Matrix Solvers
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Lecture-29
Eigenvalues and Eigenvectors for Real Symmetric Matrix

Welcome, we have been discussing on the Eigenvalues and Eigenvectors of a square matrix. And which I say it will be important if we try to understand the fundamentals, find the direct iterative method solvers, the convergence of an iterative method solvers, specially for the directive iterative solvers or the a the conditions under which the directive iterative solvers will work, or some of modifications will work on this solvers the eigenvalues and eigenvectors will be important. And we studied with the discussion that eigenvalues actually arise from the solution of ordinary differential equations or rate equations, expressed as combination of several variables.

So, we gone through an example to see how eigenvalues and eigenvectors can be computed and made few observations on them. And in the in the present class will look into more on the properties of eigenvalues and eigenvectors.

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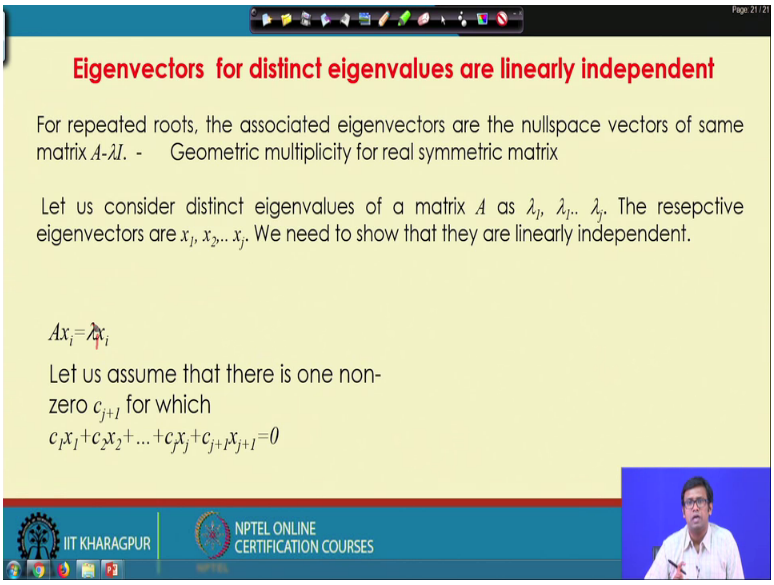

So, we discussed about distinct eigenvalues and repeated eigenvalues, as the roots of the polynomial characteristic polynomial. So, for repeated roots the part were, indeed for repeated roots the associated eigenvectors and null space vectors of the same matrix A minus lambda I. So, there is one lambda which is arising thrice or more than once sizes an example, arising more than once as a solution of the polynomial equation.

So, this when will write A minus lambda I, this particular lambda will introduce more than 1 0 pivotes in the equivalent form of the matrix. So, there are more than one dependent columns in the matrix therefore, there will be multiple Eigen eigenvectors. Because the number of dependent columnns will give us original dimension of the null space. So, geometric multiplicity and this is for real symmetric matrix, we can see exactly the number of times the value is the eigenvalue is repeated is the number of eigenvectors associated with, which equals geometric multiplicity for real symmetric matrix and null space vectors are linearly independent.

So, for one particular eigenvector when value when it is repeating all the eigenvectors associated with it, at least for real symmetric matrix they are linearly independent. Now what will try to explode is that eigenvectors for distinct eigenvalues and it is not repeating when there are distinct eigenvalues, there will be lambda 1 lambda 2 lambda 3 and none of them are repeating roots of the characteristic polynomials. The associated Eigen vectors for all the distinct eigenvalues they are linearly independent.

The proof is little convoluted, but let us see let us consider distinct eigenvalues of a matrix A lambda 1 lambda 2 up to lambda j. The representative eigenvalues are x 1 x 2 x j; we need to show that they are linearly independent. So, we have the equation A x i is equal to lambda x i, and let us assume that ; so this is x i equal to rather x is equal to lambda i x i, lambda i is one particular eigenvalue. And let us assume that there is at least one non zero c i for j plus consider j plus 1 eigenvectors and there is one non zero c i for which; c 1 x 1 plus c 2 x 2 up to c j x j plus c j plus 1 and x j plus is equal to 0.

