Matrix Solvers

Prof. Somnath Roy
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture – 07
Gauss Elimination (Contd.)

Welcome; we are continuing our discussion on Gauss Eliminations in this class we will see some of the examples of gauss elimination, which are some of the very critical examples in which the fewer terms becomes 0 and how to handle them and then will see how gauss elimination can be expressed as a matrix operation and we will try to look into dif another side of matrix solver which is called LU decomposition matrix solutions solver techniques using gauss elimination which is which we will come directly as an application of gauss elimination.

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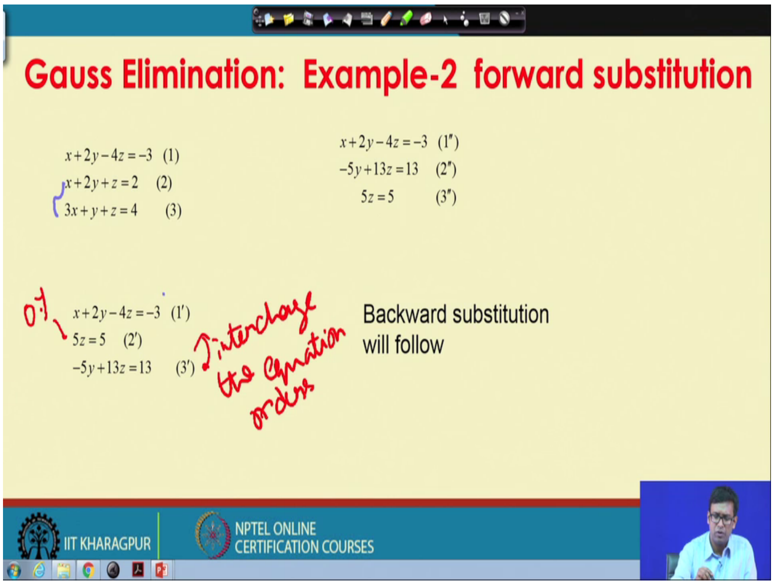

So, the this is another example one example we worked on in last class this is another example and this is the sets of equation x plus 2 y is equal to minus 4 z is equal to minus 3 x plus 2 y plus z is equal to 2 3 x plus y plus z is equal to 4. So the first idea will be eliminating the first column from the second and third equation which is x and 3 x from second and third equation; so we will subtract first second equation first equation from second equation because they have same coefficient of x and 3 times of second equation from the first equation.

And the next equation that we will get is interestingly x plus 2 y minus 4 z is equal to 3 5 z is equal to 5 and minus 5 y plus 13 z is equal to 3 so the y term of the second equation has already been vanished because it is when we subtract second equation from first equation it becomes 0. So now, we cannot subtract the coefficient of y is 0 here so we cannot subtract this from the third equation and subtract the y coefficient of the third equation.

So, what we see here is that that if I change my third equation and swap between second and third equation we get a y coefficient of the second equation which is important to have the pivot so this is a 0 0 y term coming on so we will interchange the equation interchange the equation orders and what we will get here is like this which is already given as a triangular system because x is eliminated from the second equation y and z are eliminated from the third equation.

So, for our substitution is in a way complete and backward substitution will subsequently follow in which I will eliminate z first and then I will eliminate y and I will get equation for x y and z independently; however, there is possibility of a case when this interchange is not possible and we will see a case like this.

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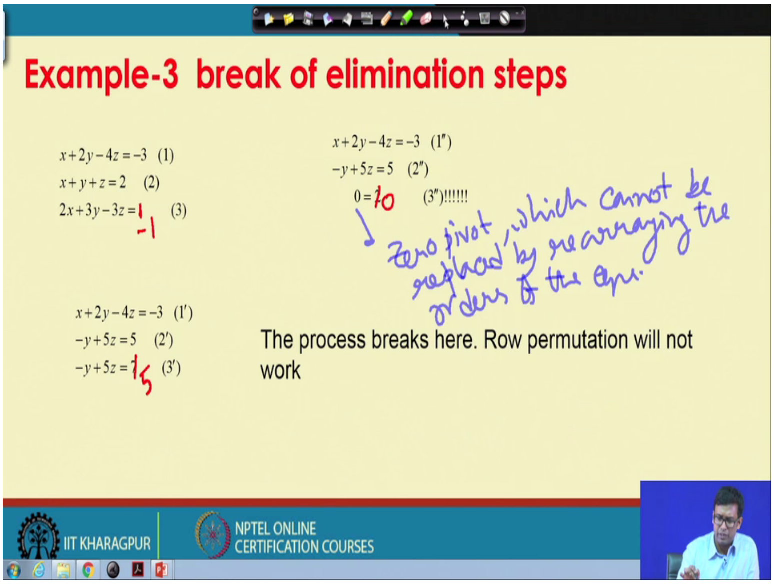

The next slide where we call the break of elimination steps these are the new sets of equation here and we subtract first equation from the second equation because the coefficient is 1 for x in both case twice of the first equation from the third equation and get the new sets of equation.

