Matrix Solvers
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Lecture – 28
Introduction to Eigenvalues and Eigenvectors

Hello all. So, in last few classes we have covered a number of direct salvers for matrix equations and as promised we will move to iterative solvers. But, before moving to iterative solvers, we will look into few definitions and few properties associated with matrix algebra. Especially, on eigenvectors eigenvalues and positive dependness of matrix and how real symmetric matrices are special in that context to look into these things in next few lectures.

And then, we will start discussing on iterative solver because, to have appreciate the application of iterative solver and their mathematics by an iterative solvers, it is important to cover this topics. So, we will start our discussion with eigenvalues and eigenvectors of a matrix.

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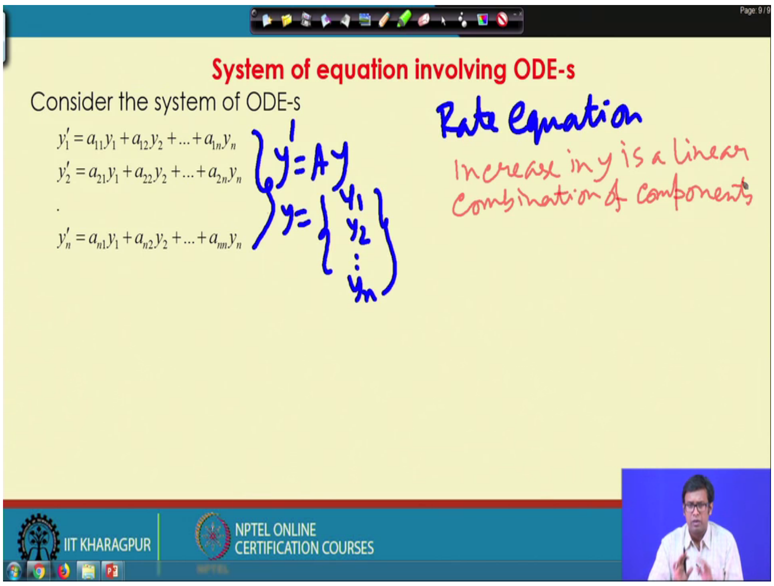

So, the idea of eigenvalues and eigenvectors comes first with ordinary differential equations which is represented as a as a matrix equation or system of ordinary differential equations, where each of the variables are given as their rate as a rate equations and rate of each of the variable is linear combinations of other variables. So, which is as the equation system given here y 1 prime. So, for example, it can be rate of change of y 1 with respect to time is a combination of y 1 y 2 to y n y 2 prime is similarly a combination of y 1 to y n and then goes a 2 by n prime.

So, all these ready weight is a linear combination of the variables y 1. So, y 2 2 y n and that is why, we can write this set of equation, we can write the set of equation as y prime equal to A y, where y is a vector now which consists y is equal to this vector y 1 y 2 up to y n.

. So, what we can see that, this is a rate equation that is rate of change of y vector is a linear combination of the values of the different variables inside y vector and we can see increase in y is the linear combination of components.

So, this example, particularly we are giving y as term as A y v prime as a rate of change of y with respect to time. And, when we will go to iterative methods, we will see rate of we will starts with the gaze value of y which will solve some equation A y is equal to 0 and this y prime is the amount by which we are going to correct the gaze value to achieve a right solution. So, it discuss it in detail in later classes.