So, there is one non zero c i; that means, c j plus 1 when we take x j plus 1 the j plus 1th eigenvector, that is linearly dependent with x j eigenvectors, this is an assumption we are making. And on top of that we are also making the posing the condition that j plus 1th eigenvector is dependent on first j Eigen vector and where the first j Eigen vectors are linearly independent. We assume that first few eigenvectors are linearly independent and there is one eigenvector which is linearly dependent on that. How can we start that we can take the first eigenvector which is a single eigenvector, and I assume another Eigen second Eigen vector Eigen vector 2 and assume that it is linearly dependent on that.

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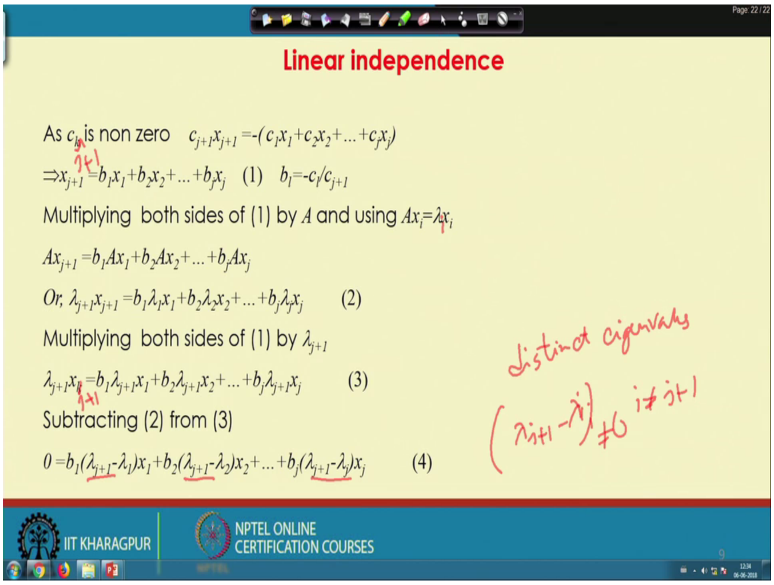

And we will see that whether these things satisfy the condition that one eigenvector can be dependent on others, but few are linearly independent. As c k is non zero, there is one c k which is non zero we can write c i plus j i c i j plus 1, so c k is rather c j plus 1; as c j plus 1 is non zero we should write c j plus 1 is non zero, c j plus 1 is j plus 1 is minus of this and x j plus 1 because c j plus 1 is non 0. So, x j plus 1 is combination of the first j i eigenvalues. Now multiplying both sides by A and we will use the fact that A x i is equal to lambda i x i, Lambda x i lambda I x i lambda i is the ith eigenvalue.

So, if I multiply left hand side by a it will be A x j plus 1 is equal to b 1 A x 1 b 2 A x 2 up to b j A x j. And then A x j plus 1 is lambda j plus 1 x j plus 1 similarly, for all other terms we will get the relationship 2. Now multiplying both sides of 1 by lambda 1 so we will get lambda j plus 1 x j plus 1 and we will get a equation lambda j plus 1 x this is not k this is lambda j plus j plus 1 x this is j plus 1, so lambda written as k.

Lambda j plus 1 x j plus 1 is equal to b lambda j plus 1 x 1 plus b 2 lambda j plus 1 x 2 etcetera. So, subtract 2 from 3 and we get 0 is equal to b 1 lambda j plus 1 minus lambda 1 plus b 2 lambda j plus 1 lambda 2 plus b j lambda j plus 1 lambda j. And lambda j plus 1 minus lambda i, i is not equal to j plus 1 is not is equal to 0.

Because the i because there distinct eigenvalues. So, in this equation these terms lambda j plus 1 minus lambda 1 lambda j plus 1 minus lambda 2 up to minus j plus 1 minus lambda j they are non-zero. And we also assume that b 1 b 2 b 3 at least all of them are non zero because c 1 c 2 to c j not all are 0 so some of them are non 0.