So, we can see that the left hand side of the second equation is at same means they got left hand side of the third equation so the there are in a sense linearly dependent on each other; so if we subtract second equation from the third equation we will see the last equation is getting 0 is equal to 2 which is which cannot be a any equation its neither an equation nor an identity.

So, this process essentially breaks here because there is no way of interchanging the equation order or there is no way in which we can find out z. So this is what we will call getting a 0 pivot here 0 pivot which cannot be replaced by rearranging the equation the orders of the equations so and we will say that the process breaks here row permutations. So permutation between the equation will not work and the process breaks here gauss elimination cannot work further and of course, because these two equations the same there is no unique solution of this equation and we can actually check that there is no solution of this set of equations gauss elimination will break here.

So, gauss elimination works if there is any solution any unique solution of this equation if there is no unique solution; that means, no solution or infinite solution gauss elimination will not work, for example if this system would have been instead of 7 if this term is 5. That means, this is minus this is minus 6 so if this is minus 1 this term is 5 then this would have been 0 is equal to 0 and gauss eliminated there is actually infinite solution there are infinite solutions here.

But gauss eliminations still would have been broken at this stage; so it cannot move any further if we get a 0 pivot which is not eliminated by row permutation.

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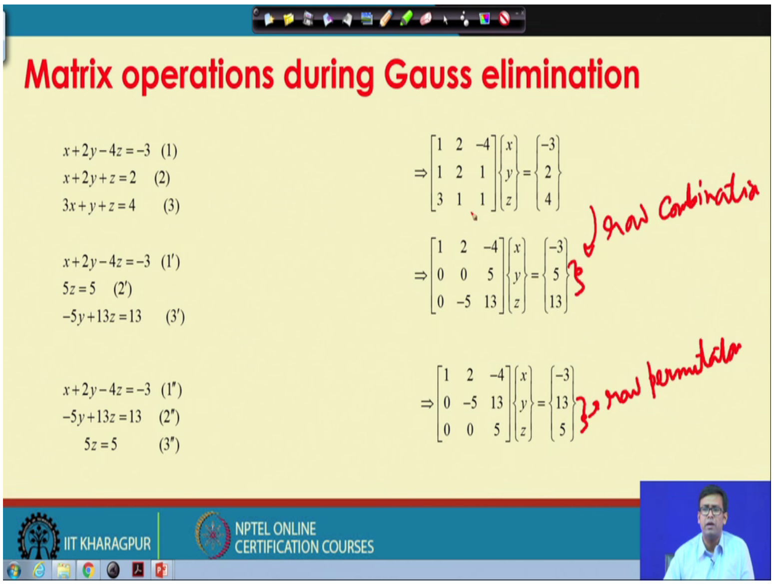

Now, we will see what are the matrix operation equivalent of gauss elimination instead of writing the equations we will write it in matrix form and see what will happen there and will take the equation system that we considered in the second equation second example previous to that row permutation worked.

So, this is the linear equation and this these are the gauss elimination steps first is the elimination step of x and then there is a row permutation equation permutation of the second and third equation which gives me the first forward substitution forward elimination. The equivalent matrix step is that this in the a matrix the as well as in the b column vector the first row and second row goes to certain steps so that they are they are changed and then there is an interchange of steps or permutation between second row and third row.

So, this is a row combination operation for second and third row and here there is a row permutation. Now what is the row combination operation multiply for multiplying for like subtracting first row from second row and multiplying first row by 3 and subtracting from third row. So it is multiplying one row by some constant may it might be 1 it might be some constant which is non zero it might be 1 might be minus 1 if we add two rows it might be 3 something and then subtracting it from another row and we can see that this can is written as multiplying the row 1 by 1 and subtracting it by row 2 can be represented by a matrix multiplication.

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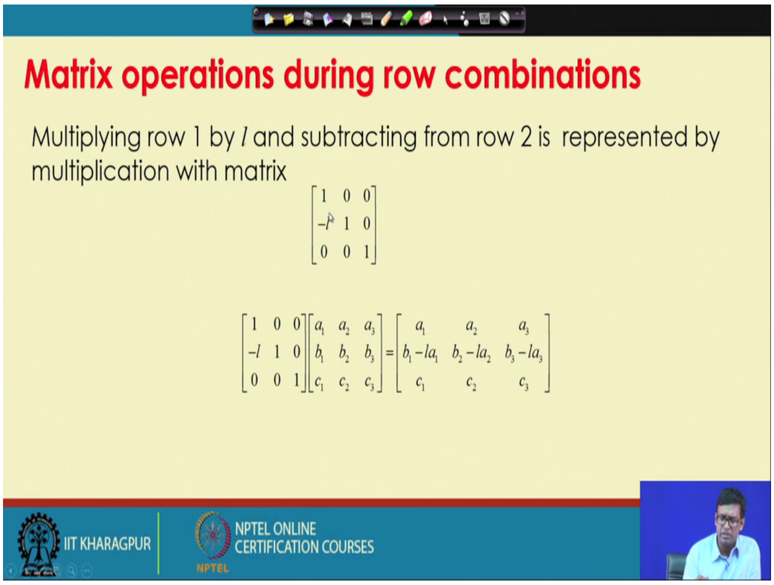