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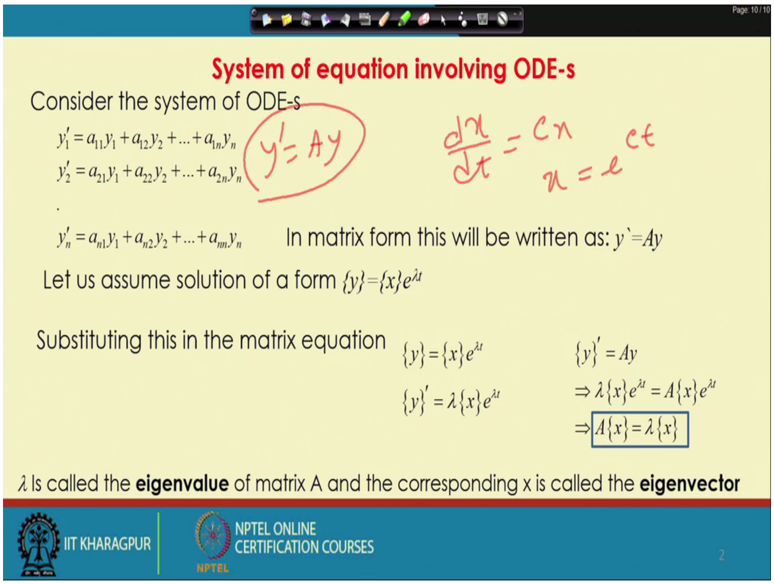

So now, we got an equation system which is basically as I wrote, can be written as y prime is equal to A y and in matrix form this will be written as y prime is equal to y A y. Now, the question will be this ignition system. So, we will start with a assume form of the solution and this will be similar as for example, if I have an equation of d dx dt is equal to c x, a single variable function relative ordinary differential equation or rate equation the solution becomes straight forward x is equal to e to the power c t.

So, similarly, we will assume a form of the solution here where the variable the vector y is 1 vector x which is coefficient, different coefficients in front of e to the power and e to the power lambda d. So, so it is a linear combination of e to the power lambda t. We can say like that program that we can say like that.

And now, if we substitute this to the matrix equation, what do we get? Y is equal to e to the x e to the power lambda t. Therefore, y prime is equal to lambda d dt of the right hand side part. So, lambda x e to the power lambda t and we have the original equation which is y prime is equal to A y. So, if we further substitute it, y prime is equal to A y and substitute y which is x e to the power lambda t and y prime which is lambda x e to the power lambda t.

We will get lambda x e to the power lambda t is a x e to the power lambda t e to the power lambda t has non 0, can be cancelled out from both side and we will get an equation A x is equal lambda x. And, this particular equation is an equation for the eigenvalues and eigenvectors of A and lambda is called eigenvalue of matrix A and the corresponding x is called an eigenvector. We will see eventually for an n by n matrix. There can be multiple values of lambda and multiple eigenvectors coming into place. We will quickly see that through an example.

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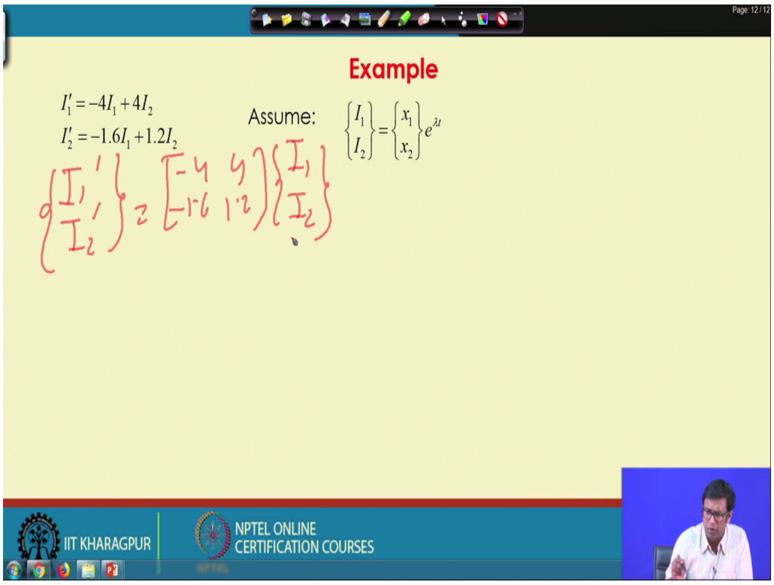