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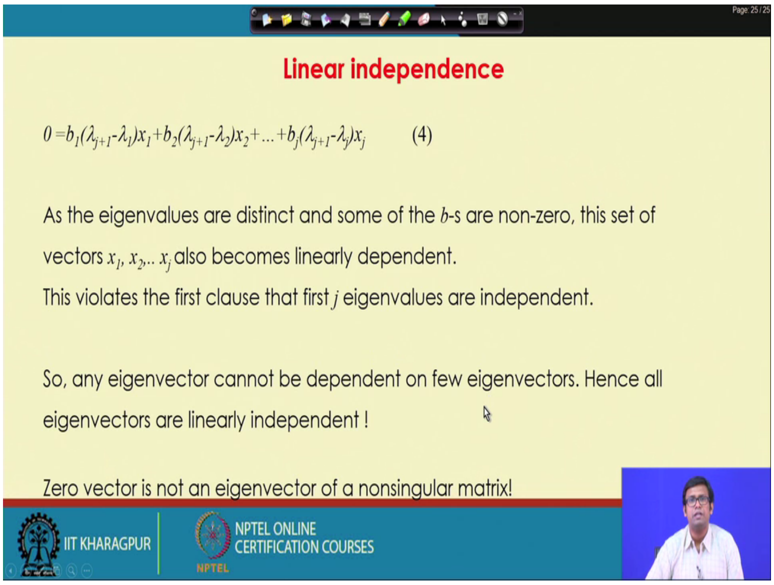

So, we get an equation where we can see that this is equal to 0, as a eigenvectors are distinct and some of bs are non zero the set of vectors become linearly dependent. Our initial condition was that first few eigenvectors we assume that they are linearly independent and one eigenvector was linearly dependent on that.

Now, we have seen if one eigenvector is linearly dependent on the previous eigenvectors the previous state of eigenvectors also become linearly dependent. And we can start with first 2 vectors and see that they are there should be linearly independent. And then we can go to the third vector and we will see that the third vector we have also has to be linearly independent with the previous 2 vectors, otherwise first 2 vectors are linearly dependent. So, the first class is violated that first j eigenvalues are independent, if I assume j plus 1th eigenvector is dependent on the not eigenvalues this is this should be eigenvectors.

If we assume the first j eigenvectors are linearly independent and then if you put then also assume the j plus 1th eigenvector is linearly dependent on first j eigenvectors, they are counter fitting. So, this is validating the first clause therefore, any eigenvector cannot be dependent on few eigenvectors. Hence, we can start from first eigenvector and go to second and then carry on and say that all eigenvectors are linearly independent and this is shown for when is eigenvectors, which has distinct eigenvalues for repeated eigenvalues the that the proof will be different, and that for real symmetric matrix we will see that all the eigenvectors will be linearly independent, but for distinct eigenvalues the eigenvectors are linearly independent.

So, 0 vector on very another important observation is that 0 vector is not an eigenvector of a singular matrix, because 0 vector is always linearly dependent. So, only if A x is equal to A is a singular matrix we can think of something like that, but otherwise 0 vector is not an eigenvalue because it is linearly eigenvector is linearly dependent on any vector set n independent eigenvectors.

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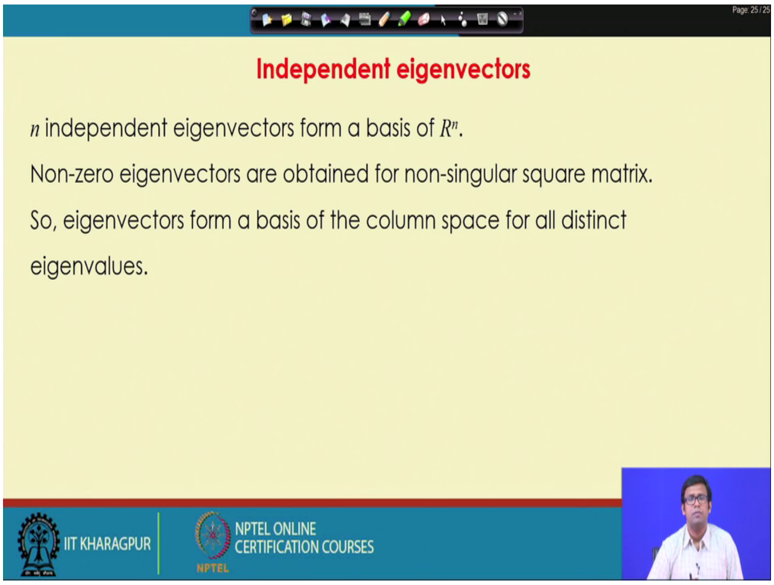

,Now if we have matrix with distinct eigenvalues there will be n into n matix square matrix. There are n distinct eigenvalues and there will be an eigenvectors which are linearly dependent on each other. Therefore, n independent eigenvectors in R n will form a basis of R n. So, the eigenvectors will form a basis on R n, non zero eigenvectors are obtained from singular matrix. So, they can form a basis. Eigenvectors form a basis of the column space for all distinct eigenvalues.