If I multiply a matrix any matrix a with this then the first row of this matrix will be multiplied by the term l and subtracted from the second row. So we can see this as the example so this matrix where its firstly, it is a initially it was an identity matrix only the row that has to be multiplied by a particular quantity and subtracted from the next row in the in the subsequent in the respective row column position there is a term which has to be multiplied and subtracted and if it is minus then this is a subtraction process if it is plus then this is an addition process.

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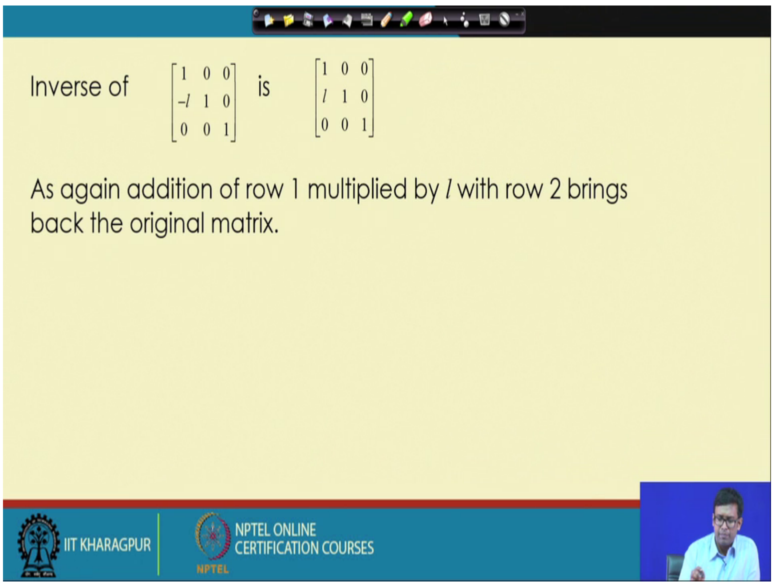

So inverse of subtraction is addition and therefore, if I the inverse of this particular matrix is the addition matrix.

So, if I subtract l times of first row from second row I get this one now if I add l times of first row with the second row in this matrix I will go back to the original matrix so inverse of this is instead of minus l I have to write l that will give me inverse of the multiplication multiplier matrix here.

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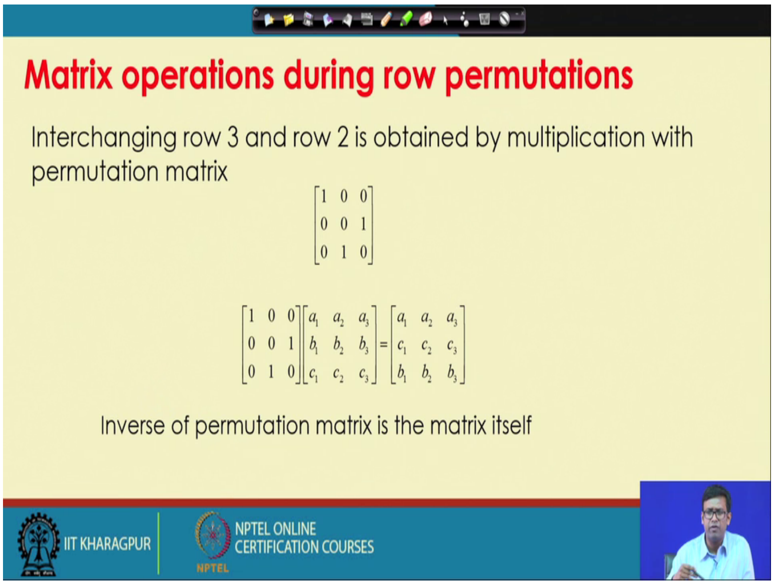

Similarly, there is a permutation matrix if we are interchanging the row 2 and 3 this will be the permutation matrix where from identify matrix the rows are also permutated you consider an identity matrix we permuted the row and we consider a.

If I multiply that permutation matrix with the original matrix it will gets it rows permutated also and inverse of the; so if I see if I multiply this with this particular matrix the row b and c will be permutated now again if I multiply this with the permutation matrix b and c will be again permutated we will get back to the original matrix so inverse of permutation is the permutation matrix itself.

If I take the permutation matrix and multiply it with a permutation matrix I will get an identity matrix; permutation matrix is itself is inverse. Now we will look into the row operations that we performed in the last class and what are the equivalent matrix multipliers there.