So, we jump to the example which is I 1 prime is minus 4 I 1 plus 4 I 2 I 2 prime is 1.6 I 1 plus 1.2 I 2. So, we assume I 1 and I 2. So, this again we can write this equation as we can assume a form of I 1 and I 2 which is x 1 e to the power lambda 2 x 2 e to the power lambda 2. So, this is again can be represented as a matrix equation I 1 I 2 prime is equal to minus 4 4 minus 1.6 1.2 I 1 I 2. Now, I will be seek solution of this equation. We will start with e to the power lambda t form and I 1 I 2 is x 1 x 2 e to the power lambda t.

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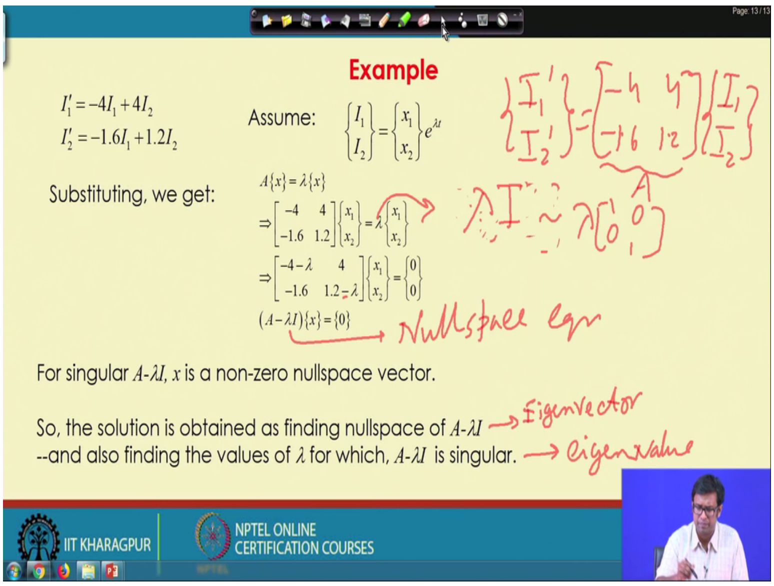

And now, if we substitute that, we will get that A x and A was as I wrote earlier. So, it is I 1 prime I 2 prime is equal to minus 4 4 minus 1.6 1.2 I 1 I 2. So, this is the matrix A here. So, we differentiate it and we will get the value equation form A x is equal to lambda x and which is minus 4 4 minus 1.6 1.2 x 1 x 2 is lambda x 1 x 2.

Now, we bring the left hand right inside matrix into the left hand side part and deduct the value lambda which is. So, here, lambda we can also write this lambda is lambda into I 0 0 I because, this will be anywhere multiply with x 1 x 2. So, is subtract lambda from the left hand side matrix and we get equation A minus 4 minus lambda 4 minus 1.6 1.2 minus lambda x 1 x 2 is equal to 0 which is e minus lambda I, sorry erased made a mistake here.

So, this instead of writing this, I think this can be written as to be correctly at this should be written as lambda into I or this particular value or lambda into 1 0 0 1. So, we subtract lambda into 1 0 0 1 from the left hand side and we get minus 4 minus lambda 4 minus 1.6 1.2 minus lambda x 1 x 2 is equal to 0 0.

So, we can quickly check that they if we get a matrix A minus lambda I x is equal to 0, now, what is this matrix? This is nothing but a null space equation. If A minus lambda e is a nonsingular matrix, A x will have a trivial solution x is equal to 0. If A minus lambda I is A, then will get nontrivial null space vectors for A minus lambda I and x will be non 0.

And in case x is equal to 0, the equation the first equation we started we it would we assume the form I 1 I 2 is equal to x 1 x 2 e to the power lambda t. This I 1 I 2 both are 0 and the equation is 0 is equal to 0. So, it becomes a trivial equation which we which we started.

