Matrix Solvers
Prof. Somnath Roy
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture – 05
Determinant of a Matrix (Contd.)

Good afternoon we are continuing our discussion on Determinants. In last class we have looked into the physical meaning of determinants, we tried to give a definition of determinants and also spend some time on discussing properties of determinants.

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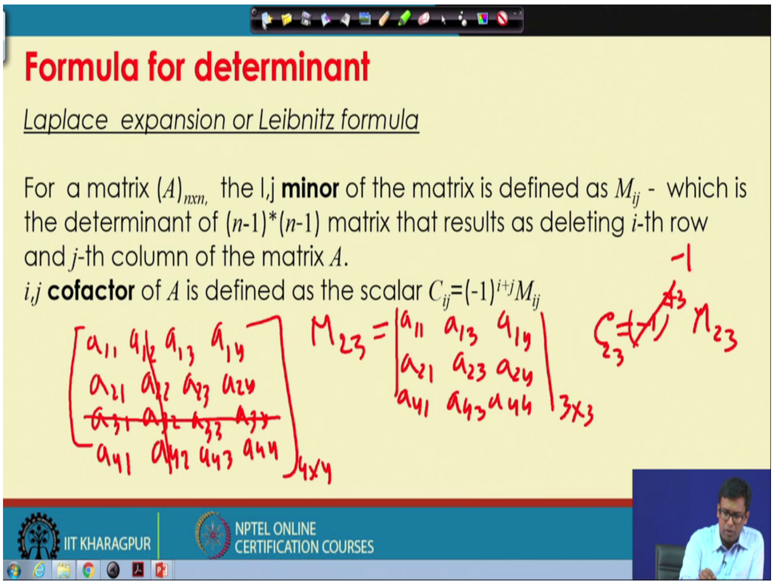

And this session we will start with the probably most awaited think right now and determinants, what is the formula for the determinants. And the formula for the determinants is usually given as Laplace expansion on Leibnitz formula given by both of the scientists independently at certain point of time, which says for a matrix A which has size n into n and as a determinant is defined only for square matrices.

The i j minor of the matrix is defined as given by the term M i j, which is determinant of n minus 1 into n minus 1 matrix that results from deleting the i-th row and j-th column of the matrix. So, for example, if I have a matrix say 4 by 4 a 1 1, a 1 2, a 1 3, a 1 4, a 2 1, a 2 2, a 2 3, a 2 4 similarly a 3 1, a 3 2, a 3 3, a 3 4 and the 4th row as a 4 1, a 4 2, a 4 3, a 4 4. Now, the minor say 2 3 minor M 2 3 in order to obtain that we have to delete the 3rd row and 2nd column so, whatever is left its determinant will be called a minor which is a 1 1, a 1 3, a 1 4, a 2 1, a 2 3, a 2 4, a 4 1, a 4 3, a 4 4.

This will be called minor 2 3 minor of the matrix and this has a size 3 into 3, whether the minor determinant of a matrix 3 into 2 that the original matrix was 4 into 4. And then comes a term cofactor, cofactor is given as C i j which is C 2 3 will be minus 1 to the power row plus column ID 2 plus 3, which is basically again minus 1 n 2 3. So, now when we calculated the minor and cofactor, that will be utilized for calculation of the determinants.

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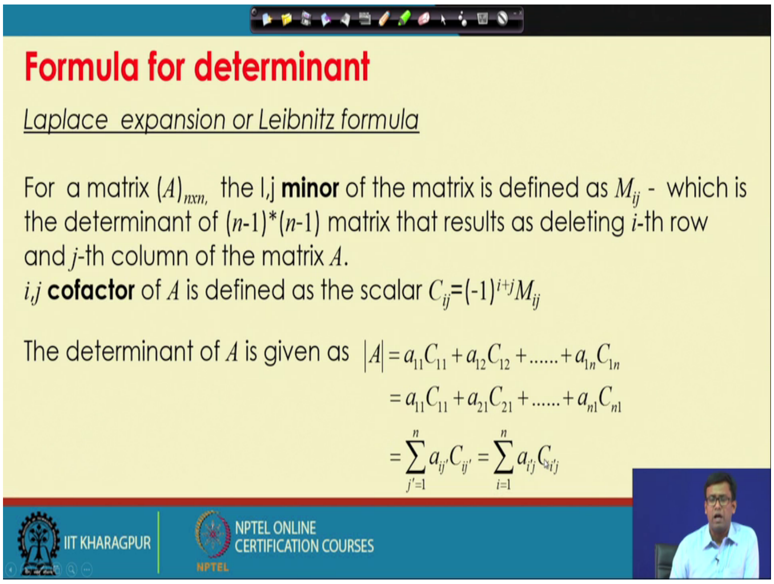

And this is given as the formula for determinant is given as summation of all the row terms into cofactor for each of the row component. So, we have to consider 1 particular row and, then calculate cofactor of each of the terms in the row and, multiply with the elements of the row. Instead of row we can do it any column also and, this summation will give us the value of the determinant.