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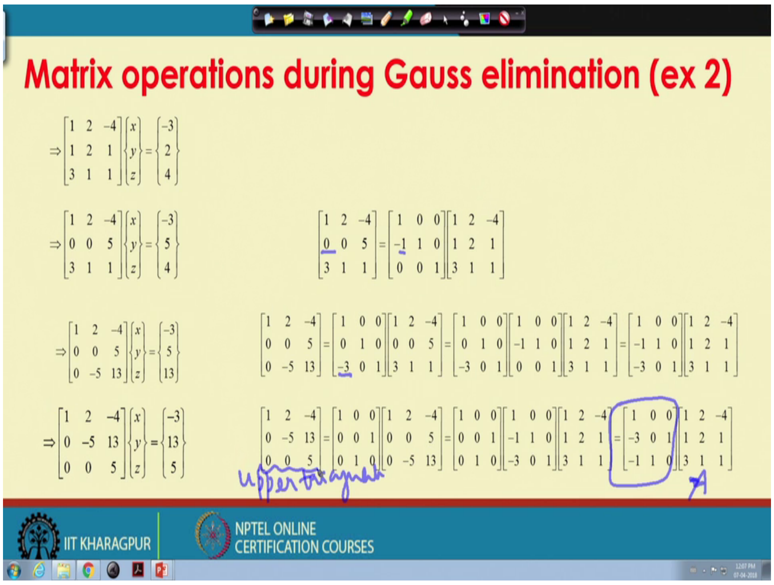

So, this is the matrix operation we read in the example 2 and first we are subtracting first row from the second row which is multiplying the original matrix by a row subtraction matrix where minus 1 is in the position of first column in the second row; that means, first row will be multiplied by minus 1 sorry first row will be multiplied by minus 1 and will be subtracted from the second row where we will get that. The next steps is doing it for the third row; third group will be multiplied a first row will be multiplied by 3 and subtracted from third row.

So, the this is the this is a matrix we got from first row operation then we multiply third row oh sorry then we multiply third row first row by minus 3 or first row by 3 so this is in the first column of third row and subtract it from third row and now we we can multiply the multiplier matrices and get that this is a final matrix which is multiplied with a for getting this particular shape and then there is a row permutation step where we do a row permutation; that means, we multiply this particular matrix this particular matrix by a row permutation matrix.

Sorry by a row permutation matrix and so this is finally, the matrix that is multiplied with my A matrix and it gives a the resultant matrix, which is now an upper triangular matrix. So there is a multiplier matrix which is multiplied with the original matrix A and now we get the upper triangular form and this is a form we are trying to we aim to get at the end of the forward elimination step of the Gauss elimination process and then there will be similarly multiplier matrix which will give me finally, a diagonal form of a from which we can directly solve.

So, now we will see an important application of Gauss elimination we have seen that Gauss elimination is a good method which elegantly can solve any equation system when there is a solution; however, it is restricted to the cases where there is no solution or there is infinite solution, but if there is unique solution Gauss elimination can handle it the next restriction on the Gauss elimination process comes to the fact that this has very large number of operations needed for cases where the matrix sizes are large enough.

For example for 10 to the power 6 into 10 to the power 6 case it needs around 10 to the power 18 operations which will take enormously huge time in even a very first supercomputer. So Gauss elimination is restricted usually to smaller matrices even when we are using computer program, but there is a variant of Gauss elimination which can probably work better for certain matrices and even can be taken forward to solve large matrix system.

For that we will look into an application of gauss elimination which is called LU decomposition. So if we see what are the matrix operation performed during row combination we have seen that multiplying first row by l and subtracting 5 from 2 needs a multiplication with a matrix like 1 0 0 minus l 1 0 0 0 1 so the row which has to be multiplied by l and subtracted from that particular row in the same row column position or column row position. This is multiplied l will be multiplied with first row first row at A matrix and subtracted from second row in the first column second row position minus l will be there and this we have seen earlier.

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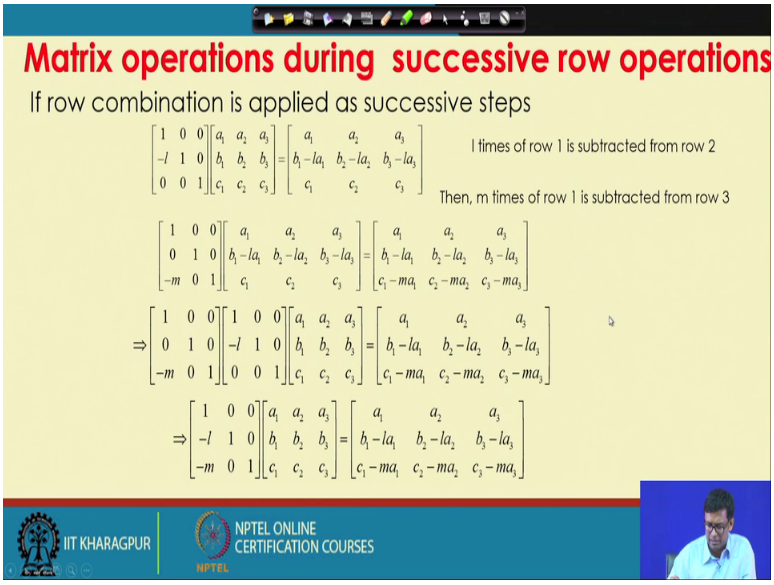

So, if we think of successive row column row operation and which we actually do in a Gauss elimination process that the row operation multiplying 1 row by 1 particular constant then subtracting it from other we do it for several times. So we do it for one particular row we again do it this is l times of row 1 subtracted from row 2 and then in gauss elimination what we will do we will also multiply something with first row and subtract from the third row.