So, there are n independent eigenvectors these n independent eigenvectors are basis for column space also. And now we can see that as we got a basis for column space which are eigenvectors, we can utilize the eigenvectors to get a diagonal form of the matrix A or we will do some regulation with the eigenvectors are some transformation of the matrix in vectors and will get a diagonal from out of them.

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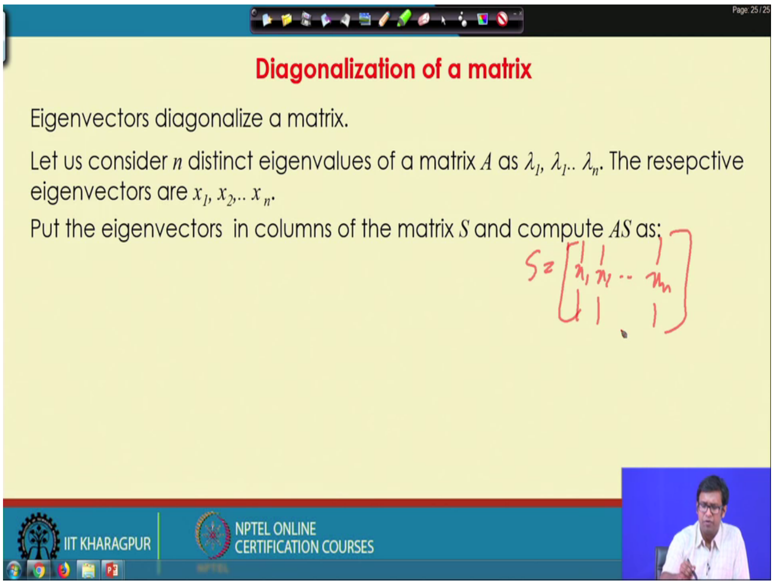

Eigenvectors can diagonalizable a matrix, let us consider n distinct eigenvalues of the matrix A, which is lambda 1 lambda 2 up to lambda n. The respective eigenvectors are x 1 x 2 x n. So we will put the eigenvectors as the columns of a matrix S, so it will be like S is equal to x 1 is x 2 and these are the eigenvector up to x n, these are the eigenvectors, this is the matrix S. Now if you compute A S, A S is equal to A x 1 x 2 x n.

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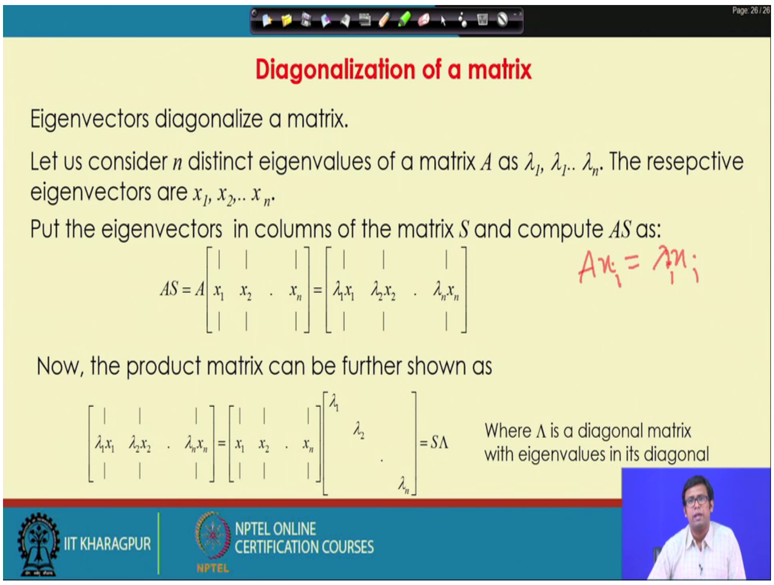

And A multiplied with each eigenvector will be A x because we have the formula A x is equal to lambda x, for each Eigen value lambda A x i is equal to lambda i x i.