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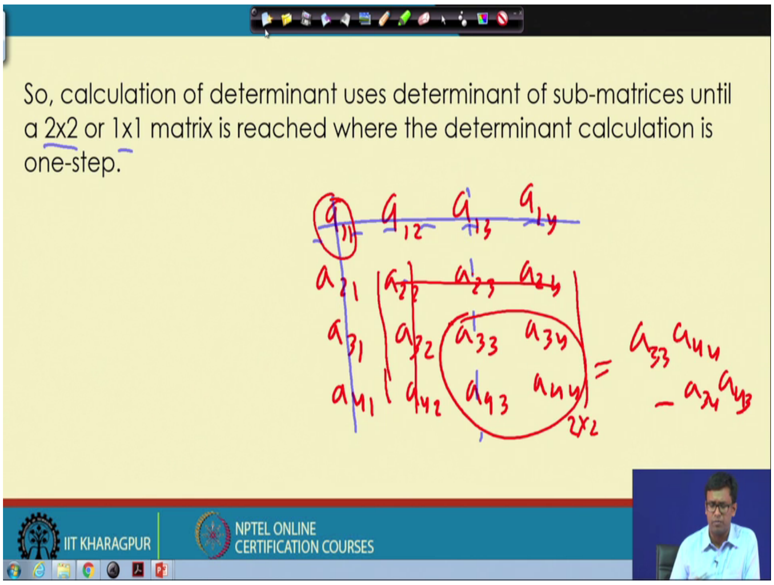

Now, if we think of calculation of determinant. So, needs calculation of further determinants for example, the earlier example that we have given a 1 1, a 1 2, a 1 3, a 1 4, a 2 1, a 2 2, a 2 3, a 2 4, a 3 1, a 3 2, a 3 3, a 3 4, a 4 1, a 4 2, a 4 3, a 4 4 in order to calculate its determinant first I have to delete say for example, first row and first column and, I get a minor matrix. And then I have to calculate the determinant of this minor matrix and, multiply with i j minus 1 2 the power i plus j and, multiply with this particular row which is 1 1 element. And then I will take each other term of the row and calculate that cofactor minors and cofactors subsequently.

So, 1 determinant needs calculation of several other determinants and, determinant of the sub matrices for this 4 by 4 matrix it needs calculate of this 3 by 3 matrix may be then we will delete for example, later we will delete this row, this column and this particular row and, then calculate matrix which will have this column, this column, this column and its determinant. So, calculation several determinant therefore, if we start with the large matrix in my like 100 by 100, we have to calculate determinant cofactors of 100 terms of each row, one particular row or column.

So that means, you have to calculate 199 into 99 matrix determinants, again each 99 into 99 matrix will lead calculation of 99 98 into 99 size matrix determinant therefore, there will be loop of calculation of determinants and finally, we will reach to a smaller matrix like which will have 2 into 2 or 1 into 1 size for example, if I think of calculating determinant of this particular matrix, I have I will first delete this column, this row and I will get a smaller matrix like this.

Which is now 2 into 2 and therefore, is determinant can be calculated easily a 3 3, a 4 4 minus a 3 4. a 4 3. And then these determinants will be plugged back determinant of this multiplied by minus 1 to the power 1 plus 1, which is 1 into this will give me 1 component of the determinant of this 3 by 3 matrix, then I will take this row element and the cofactor, these row element and the cofactor and finally, get 3 by 3 matrix.

So, we will get all the 3 by 3 matrix determinant which are cofactors of 4 by 4 matrix determinant and so, on we can finally, calculate go back to this 100 by 100 matrix here considering. So, this calculation is not very straight forward one especially for large matrices and, we have to use something like a computer program for large matrix. And we will see later that all large matrix calculation, even for solving large matrix equations, we need to write any form of a some form of computer program ok. Now, we will see some of the application of this determinants at least why this complicated calculation will be needed at certain cases.

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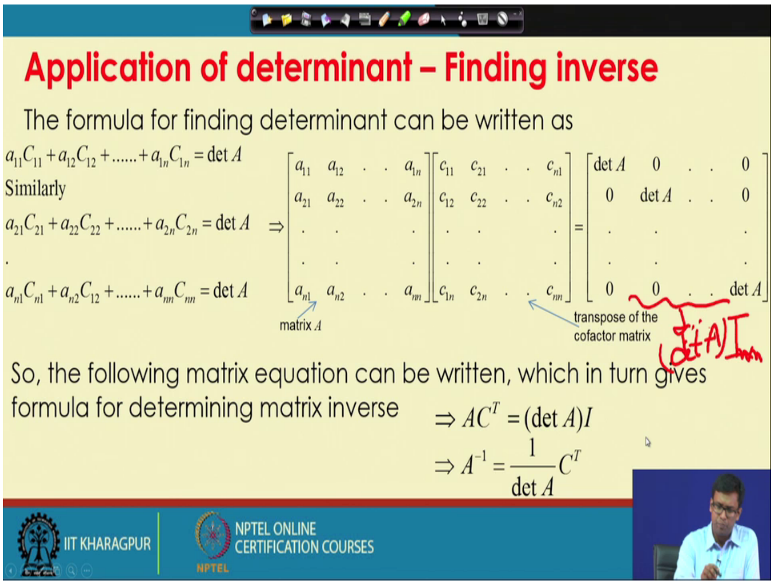

And one of the application is finding an inverse. So, if we write the formula for finding determinant, which we discussed in last slide is that the element in one row into it is cofactor the next element in that row into the cofactor so, on. For each row I can get same formula like this, the elements of the multiplied by the cofactor gives the determinant. And therefore, we can get in if there is n by n matrix n equations for n each rows, where the right hand side in all the cases are determinants.

And this equation now can be expressed as a matrix form, where a is the original matrix matrix a, the cofactors are ordered along like know in a transpose manner that is c 1 1, c 2 1 up to c n 1 comes in one row of the matrix and c 1 1, c 1 2, c 1 n which are ideally row cofactors of each row of a 1 1, 1 2 comes as 1 column. So, this we can say transpose of the cofactor matrix and, this gives us a matrix determinant diagonal matrix, this is a diagonal matrix each entries determinant of A.