So, n times of row 1 is subtracted from row 3 will be a process like this and in Gauss elimination I really I should get these 2 terms to be vanished. Now the next step will be that you multiply something with this row and subtract from the third row; so which is which is like so this is this is what you get after the first row operation like this these two other multipliers and the third operation is and if we multiply them because these are independent these are independent operations on each row the kind of add up the multiplication matrix adds up and you can try it yourself we will see that this is the product of the multiplier matrices here.

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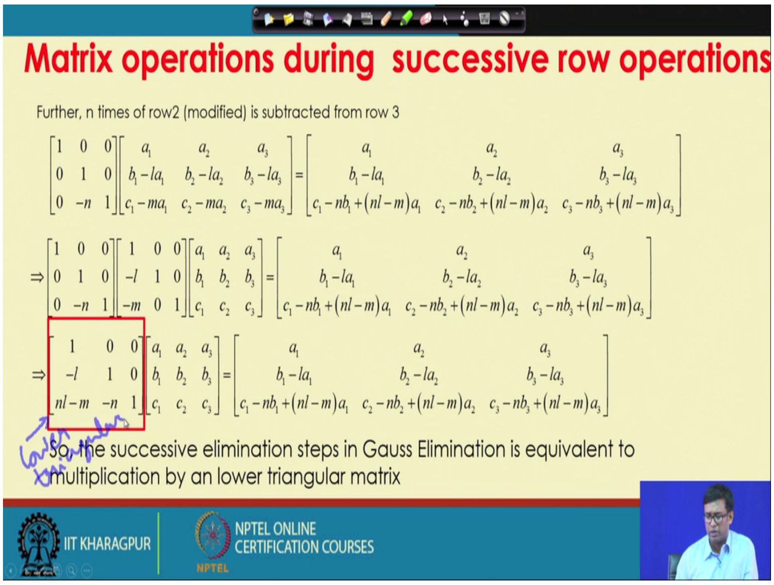

The next step will be n times of row 2 like row 2 has to be multiplied with something an element and subtracted from row 3; so n times of row 2 which is the modified row 2 which is subtracted from n 3, but modified row 2 is modified by the by as the terms of row one is added to them and row 3 is also modified. So this will be with the new matrix you use a multiplier matrix and you get this particular term.

So, if I see what is the final multiplier matrix that is multiplication of this 2 matrices and this is a matrix like this interestingly when we carry on the separations first we which is like a Gauss elimination operation. We multiply something with first row subtract from second row, we multiply something with first row subtract from third row, we multiply something from and so on then we multiply something with second row subtract from third row and so on so we carry out this operations we get a lower triangular multiplier matrix; this matrix has a lower triangular form.

So, successive elimination steps is in Gauss elimination is equivalent to multiplying the main matrix by a lower triangular matrix and we have seen that inverse and now will see what is the inverse of that actually and inverse of a multiplication matrix which is a row operation, multiplication matrix is only you substitute the negative off diagonal term by a positive off diagonal term this is for single step multiplier, not for the final multiplier matrix.

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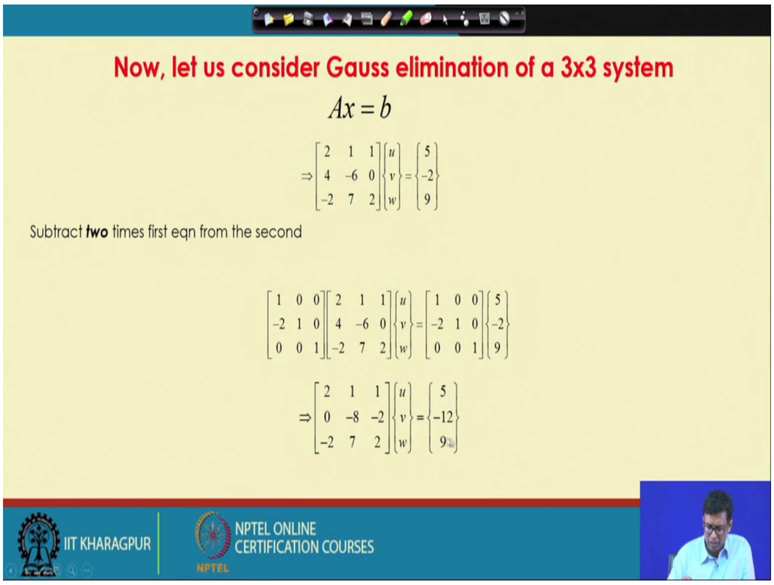

Now let us consider Gauss elimination of a 3 by 3 system A x is equal to b and we get this particular matrix equation we will subtract twice of the first equation from the second equation which is multiplying by this particular matrix. The next step is and we get a final matrix like this a new matrix like this matrix equation and note that B vector is also changing during these steps.