So, will seek form of which will make A minus lambda I singular so that we can have non zero. Now, this is interesting in a sense, now we are trying to find out a singular matrix and we will find it is null space which will be the solution in this particular case. So, we write for singular A minus lambda I x is non 0 x is non 0 null space vector.

So, the solution is obtained as finding null space of A minus lambda I and the null space is non 0 only when A minus lambda is singular. So, it is also finding the values of lambda for which A minus lambda is singular. So, have to find a value of lambda for which A minus lambda is singular and this value will be called as the and for this lambda, the null space of A minus lambda I will be eigenvector.

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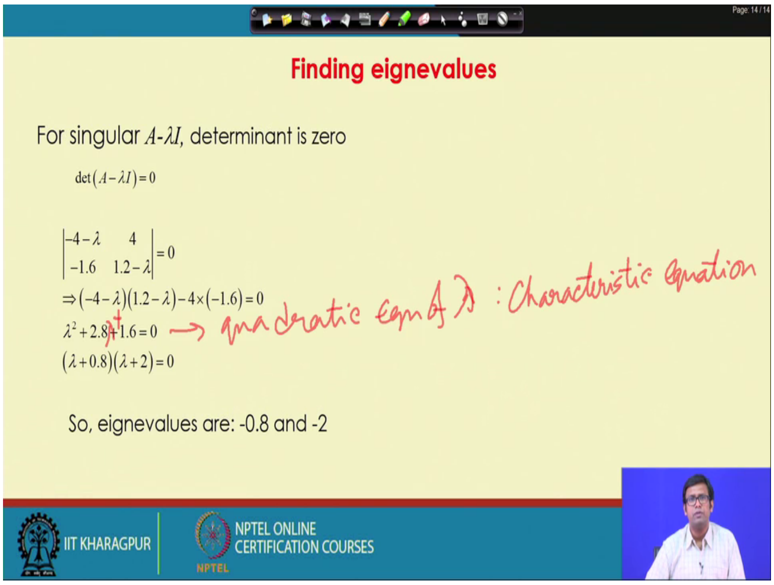

So, will look into the next few steps that for singulars A minus singular A minus lambda, it determinant is 0 and that will give us a equation from which we can find the values of lambda for which a minus lambda is equal to 0. So, determinant of A minus lambda is equal to 0 which gives us minus 4 minus lambda 4 4 minus 1.6 1.2 minus lambda is equal to 0.

So, minus 4 minus lambda into 1.2 minus lambda plus 4 into minus 1.6 is equal to 0 which is lambda square plus sorry this is 2.8 lambda square plus 2.8 lambda plus 1.6 is equal to 0 and this is a which we will solve, we will get lambda is equal to lambda plus 0.8 and lambda plus 0.2 is equal to 0. This is a quadratic equation of lambda.

Or for higher dimension of A, for 3 into 3, for n into n A, this is a higher order polynomial of lambda and this is called characteristic equation, which is polynomial of lambda is equal to 0. And we have to find out the roots of the polynomial. So, we get lambda plus 0.8 into lambda plus 2 is equal to 0 which gives and each of this lambda is the eigenvalue and we get the eigenvalues which is minus 0.8 and minus 3.

So now, the next steps will be to find out the eigenvectors. So, what will do that, we will take this matrix, go back to the again, go back to the previous slide we will. So, we got minus 0.8 and minus 2 is equal to 0. We will take this value minus 0.8, substitute it and substitute it in this equation for example, minus 0.8 here.