So, we can say that this matrix the right hand side matrix is nothing, but determinant of A which is a scalar sorry determinant of A which is a scalar and, this multiplied with an identity matrix of size n into n. So, we can further say the following matrix equation, this matrix equation can be in which is basically expression for determinant of A matrix using each row and it is cofactors and, this will give a this equation is matrix A into transpose of the cofactor matrix C transpose is determinant of A into an identity matrix A into C transpose is determinant of A into identity matrix.

So, we can divide both side by the determinant A and can write a into C transpose by determinant of A is an identity matrix, A into anything which gives an identity matrix is the inverse of A therefore, we can write A inverse is the transpose of the cofactor matrix divided by determinant of A. So, this is one way we can utilize the matrix determinant to find out matrix of an equation.

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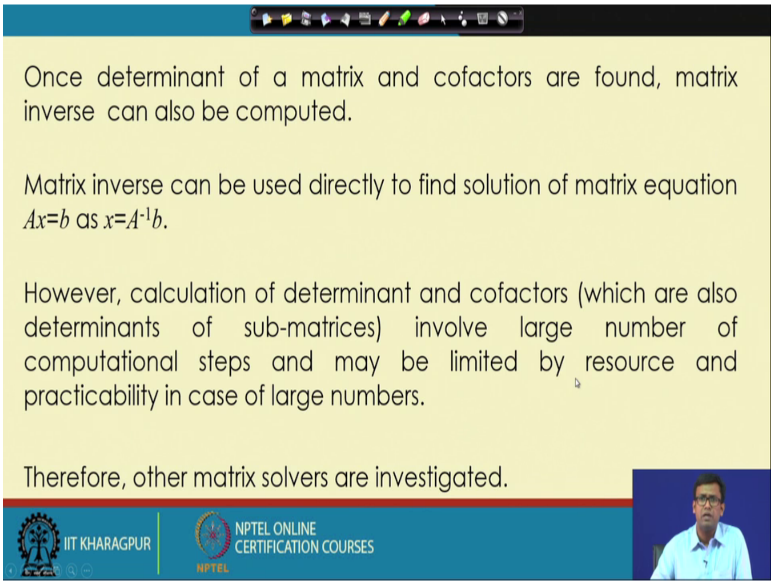

And now sorry now, once we can find out the determinant and cofactors of course, cofactors needed to find the determinant; we can find the inverse of the matrix. And once matrix inverse is found out now, we can have our probably first proposed matrix solver that x is equal to b system equation system is given for A is a large matrix x column has large number of vector. So, that you can solve them directly x is equal to A inverse b, where a inverse is calculated through determinants and cofactors and, x is now the solution.

So, this is a 1 matrix solver technique, where find out a inverse and multiply with b to get the solution x. However, calculation of determinant an cofactors have seen for large matrices, involve large number of computation because, each cofactor comes from a sub matrix which, whose determinant again cannot be directly calculated if it is not 2 into 2 type of matrix. Again we have to get cofactor of the sub mattresses and, calculate its determinant and go up to a 2 by 2, or 3 by 3 smaller matrix.

So, it involves large number of computational steps and, sometimes this calculations multiplication of so, many numbers finding out, number of determinants and adding them up after multiplying with the row, row elements this needs very large number of computation and, sometime it may be limited by the practicability or the resource for a simple desktop computer, I might not be able to solve a 1000 by 1000 system by finding inverse in for example, in couple of hours it might need very long time for the calculation.

So, these are not very elegant of solving matrix equations and, mostly restricted due to the resource limitation and also practicability of the solution and, that is why we need to investigate other matrix solvers. So, again the determinants can be applied to find out matrix solution of matrix equation in another method also.

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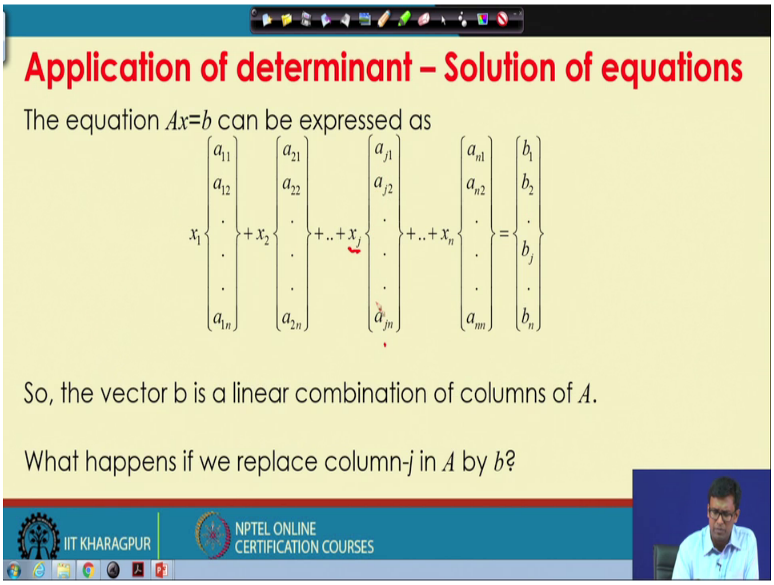

$$x_1\begin{Bmatrix} a_{11} \\ a_{12} \\ \cdot \\ \cdot \\ \cdot \\ a_{1n} \end{Bmatrix} + x_2\begin{Bmatrix} a_{21} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \\ a_{2n} \end{Bmatrix} + .. + x_j\begin{Bmatrix} a_{j1} \\ a_{j2} \\ \cdot \\ \cdot \\ \cdot \\ a_{jn} \end{Bmatrix} + .. + x_n\begin{Bmatrix} a_{n1} \\ a_{n2} \\ \cdot \\ \cdot \\ \cdot \\ a_{nn} \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ \cdot \\ b_j \\ \cdot \\ b_n \end{Bmatrix}$$

And which we going to discuss that will gives us a very famous rule of solving matrix equation called Cramers rule. So, if I have a equation x is equal to b, we have discussed this can be expressed, as summation of column vectors multiplied with coefficients x 1, x 2, x I, x j, x n. So, a linear combination of column vectors gives us the solution vector the right hand side vector b.