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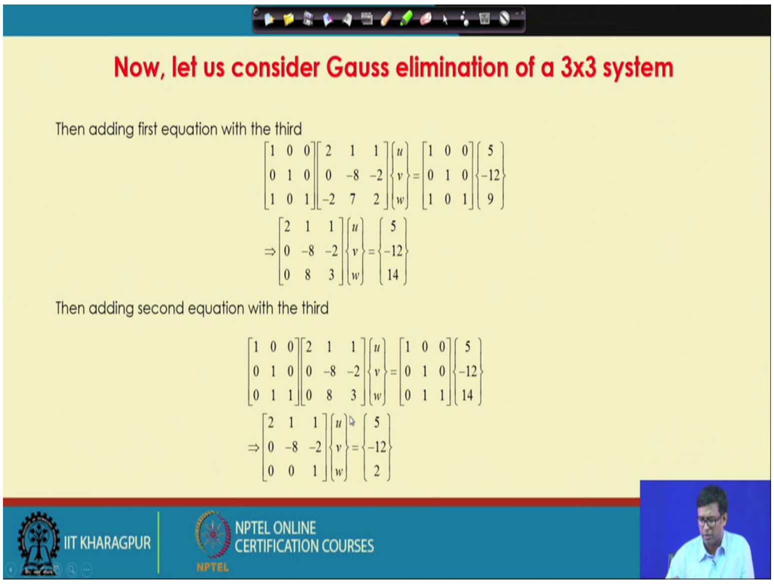

The next step is adding the first equation with the third; so with this particular matrix which is a new matrix now I added the first equation with the third and this is the multiplier matrix we get a matrix like this. The next step will be at the second equation with the third and we multiply it with another matrix and we will get this particular matrix form.

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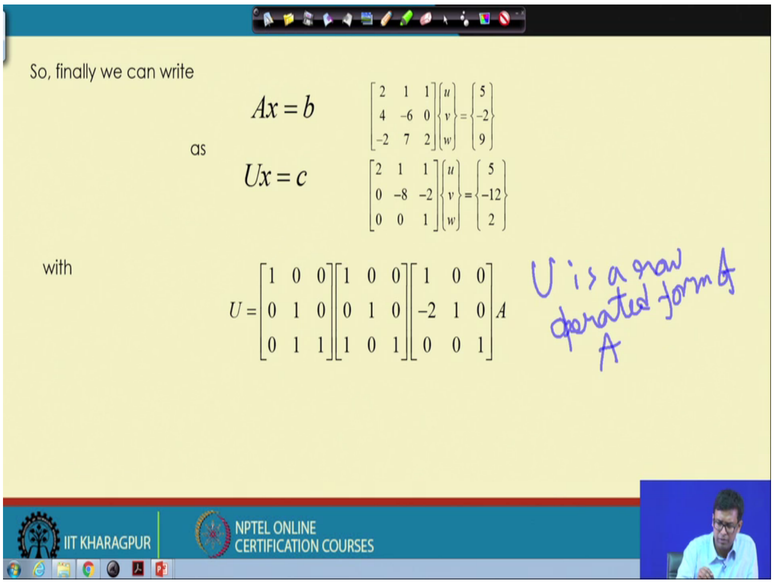

So now we can finally, write that A x is equal to b which was my initial matrix equation is changed to U x is equal to c where U is an upper triangular matrix a is multiplied with certain matrix earlier seen this important that this is a lower triangular matrix which is multiplied of a to give me U.

But U is an upper triangular matrix. So and then if we get this then we will do certain back substitution terms to get a diagonal matrix; however, we can see that this matrix U this matrix U is few matrices multiplied with A or if I write it down so we can say that U is a or U is a row operated form of a and we can say that U is equal to something into A and that multiplier is also is a lower triangular form; there is a lower triangular matrix which is multiplied with A gives me an upper triangular matrix U is a upper U is a upper triangular.