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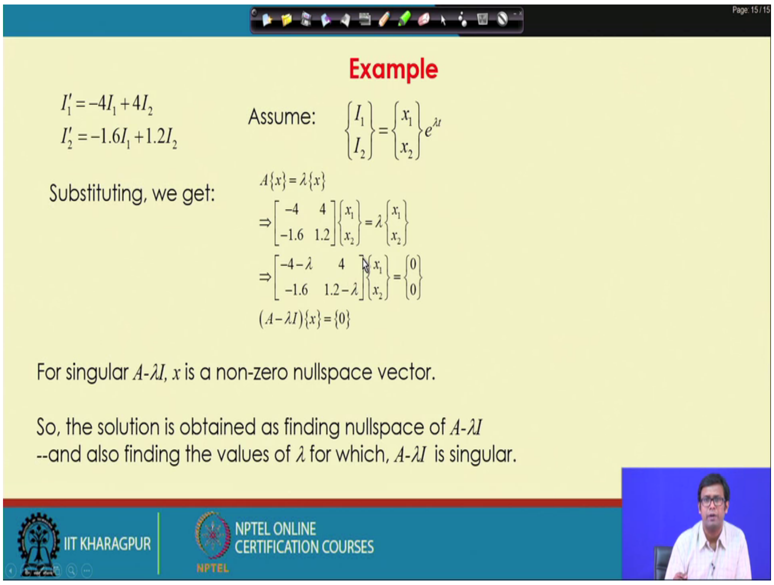

So, minus 3.2. So, x 1 x 2 is equal to 0 which is the null space equation because, this matrix is singular when we have the value lambda is equal to 0.8 minus 0.8 or lambda is equal to minus 2. So, ways of substitute lambda is equal to minus 0.8. This is the singular matrix. So, x is equal to 0. We will have one non zero x value for one particular value of lambda and that will be the eigenvector. So, we will look into it.

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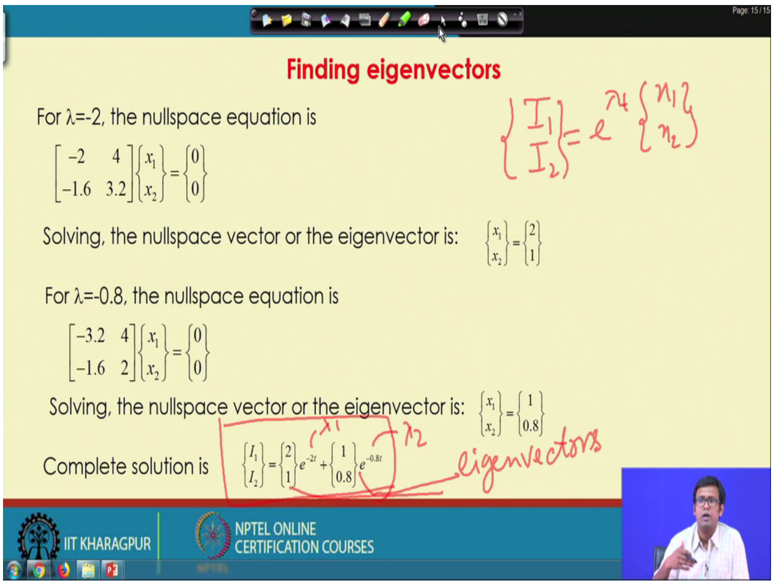

So, we go we go ahead with eigenvectors minus 0.8 and minus 2 and the null space equation for lambda is equal to minus 2 is 2 4 minus 1.6 3.2 x 1 x 2 is equal to 0 and we can quickly see that. Now, the columns are dependent on each other. The second column is nothing but first column into minus 2. So, it will have a rank which is not a full rank, matrix rank will be 2 minus 1 and 1. So, you should have get 1 null space vector and solving the null space vector is 2 1.

. Similarly, for lambda is equal to minus 0.8. The second eigenvalue we get the equation minus 3.24 minus 1.6 2 x 1 x 2 is equal to 0 and we can also see that this is in also that columns are dependent on each other. So, it will have a rank 1. So, we will have n minus R which is 1 null space vector here. So, the eigenvector comes as x 1 x 2 is equal to 1.8. Therefore, the complete solution, we assume the form of the solution which is I 1 I 2 is equal to e to the power lambda t x 1 x 2 when x 1 x 2 is a eigenvector.