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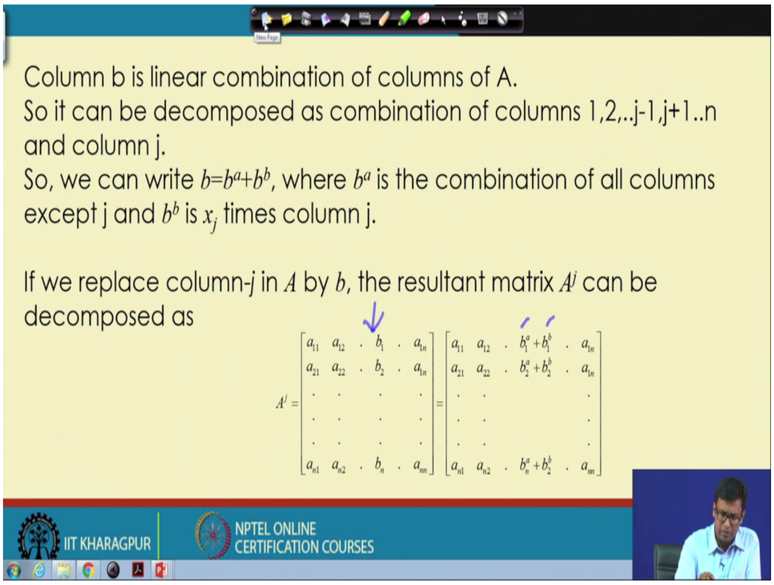

So, b is the linear combination of columns of A. Now what will happen if I take away one column of A and replace it by b or I have the matrix equation x is equal to b, I take j-th column and I take this j-th column and substitute it by b, what happens to it. And we can see column b we know that column b is the linear combination of columns of A that is why how the matrix equation is there. So, b has components along all the vector column vectors of A. So, there are combination b can be expressed as combination of column 1 like column 1 into a constant plus column b into a scalar plus columns 3 into a scalar plus column 3 into a scalar up to column n into a scalar.

So, we can write it is a combination of column 1 2 j minus 1 column j plus 1 to n and also column j, combination of all columns basically. So, we can decompose b into 2 parts one is a another is b where a is combination of all columns except j and, b is contribution of column j and, what is the contribution of column j. If we go back to a previous slide, we can see contribution of column j vector is nothing but x j, x j amount of columns j is multiplied to get b 1 part of b.

So, contribution of column j is x j. So, now, we can say this is linear combination columns except j and, there are different x 1, x 2, x n different amounts, or scalars are multiplied with other column vectors and x j is multiplied with column j. So, if we replace column j into A by b the resultant matrix now, can be decomposed as or can be written as that a is the j-th the j-th column is now replaced by b 1 b matrix. So, b matrix has 1 point b a and b b.

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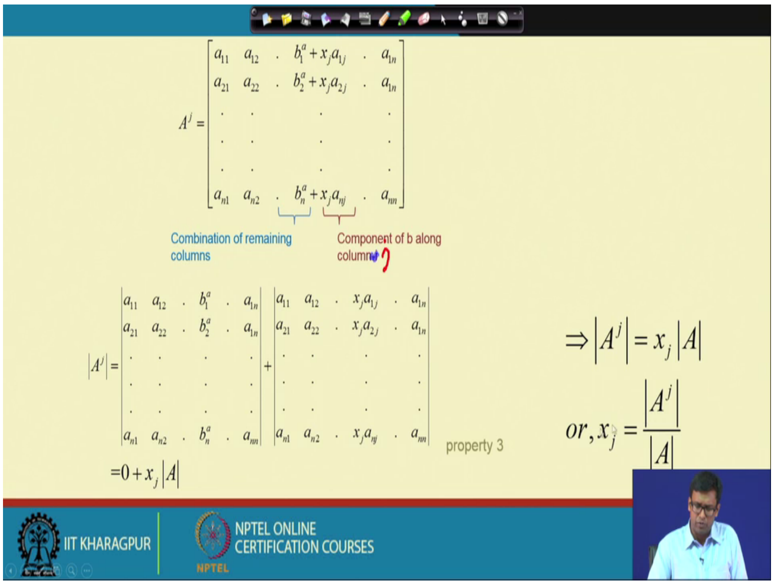

And now if we see wait a second, if we write a the a has one part which is b a and another part which is x j into a 1 j x j into a 2 j x j is basically the scalar multiplied with the column j and therefore, if I try to take it is a has two parts one is like the j-th column of a has two parts 1 is combination of all other column and, another is combination of the component of b along column j. This is not column I, this should be column j, this should be column this should be column j.

So, if I try to take its determinant and, we can we have discussed about different properties of determinant, that the determinant can be expressed as, if we decompose column into a row into two part the determinant also can be decomposed accordingly therefore, the determinant will be 1 1 determinant one part of the determinant is due to the combinations of all other columns except j and, other part is due to x j.

And so, this is property 3 which we discussed in the properties of determinants. This part is a linear combination of all columns in this matrix, because b a is combination of all other column. So, this is as linear combination of all other columns the determinant must be 0 and, we have seen it earlier. And the right hand side is a multiplication of x j with one particular columns. So, we can take x j out of it and this will be nothing, but the original matrix A.