So, A x 1 is lambda 1 x 1 A x 2 is lambda 2 x 2 A x n is lambda n x n. So, as will be same column matrix S with the eigenvectors, but each of the column multiplied with different eigenvalues. Now the product matrix can be further shown as lambda 1 lambda x is equal to x 1 x 2 again; we can take eigenvalues and show that eigenvectors is multiplication with eigenvectors into a matrix which is the diagonal matrix and we only have the eigenvalues in the diagonals. And what we can write as; S into lambda where capital lambda is a diagonal matrix with eigenvalues in it is diagonal.

So, what we get we got A S is equal to S into capital lambda, where lambda is a matrix with the diagonal eigenvalue as a diagonal matrix. So, the diagonals are eigenvalues only.

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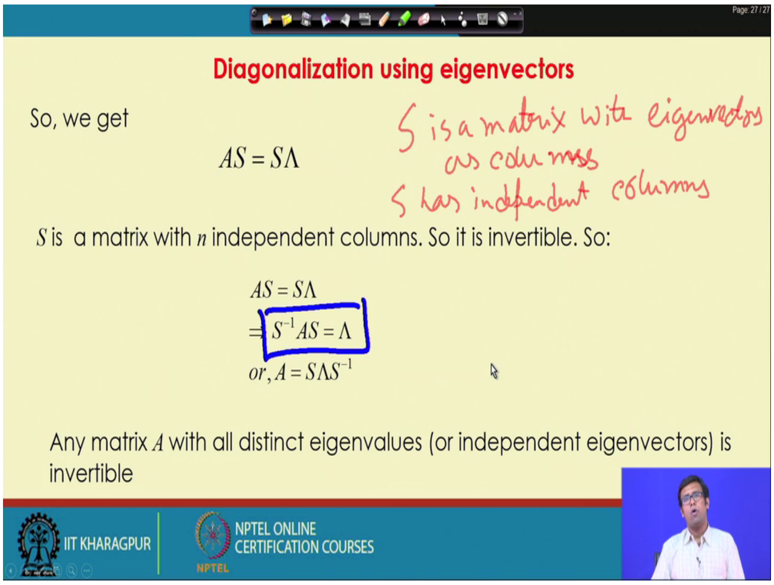

So, we get A S is equal to S lambda, and now S eigenvalues are independent. S is a matrix with eigenvectors as columns, we are considering eigenvalues Eigen eigenvectors of distinct eigenvalues. So, I assume that all the eigenvalues of a are distinct eigenvalues therefore, the eigenvectors are independent as we have just seen earlier. So, S has independent columns. So, we will see that as a matrix S has a is a n into n matrix with n independent column it should be invertible. So, I can multiply the left hand side and right hand side by S inverse and when write S inverse as is equal to lambda.

So, this is how we can diagonalize S, A we have to multiply A pre multiply by inverse of the matrix which has columns as the eigenvectors and post multiply by it by that matrix again and we will get lambda, which is matrix where that eigenvalues are only in the columns or we can write a is equal to S lambda S inverse. Now what is recognize any matrix a we will have all distinct Eigen values or independent Eigen vectors if it is invertible. Any invertible matrix can be expressed as a diagonal form; we can get a diagonal we can diagonalize the matrix from matrix operation.

Therefore, we can get a diagonal form of any matrix which is invertible, and the diagonal from the, what will be in the diagonals, diagonal values will be in the diagonals. So, if a matrix has can reach diagonal form that; that means, the matrix is invertible in that case; it will have distinct independent eigenvectors are distinct eigenvalues rather the vice versa. In case a matrix is independent eigenvectors so the matrix is distinct eigenvalues we can get a diagonal form of the matrix. And we the only get diagonal form of the matrix diagonal form means the there, there is a diagonal and all other terms are 0. So, only the pivot terms are existing, pivot terms all pivot exists for a matrix which is invertible.