So, we will get 1 solution which is e to the power 2 t into 2 1. Another solution will be e to the power 0.8 t into 1.8 and both are the solution linear equation. So, the final solution will be the linear combination and that is why, we can see the complete solution is I 1 I 2 is 2 1 e to the power minus 2 t and this is 1 eigenvalue and e 1.8 e to the power minus 0.8 t. This is another eigenvalue and these are the eigenvectors so far, so, good.

But, we can do is that we can further 80 equation, you can find the eigenvalues and eigenvectors and the final solution will be a combination the eigenvalues and eigenvectors and it can be any linear combination of that here we are putting to 1 1.8. But, we can use some other. We can multiply them by some constants. So, it can be c 1 into 2 1 e to the power minus 2 t c plus c 2 into 1.8 e to the power minus 0.8. It is the linear combination of the eigenvectors multiplied each 1 multiplied with the eigenvalue.

So now, the process though it is solution of a rate equation which involves multiple variables. This process also gives us something regarding a matrix that is the eigenvalue of a matrix which is something which subtracted from the diagonal elements of the matrix gives us a singular matrix. And the solution x which is which is the null space and eigenvector x, which is the null space of that singular matrix which is the eigenvectors.

So, we get eigenvalue and eigenvector and they have several other application. Apart from solving rate equation, rate equation is one very important thing they have other applications as. So, there are several other properties of matrix which are based on the eigenvalues. So, and eigenvectors, will discuss them later.

However, rate equation is one of our primary drive here because, we are we will soon start working on iterative methods which is we will discuss which is singular as the rate equation change of value of one variable is dependent on the other variables. So, this property of eigenvectors and eigenvalues are this particular usage of eigenvalues and eigenvectors in the back is important as the backbone of the iterative methods. At least, few of the direct iterative methods which will I say, it will come to that later.

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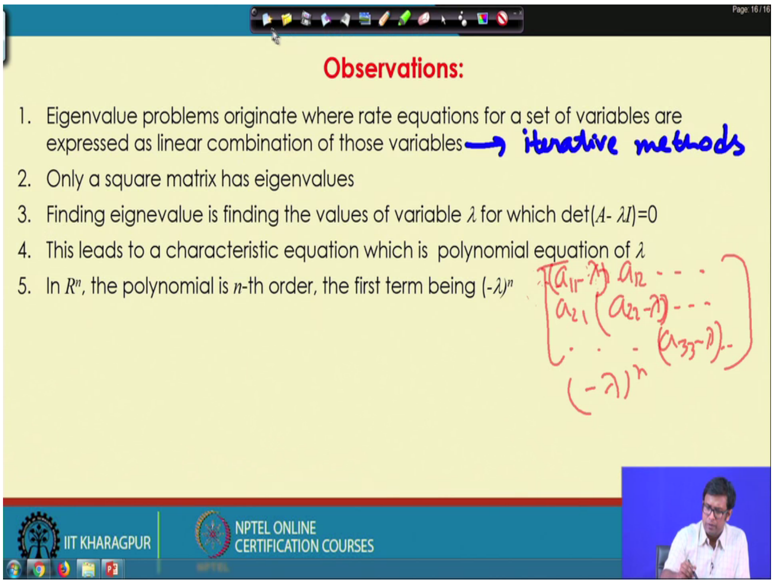

So now, we can make a very important observations. In order to obtain the eigenvalues and eigenvectors or through the process that we follow obtain the eigenvalues and eigenvectors, eigenvalue problems originate where rate equations for a set of variables are expressed as linear combinations of those values and that is why, I as I said to make this point earlier. So, they are related with iterative methods because, the set of the change of one variable will be linear combination of other variables at least for the direct iterative methods.

So, the next observation is, only the square matrix has eigenvalues. This is because, if I have n variables whose rate is dependent on those n variables only. So, we get n questions and n unknowns and that that is why, this will give me a square matrix system. So, only a square matrix will have eigenvalues finding eigenvalue is finding values of variable lambda for which determinant of A minus lambda is equal to 0. Because, the lambda we have to solve the equation A minus lambda I x is equal to 0 where x is equal to non 0 variable.

