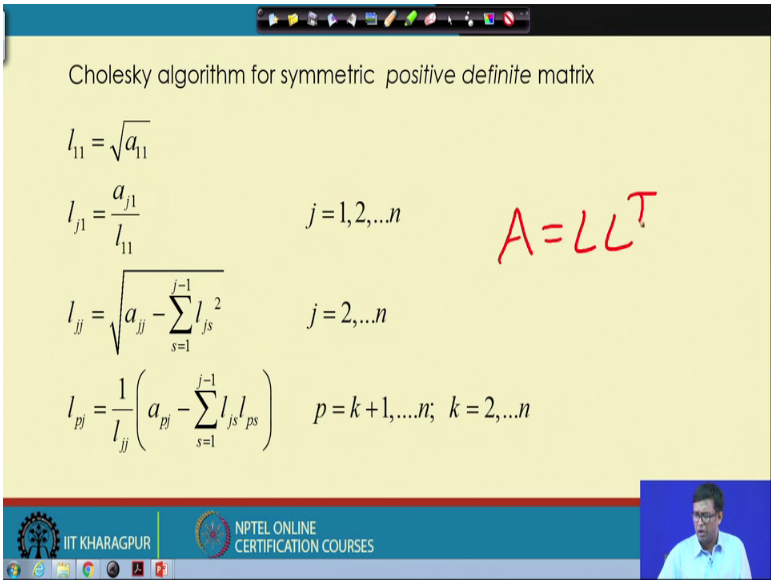

So, this algorithms are relatively faster algorithms, and has been utilized in number of cases for getting right solution of A LU is equal to x instead of A is A x is equal to b, we are solving L U x is equal to b, and this algorithms are useful for that. There are few other applications of LU decompositions also, so these are the algorithms for LU decomposition. And next few minutes will spend on seeing the applications of LU decomposition, where the L and U matrices I am getting using this algorithms can be utilized.

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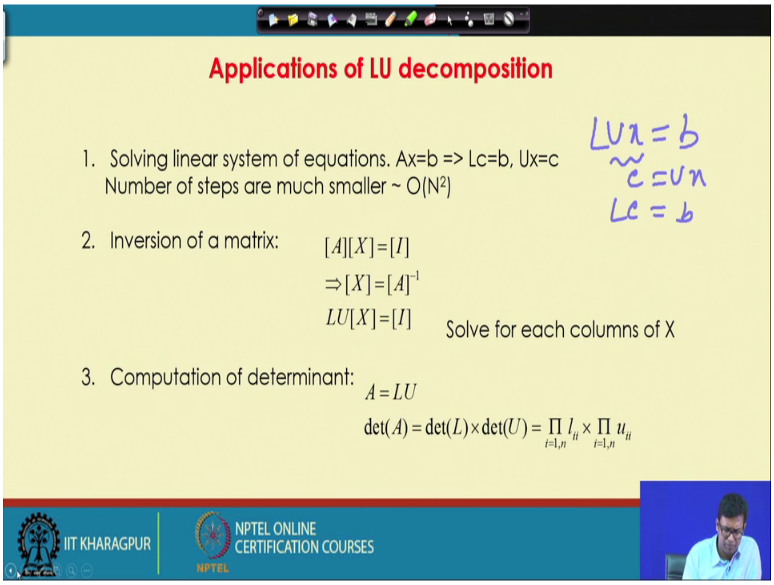

So, one of the application of LU decomposition is instead of solving A x is equal to b; we have seen it earlier, we have to see LU x is equal to we have to sorry will solve L LU x is equal to b, which is U x is equal to c and L c is equal to b. And if we already have LU decomposition, the number of steps are much smaller compare to gauss elimination process, and the number of steps are N squared. And that is why we discussed about the algorithms for LU decomposition, which can give us this decomposition, we awarding the typical gauss elimination process.