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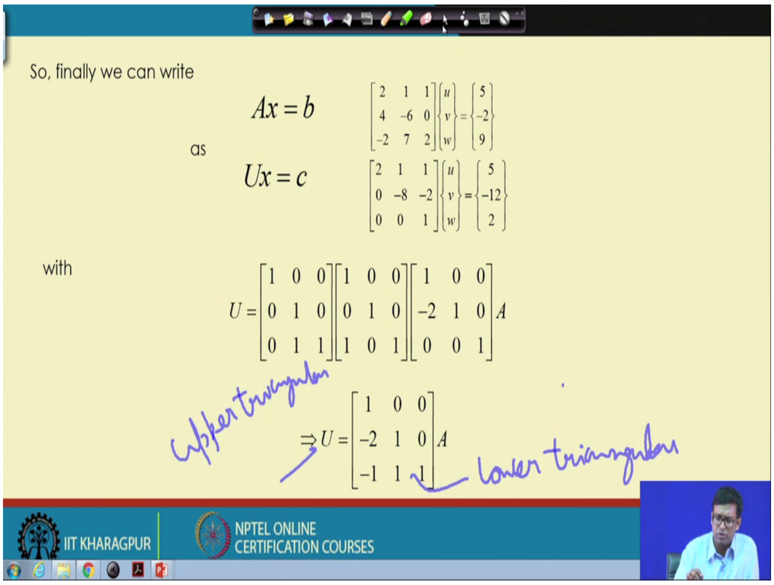

So, we get that upper triangular shape by multiplying some row operation matrix with a and the final row operation matrix is of lower triangular shape. If and this is done when there is no row permutation row permutation will gives me something different result, but let us let us look into this case only first; so this is what we got.

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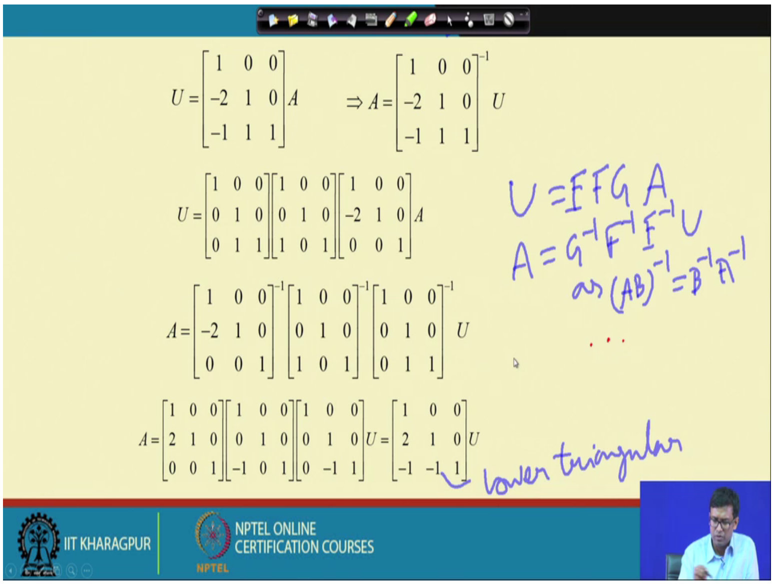

A is the lower triangular matrix multiplied with U is a lower triangular matrix multiplied with A is at cellular over triangular form and we can write A is equal to inverse of this matrix into U and further these are the matrices which are been multiplied to U so it is like U, U is equal to matrix E multiplied with matrix F multiplied with matrix G into A.

So, I can write a is equal to G inverse F inverse E inverse U; we are using the property that as A B inverse is equal to B inverse into A inverse we are using this property. So if I write that U is equal to inverse of this matrix; now this is very easy to invert this matrix minus 2 will be plus 2 we will get the inverse the this one will be all the off diagonal terms will change their sign this one will be minus 1 I will get an inverse, this one will be minus 1 I will get an inverse; so I can write that this is the inverted form of the matrix and if we multiply them we get another lower triangular matrix here.

So, this is a lower triangular matrix; so what do you have seen here that a comes as a multiplication of the lower triangular matrix which is inverse of the matrices needed for Gauss elimination step and their product of inverse of the matrices and the U which is the matrix after the forward substitutions all the forward substitutions steps of gauss elimination is over or which is a upper triangular matrix.

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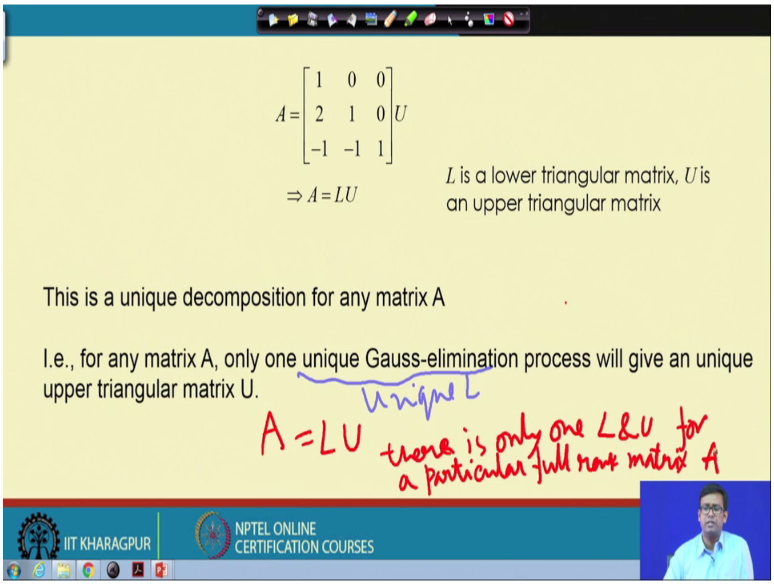

So we can write we can write A is equal to a lower triangular matrix into upper triangular matrix or A is equal to LU where L is a lower triangular matrix and U is an upper triangular matrix.