So, the determinant of A j which is j-th column replaced by b, will be 0 plus x j into determinant of A. And we can now write determinant of a j is x j into determinant of A therefore, the x j will come out as determinant of A j divided by determinant of A. And we can do it for column 1, column 2, column 3 up to column n and can get different values of the solution of like x x 1, x 2 up to x n.

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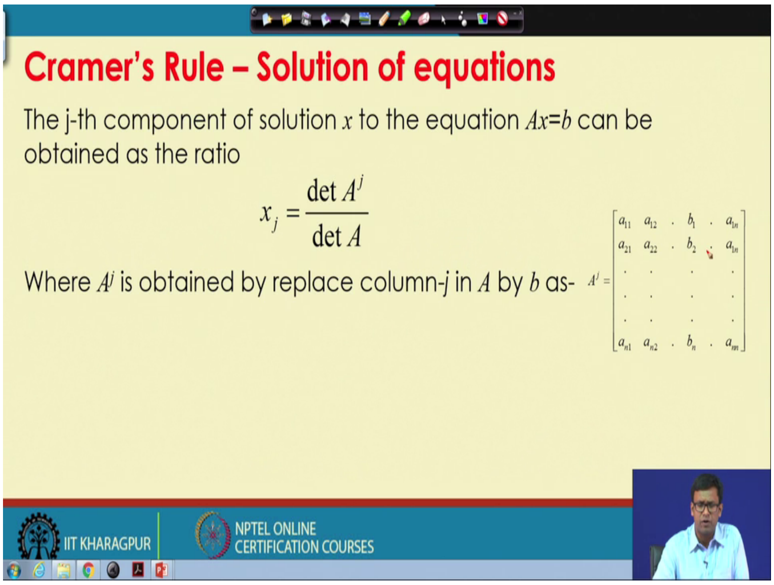

And this is given as very famous Cramer's rule. The j-th component of solution x to the equation A is equal to b can be obtained as the ratio x j is equal to determinant A j, where A j is by determinant of A where, A j is replaced by column j in A where A j is obtained by replacing the column j of A by the vector b as this is the form of A j. Now, this has been utilized for certain cases at least for not extremely large matrix, but for some larger matrices to find out the solution. And it directly give the solution of x is equal to b.

However if A has determinant 0, then it will not give us any solution. And there can be cases when A is a singular matrix will we of course, Cramer's rule we will say that the solution is either infinity something by 0, or it might be determinant of A j is also 0 so, it comes to a 0 by 0 form.

So, Cramer's rule cannot give us solution in case of determinant of a is 0. And we have to see what actually happens to the equation system, when determinant of A is equal to 0, or the cases when the equations will not have a solution or maybe we can will see some of the cases, where the equations have infinite solutions and all these cases link with determinant of A is equal to 0.

So, one very important utility of the concept of determinant is to identify an equation system, which has no solution, or which has a unique solution, or which has infinite number of solutions. So, we will have a quick discussion on the cases, when we have different type of solutions like what will be we can call existence, or uniqueness of solutions of x is equal to b and calculation of determinant of A will play a very vital role of that.

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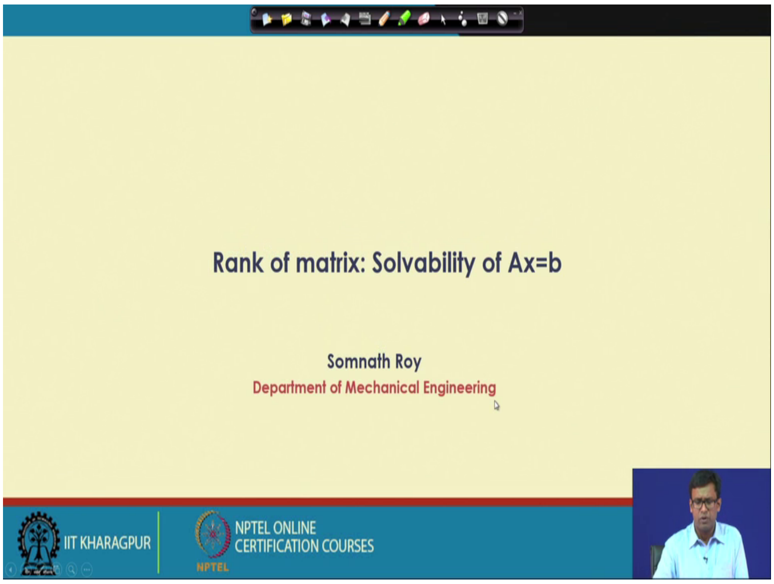

So, quickly look into a concept called rank of a matrix which is which comes from again calculation of determinants we will see and, solvability of the equation x is equal to b, by the term solvability will see the case whether they equation x is equal to b is at all solvable, whether the solution gives us an unique solution which is desired in many cases, or there are cases where multiple solution existed in terms of particular solution you will see, infinite number of solutions exist in certain cases.

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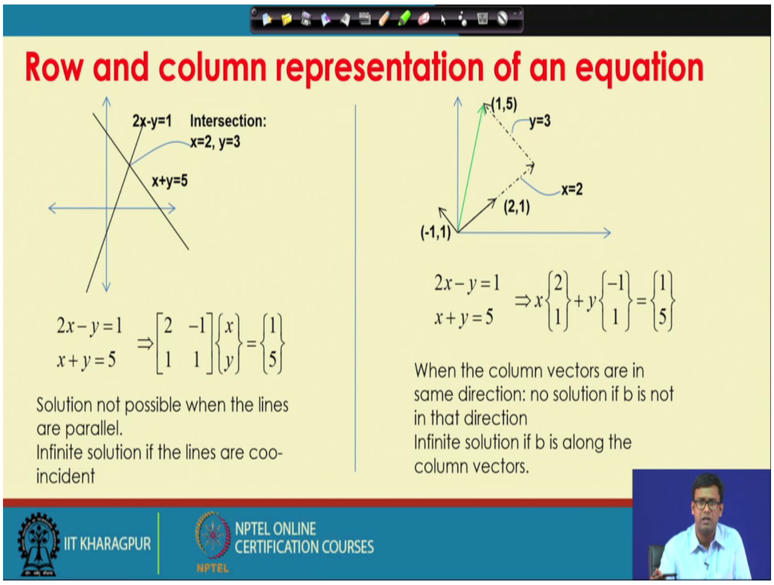