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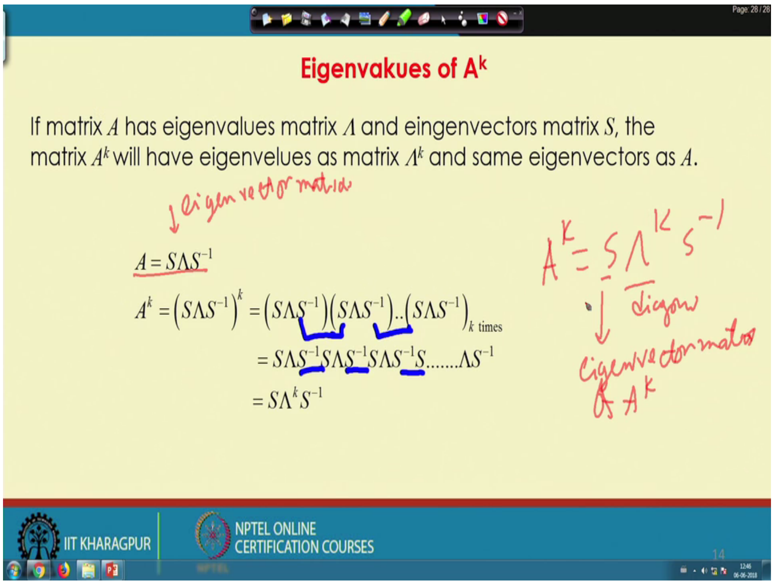

So, if the eigenvalues are distinct the matrix must be an invertible matrix. And now this is a actually an important concept, but and we will see the utility later. That if a matrix A has eigenvalues matrix lambda capital lambda and eigenvectors matrix S. And now we take the matrix A to the power k we multiply the matrix k times, and this will have eigenvalues as the matrix of as the eigenvalues of lambda to the power k. So, lambda 1 to the power k lambda 2 to the power k up to lambda n to the power k will be the eigenvalue matrix, and same eigenvalues as A. So, we will see a is equal to A S S inverse. So, we put a to the power k S lambda S inverse, which is S lambda S inverse into S lambda a inverse k times.

And all S S inverse into S, S inverse into S they cancels out. S inverse into S S inverse into S S inverse into S cancel, but the first S and last S inverse exists and lambda exist. So, it will S lambda to the power k S inverse. Now this can be a quick exercise the lambda is a diagonal matrix. So, what will be lambda to the power k ? Which is which can be very easily checked what will be the lambda to the power k. So, A to the power k can be very easily expressed as S a lambda to the power k S inverse. And this also says; because lambda to the power k and point to be noted is lambda to the power k is also a diagonal matrix, so you can check it quickly.

Because A to the power k, because we can write A to the power k is equal to S lambda to the power k S inverse which is a diagonal matrix, and it will have the eigenvalues of a to the power k. S and S is eigenvector matrix of A to the power k. Because a is equal to S lambda S inverse in this case S is eigenvector matrix, and lambda is the is a diagonal matrix. Similarly, we get a diagonal form using S k lambda k S inverse therefore, S is also the eigenvector matrix of a to the power k.

So, A to the power k and A we have same eigenvectors and eigenvalues of a to the power k will be the diagonals of the matrix lambda to the power k, which you can easily check what will be this is this is very, very simple, but please check it; that what are what is this matrix what are the diagonal components of this matrix.

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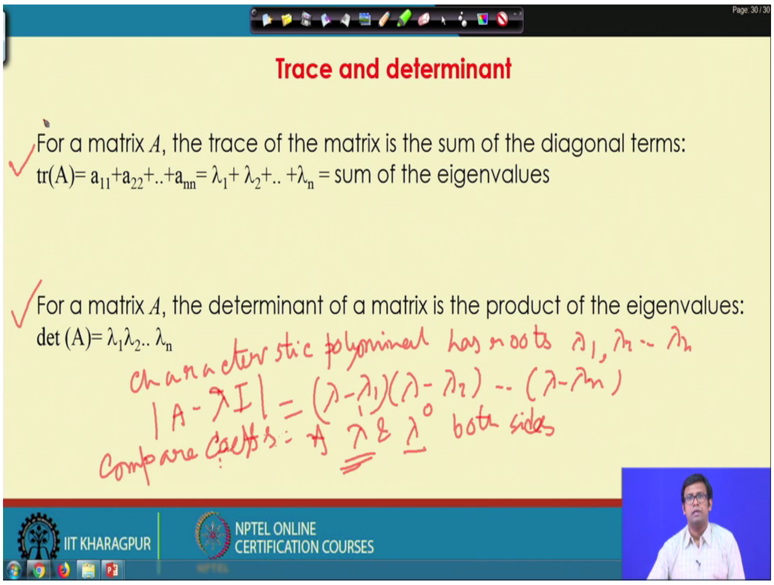

Now, we get couple of important definitions, what is a trace of a matrix? For a matrix say that trace is the sum of the diagonal terms. Trace of a is a 1 1 plus a 2 2 up to a n n, and this is equal to sum of the eigenvalues. Similarly, for a matrix A the determinant of a matrix is the product of the eigenvalues, determinant of A is lambda 1 lambda 2 up to lambda n. And this can be very easily shown by that that term that characteristic polynomial has roots lambda 1 lambda 2. So, determinant of a minus lambda I is equal to lambda minus lambda 1 into lambda minus lambda 2 lambda minus lambda n.