Gauss elimination steps are the steps through which I am getting U finally, and this is the inverse of this matrices is giving me multiplier matrices with A is giving me L. So L and U are very unique with gauss elimination and as gauss elimination is kind of direct process there is only one gauss elimination process possible for one particular a matrix then LU decomposition is also unique for any matrix. There is only one particular combination of L and U which can be obtained for a square matrix through gauss elimination that is for any matrix A only one gauss elimination process there is only 1 unique gauss elimination process and that will give an unique upper triangular matrix U and this unique gauss elimination process will give me an unique L also; this gives me an unique L also.

So, if I write a is equal to LU for when we I can do that is decomposition A is equal to L U there is only one L and U for a for a particular full rank matrix there is only one lower triangular matrix and only one upper triangular matrix whose product can be written as A is equal to LU. For any A this factorization is unique there is no other possible combination for any particular matrix A. Now this LU decomposition can be taken one step forward and this is important that this LU decomposition is unique to any matrix A if the matrix can be solved using gauss elimination there will be only one LU decomposition possible.

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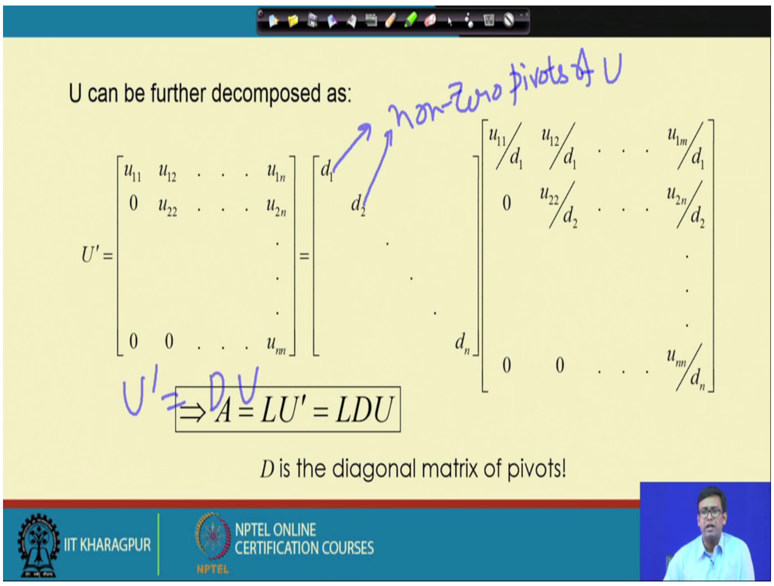

Now, this LU decomposition can be taken one step further which is U can be finally, further decomposed as a diagonal matrix and another upper triangular matrix so we will take d 1 out of it and divide all the terms of the first column by d 1. We take d 2 out of it and divide all that terms of the second column second row by d 2; sorry first row second row so on. So and this is possible because the pivots are non zero this these are basically the pivots non zero pivots of U if any of these pivot is 0 the process would have stopped here; so we can then write U is combination or U prime.

If this is the particular matrix U prime we can write U prime is the multiplication of a diagonal matrix and another upper triangular matrix; so upper triangular matrix can be further decomposed into a diagonal matrix and then upper triangular and then we can write A is equal to LU prime or A is equal to LDU instead of LU decomposition we can also have a LDU decomposition note that this U will be different than U prime; so this is also a unique decomposition of any A matrix A can be written as LDU also and d is the diagonal matrix of pivots and d there is no 0 element in the diagonal of d, d has to be nonzero or pivots have to be nonzero here.

So, this is how gauss elimination can decompose an A matrix into matrix into LU matrices or LDU matrices. Now there are certain advantages of LU and LDU decomposition and they can be utilized in several purpose using solving of the matrix equations and this solutions are usually faster and simpler than the standard Gauss elimination process and also apart from Gauss elimination there are some other applications of LU decomposition also some we will see in the next class and some of the applications of LU decomposition we will come at the later stage of this course when we will see look into iterative solvers; which is not like directly solving x is equal to b going through certain iteration starting from a gains value.

And how the iterative solvers can be speeded up we will see that LU decomposition will come something like a pre composition pre conditioner which will increase the speed of the convergence of iterative solvers later, but in next class we will see some interesting utilization of LU decomposition for solving Gauss elimination solving matrix system as well as for finding determinants or for finding inverse of one particular matrix.

Thank you.