So, we go to the equation systems and, we start with looking into representation of the equation system, or when a physical system or a vector algebra can be represented as an matrix equation. So, we start with the most primitive form of equally combination of a linear equation system, which is 2 straight lines crossing each other as equation one as equation 2 x minus y is equal to 1 another has equation x plus y is equal to 5 and, if we solve this equation we get the intersection point.

Now, and this is the equation represented in a matrix form. Now, solution is not possible if the lines are parallel, there will not be any intersection point, if one line is coincident on other at any point on that line will be a solution. So, there will be infinite solutions. So, in case the lines are parallel there is no solution in case the lines are incident on each other, which is basically in both the cases lines are exactly same slope. The first case lines are parallel no solution, lines are coincident infinite solution and, if the lines are of different slope. Then definitely they will meet at certain point, they will intersect each other at certain point and we will have a solution.

So, no slope same slope is no solution, or infinite solution and in different slope gives us an unique solution and, this is a row wise representation because, each equation gives me the members of each of the rows. If we look into the column wise representation, it is basically 2 vectors a linear combination of 2 vectors, first vector is a column 2 1, second vector is a column minus 1 1 and, I multiply 2 1 by two times and, then I add minus 1 1 by 3 times, I get the 0.15. So, 1 5 is the x is equal to 2, or is equal to 3 is the coefficients multiplied with each of the columns, which is giving me the resultant b column.

And in which case there will be no solution, or infinite solution when the column vectors are in the same direction, then there cannot be any solution, if b is in a different direction for example, if this is 1 5 is the b vector and, I have 1 column 2 1 another call another column vector for to know way I can add this to have a linear combination and find out b. So, if 2 the column vectors are in one particular direction and the b vector is an another direction there is no solution.

If the b vector is in the same direction of the column vectors, if column vectors are in same direction and b vector is in the same direction, there are infinite solutions. If the column vectors are at two different directions for any b vector I can express it as combination of these 2 column vectors, linear combination of these 2 column vectors and I can get a solution.

So, again here the solution when the column vectors are in the same direction, if b is on that direction, there is infinite solution otherwise there is no solution.

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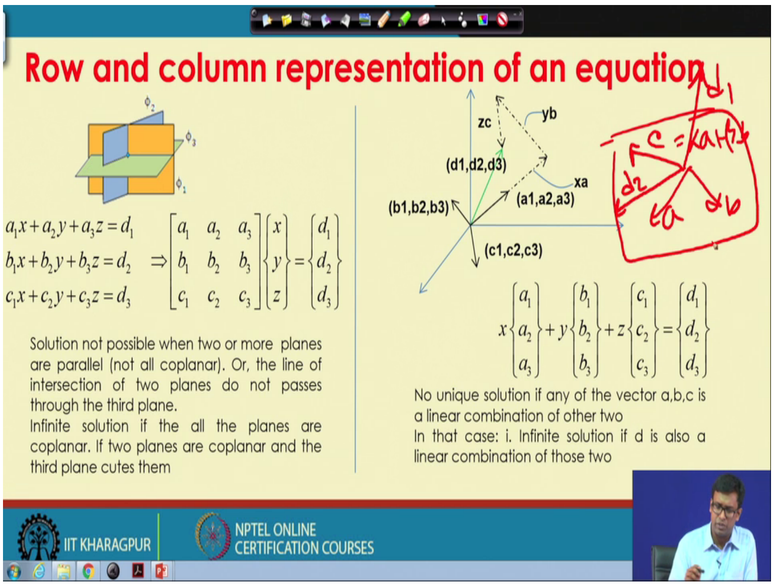

We can have the same thing represented for a 3 d case, in a 3 d the row based equation system will give you us equation on 3 different planes. If the planes intersect at 1 point there is a unique solution. If they do not intersect at one particular point there can be two cases, if the two or more planes are parallel, but they are not coplanar. So, that this planes are never intersecting, will have no solution, or lie or there can be a third case there are not coplanar, but line of intersection of they are not parallel.

But line of intersection of 1 plane or 2 planes is cute to the third plane and then they are also, they will not be any solution. And if all the planes are coplanar any point on that plane will be a solution, there will be infinite solution. Similar thing we can say that will happen for combination of vectors, that if I add 3 vectors a 1, a 2, a 3 I will get the third vector there is no solution, when like if a b c is a linear combination the other 2 there will be no unique solution for example, if a b c 1 vector is combination of others, then there will be basically two independent vectors which is creating a plane. So, for example, we can have a case like this like this is vector a, this is vector b and this is the plane containing vector a b in which c is a member.

So, c is basically alpha a plus beta b. Now, if d is orthogonal to that, if d is here there is no combination of a b c which can form d there is no solution, or if say this is d 1, if d is again on the plane, then there is a infinite possible combinations which can give us the solution. So, we can see that there are cases of no solution in infinite solution in this cases of some similarity, if the plane are parallel, if they are co coplanar, then no solution. If they are coplanar then infinite solution, if they are parallel, but never meets then there are they have same angle of inclination, but never meets they are parallel they are infinite solution.

Two if the third vector is combination of first vectors, then there are infinite solution. If the solution vector is also combination of first 2 vectors, if not there are no solution.