And will see that if we try to explain this lambda 1 lambda 2 lambda n and compare the powers in both sides, compare powers of lambda n lambda to the power 1 and lambda to the power 0 both sides. And then it will follow that comparing the power of lambda to the power 0 will follow the determinant loop, and comparing the power of lambda compare the coefficients rather compare the coefficients of lambda to the power 0 will give you lambda 1 to lambda lambda 1 lambda 2 up to lambda n this multiplication is determinant of A.

Similarly, comparing if you write the characteristic polynomial and I will show you the characteristic polynomial in the next class. And comparing the powers of lambda coefficient of lambda, which is lambda 1 plus lambda 2 up to lambda n will give you that if you expand this and try to write it down the trace of the matrix A, a plus a 2 2 up to a n n is equal to lambda 1 plus lambda 2 up to lambda n minus some of eigenvalues. So, if we know the trace and the determinant at least we can find out what is or if we know that eigenvalues, we can find out what is the trace of the matrix and what is the determinant of the matrix.

And there is also interesting because we will we at least see about similarity transformation etcetera. The linear transformations in which eigenvalues do not of a matrix do not change, in that case is the determinant trace of the matrix will also not change. So, if we rotate the matrix, rotate the co ordinate frame the matrix will change where the sum of their diagonal elements will not change, as well as the volume encompassed by all the vectors of the matrix that will also not change. So, determinant and trace are uniquely determined by the eigenvectors eigenvalues.

Therefore, with transformation they will do not change and we can with coordinate rotation they do not change transformation like coordinate, rotation. And we can say that determinant trace and trace are invariants of the matrix. So, it do not vary with the coordinate rotation.

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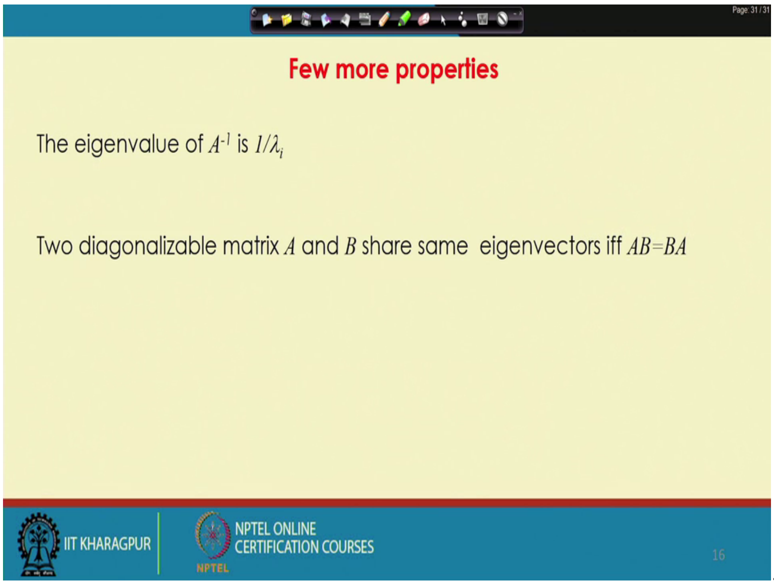

There is one more few more important property that eigenvalues of A inverse is 1 by lambda and this things can be very, very easily proved from just small exercise some of this these problems may come in exams. And you can check it yourself, these are very straight forward, but this will be a good exercise if you check these things yourself.

Eigenvalue of A inverse is 1 by lambda and if there at 2 diagonalizable matrix A and B, which share the same eigenvalue this will happen if A B is equal to B A. If A B is equal to B A then A and B will have same eigenvectors and same eigenvalues same eigenvectors. So, these are few important properties of eigenvalues and eigenvectors. And there are few more important properties for a class of matrices, and these classes called real symmetric matrix there are some important properties for another class of matrix, which is called positive definite matrix.

From the real value matrix which have symmetrics eigenvalues and eigenvectors will give some more important properties, which will discuss in the next session.

Thank you.