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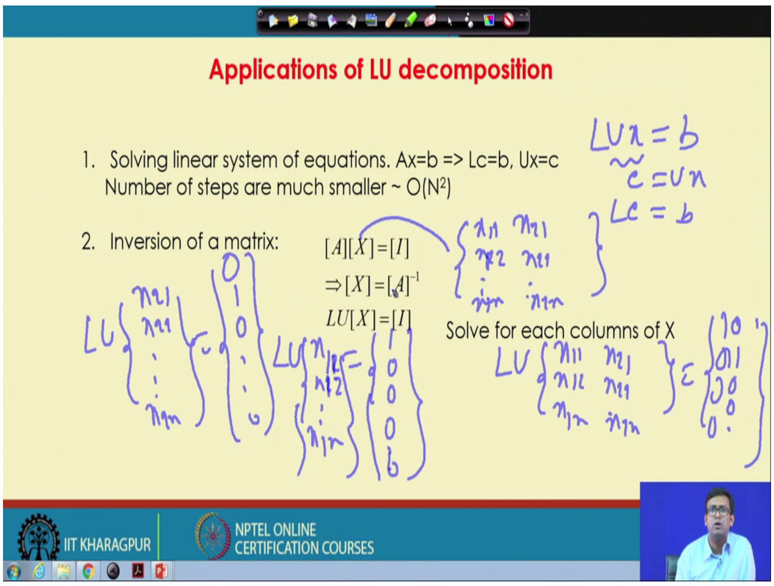

The other application is inversion of a matrix. So, we have to find out A inverse now, we consider a matrix X X is a X is not a column vector, rather X is a matrix sorry. This X is a matrix, which has multiple columns x 1 x 2 x 1 1 x 1 2 x 1 n then x 2 1 x 2 2 x 2 n matrix like that. So, we multiply A with X, which is gives us identity matrix, therefore X should be A inverse. And now we write instead of A we write LU. So, we get LU into x 1 1 x 1 2 x 1 n x 2 1 x 2 2 x 2 n and so on is equal to an identity matrix 1 0 0 0 0 1 0 1 0 0 so on.

Now, we solve for each column of x like I solve LU x 1 is equal to 1 0 x, so x 1 x 2 x 1 1 2 x x 1 1 x 1 2 x 1 n 0 0 0 I solve the matrix equation like this. Then I and I will get what is x 1, x 2, x 3, because the solution of LU x is equal to b is simpler than solving A x is equal to b. Now, I again solve another matrix, LU matrix equation LU into x 2 1 x 2 2 x 2 n is equal to and this is a second column of identity matrix 0 1 0 0 0.

And I solve what is x 2 1 x 2 2, I get the second column of the x vector and so on, we do for we solve n system of equations like n matrix equation, LU x is equal to identity, and get each columns of x. So, finally we will get all the columns of x, and this the combination of all the columns will give me what is A inverse.

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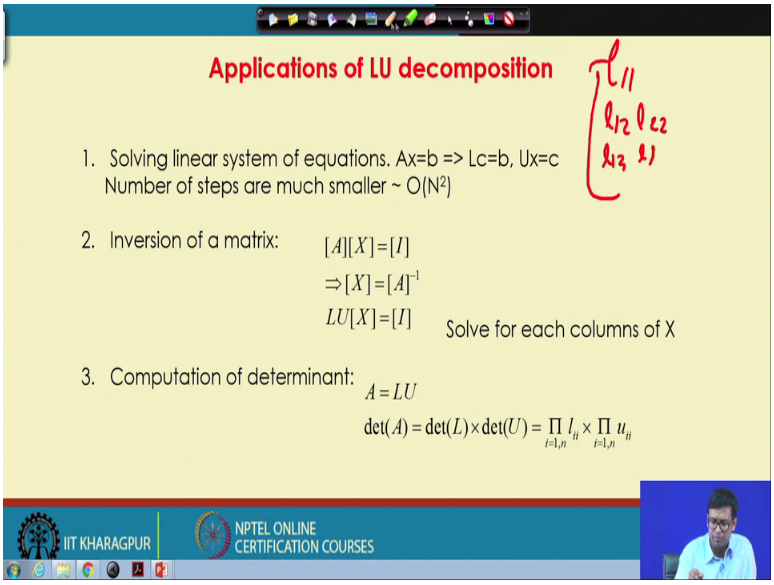

The third application is also very important application, which is computation of determinant. We know A is equal to LU, so determinant of A will be multiplication of determinant of L, and multiplication of determinant of U.