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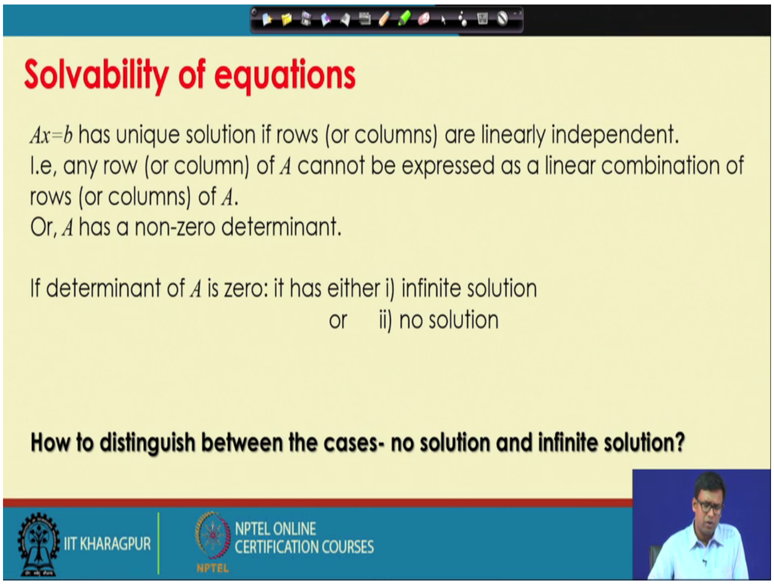

So, now, we will see that what we call solvability of the equation A x is equal to b has unique solution, if rows or columns are linearly independent, all the planes are in different, then angle of inclination all the lines are in different angle of inclination. All the no column can be expressed as resultant of combination of other column. So, that is what is called that they are rows, or columns are not linear combination of any row column is not a linear combination of other rows or columns of A. And in this case the matrix is non singular and A has a non zero determinant.

Now, if A has a 0 determinant then, there will be either infinite solution or no solution and, now we have to see how can we distinguish between the cases infinite solution, or no solution.

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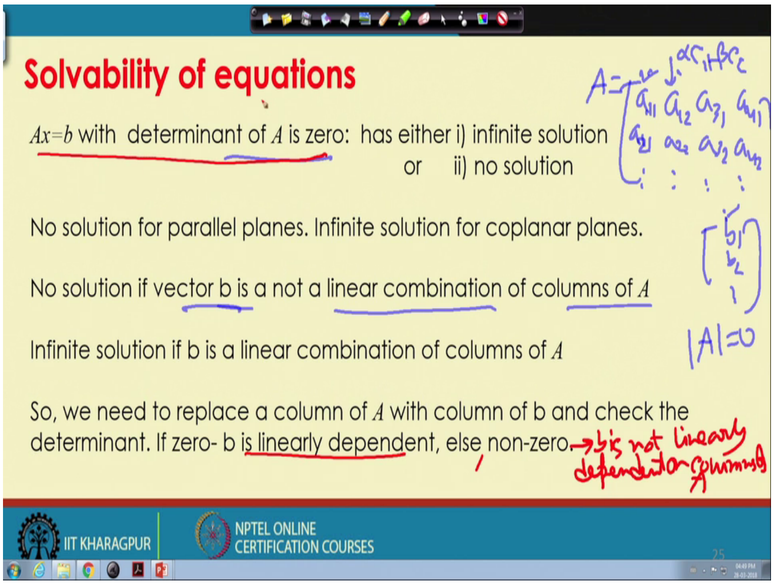

We look into x is equal to be where determinant of A is zero; that means, one row is at least one row is linear combination of other rows, at least one column is linear combination of other columns and, A has can have infinite solution can have no solution.

No solution in case of parallel planes infinite solution in case of coplanar planes in 3 d, no solution, if vector b is a linear combination of column A if vector b is also like one of the column is linear combination of other columns of A vector b is not a linear combination column A no solution. If vector b is also linear combination of column A then, there are infinite solutions. So, we need to replace we have to check whether vector b is a linear combination whether vector b is a linear combination of columns of A.

And this things are happening, when determinant of a is zero; that means, a at least one column of A or more than one columns of a are linear combination of some other columns of A. Now, if to see whether b is also a linear combination of A. So, one column if a is linear combination of other columns of A; that means, determinant of A is zero. Now, if we replace that column by column b, if column b is linear combination of all other columns of A, the determinant will still 0. If it not so, then b is not linear combination of other columns.

So, we need to replace a column of A with columns of b and check the determinant and 0 b will see that it is linearly dependent. Now, it might happen say I have 4 columns of a a

1 1, a 1 2, a 2 1 so on, a 1 2, a 2 2 so on a 3 1, a 3 2 so on and a 4 1, a 4 2 and so on and this is my b 1 b 2 so on.

Now, may be this is equal to alpha c 1 plus beta c 2. So, I take this as a determinant of A is equal to 0. Now, if I replace first column, if I replace the first column by the column vector by the b vector, the determinant will be still 0. So, I will see that the determinant is still 0, I cannot make a decision, if I replace third one the determinant will be still 0 only if I replace the second column by b the determinant is non zero because, b is not a co linear combination of 1 3 and 4.

So, I have to check with each column, I have to replace each column by determinant of b and see whether the determinant is non zero. If I see all the determinants are non zero, then I will say that zero b the zero determinant is linearly the b. Is linearly dependent on b if non zero, then I will see say that b is else non zero implies that b is not linearly dependent on columns.

And then b cannot be expressed as a linear combination of columns of n therefore, there will be no solution. In the other case when b is linearly dependent, there will be infinite solution because, unique solution case has been ruled out as the determinant of a x A is already zero. So, we will go to the next slide.