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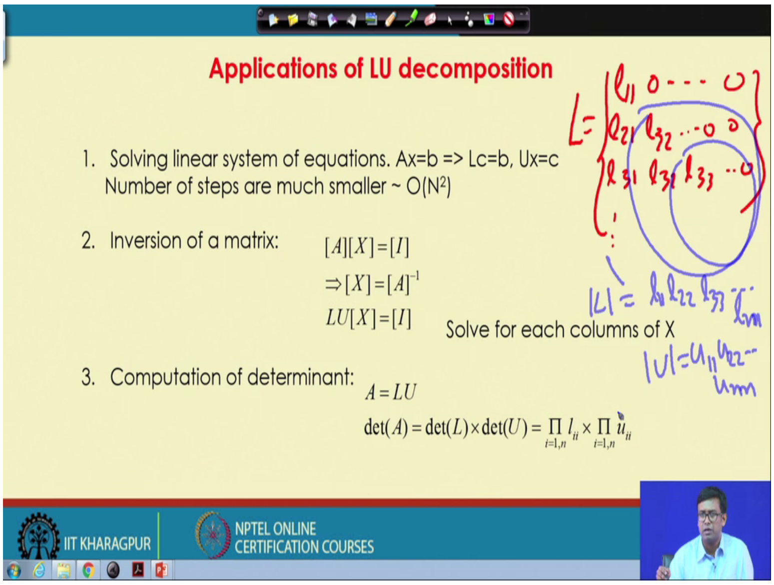

Now, L is a lower triangular matrix, so it has l 1 1 1 l 1 2 L 2 2 l 1 3 l sorry l 1 1 I am sorry. So, L will be l 1 1 0 0 0 l 2 1 l 2 2 0 0 0 l 3 1 l 3 2 l 3 3 0 and so on. So, if I try to find its determinant is will be l 1 1 into this particular cofactor, whose determinant will be l 2 2 into this particular cofactor and so some on. So, we will finally, find out the determinant of L is nothing but product of l 1 1 l l 2 2 l 3 3 finally l n n. So, product of the diagonals is determinant of a lower triangular matrix, similarly for an upper triangular matrix U also products of the diagonal will be the determinant.

So, if I write A is equal to LU, and if I already know the decomposition A is equal to LU too some of the algorithms we have discussed earlier. Determinant of A is determinant of L into determinant of U, which is product of the diagonal terms of L into product of the diagonal terms of U. And in very simple way, I can find out determinant of the matrix A. Usually we have seen that using the minor cofactor rule finding determinant is very complicated, and lot of nested loops are also needed, when you are computing if the complexity is high, but here the computation of determinant becomes very simple.

So, this is one of the one of the important applications of LU decomposition. And I will said that LU decomposition will further have some applications, when you will discuss about pre conditioners for improving performance of iterative methods. So, this is one important part of the matrix server course that LU decomposition exists uniquely for any full rank matrix A. And they have certain applications in solving the matrix equation x is equal to b faster or finding out inverse of finding out determinant of the matrix A. And also in more advanced solvers for improving their performances incomplete LU decomposition or we will do first few steps of LU decomposition, and then utilize for improving the performance of iterative servers, they will be utilized.

So, this is one broad area of Gauss elimination. Another area of gauss elimination which will look into it is finding inverse of the matrix. And what we one hint we already got from inversion of the matrix using LU decomposition, that if we can solve an equation A x is equal to I, the X will give me inverse of the equation. So, if we in order to solve this equation, if we perform gauss elimination steps on A, and finally diagonalize A, similar steps performed on A I will give me inverse of X.

So, Gauss elimination will be done on matrix A, and finally we will get diagonal form of A. And we will perform similar steps on I, and get a get the forms of I transformed following the same steps through which A has gone, and we will see I will give is giving us the inversed of A, and this is called a Gauss-Jordan technique, which will discuss in the next class.

Thank you.