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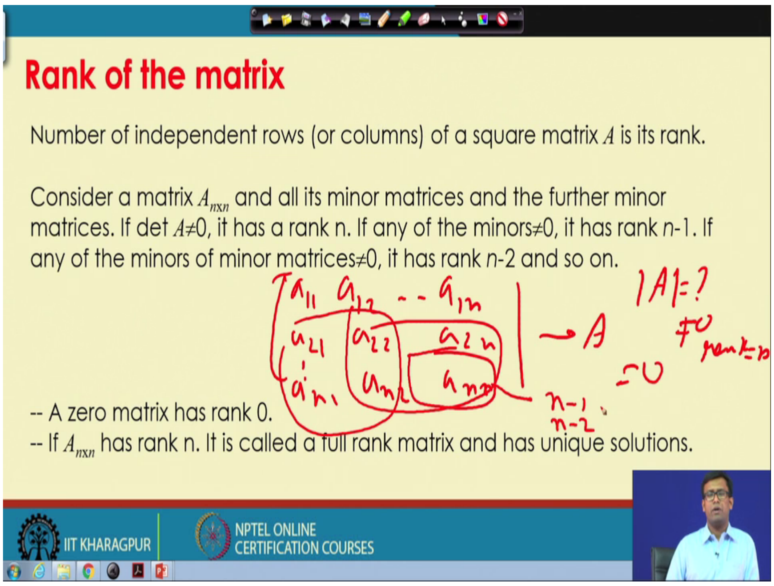

We define a term rank of a matrix, which is rank of number of independent rows or columns of a square matrix, consider a matrix n into n size and all its minors and further minors etcetera. If determinant of a is non zero, it has rank A in A. In case determinant of A is 0 and any of the matrix of the minors is not 0, it has rank n minus 1, if determinant of A is 0 and first minors are 0, we have to see the minors of minors, If they are non zero then will say rank is n minus 2.

So, this is basically a way to find out what are the number of independent rows and columns and, A zero matrix will always have rank 0 and A n into A and we if it has rank n we call it a full rank matrix and, we will say that it will have an unique solution. So, how will you calculate rank for example, I have matrix a 1 1, a 1 2, a 1 n, a 2 1, a 2 2, a 2 n, a n 1, a n 2, a n n.

I will check what is this is a whether determinant of A, what is determinant of A, if it is non zero rank is if it is 0, then I will consider the minor matrices like this. If they give a determinant non zero rank is n minus 1. Otherwise we will take a smaller minor matrix sub minor matrix, if that gives non zero determinant then rank is n minus 2.

So, on and we can find out what is the final number of independent rows and columns and that will be rank. If all the elements of a matrix is 0, it will automatically have rank 0. Otherwise it will have at least rank 1 at least there is one non zero component. And if n into n has rank n, then we call it to be a full rank matrix which will always give a unique solution.

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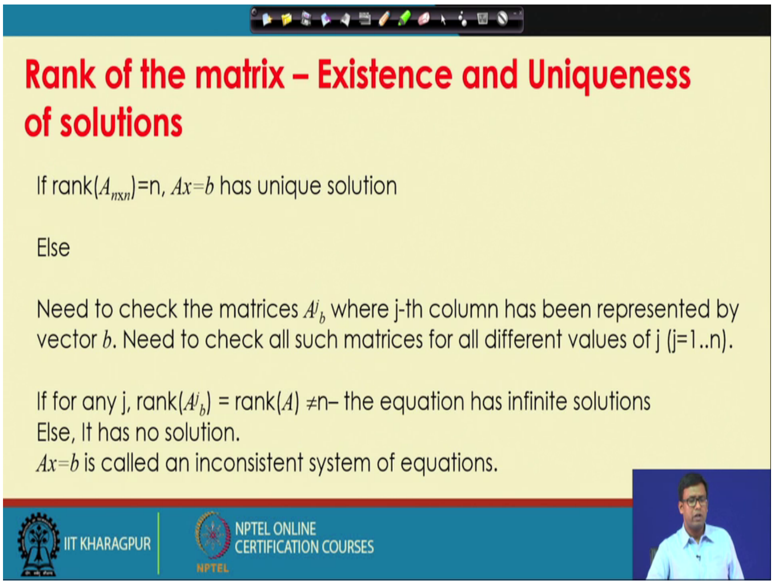

So, if a rank of A is n into n is A A x is equal to b will always have a unique solution, if not so we will have to check the matrix A j b, which we have done while finding out solution of the matrix, column j the j-th column A is replaced by b, we have to check A j b where j-th column has been represented by b. And we have to check all such matrices for different values of j. Now, if for any of j the rank of A j b is is equal to rank of A which is not equal to n, we have ruled out the rank of A is equal to n, there is no unique solution in that case rank is not is equal to n there are infinite solution, else there it has no solution.

And we call if there is no solution, then x is equal to b is called an inconsistent system of equation this equation system is not consistent therefore, there is no solution. So, these are the very important three cases if A is full rank matrix that is unique solution if A is not a full rank matrix, but rank of A is same as rank of A j b at any column like j is replaced by b, we say has infinite solution. If A is not a full rank matrix, but rank of A j b is greater than rank of A, we will say there will be no solution.

So, now we will start looking into cases, what happens how will be when x is equal to b solution of x is equal to b, when there are infinite solution, or there are unique solution of course, no solution cases and usually been ruled out, but there are some methods. Where you can find out the best solution, or the most optimal solution when no solution actually exist, there what is the most optimized solution, or what is the nearest solution and we call that base solution. We will start looking into different solution methods in the later classes.

Thanks.