Therefore, A minus lambda I must be a singular matrix which needs determinant of A minus lambda is equal to 0. This will lead to a characteristic equation which is a polynomial equation of lambda if and we have that like, if I express determinant of A minus lambda I, I will get a polynomial of lambda.

And, if A is an n into n matrix, the polynomial will have degree of n. So, order of n will be equal or length of n is equal to the degree of the polynomial n degree of the polynomial n in R n the polynomial. If a columns of a belong to R n, the polynomial is a nth order and first term of the polynomial will be minus lambda equal to the power n and why is it?

So, because, whatever we get into this situation, this equation has lambda 1 lambda 2 lambda n at the roots of this equation. This equation can actually be written as a 1 1 1 minus lambda is here then a 1 2 etcetera then a 2 1 1 into a 2 2 minus lambda and then the remaining terms then a 3 3 minus lambda and the remaining terms. So, when we will try to find out the determinants, we will get term minus lambda whole to the power n which is the first term of the polynomial equation.

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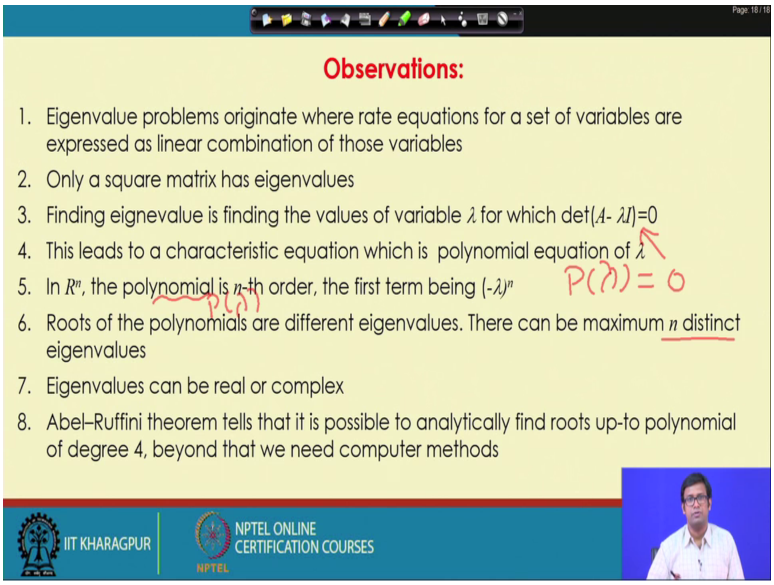

Roots of the polynomial are different eigenvalues. So, because we are solving, if we get a polynomial P lambda, how we are solving? If this polynomial is P lambda, we will solve P lambda is equal to 0 which is nothing but A minus lambda is equal to 0. A minus lambda is a polynomial of lambda which is equal to 0. So, we have to find roots of the polynomials and they can be they will give different eigenvalues and there can be maximum n distinct roots. Because, there is a nth order n polynomial.

So, it is also to be mention that, there can be n distinct maximum n distinct eigenvalues can be less than n eigenvalues. Also, then some of the eigenvalues are repeating. For example, I have an equation where 2 and 2 both are eigenvalues. So, there is only 1 eigenvalue, both eigenvalues are 2, there is 1 eigenvalue which is 2 which is repeating 2 times lambda minus 2 into lambda minus 2 is equal. It is the polynomial is lambda square minus 4 is equal to 0. For example, I got something like that.

So, there can be less than n eigenvalues, then some eigenvalues have been are repeating are the repeating roots of the polynomial eigenvalues can be real or complex. So, because it is a polynomial equation and we will look into fundamental theorem of algebra later which says that, there can be complex roots of polynomials and we know that from on in event from quadratic equations, we can see that there can be complex roots of a quadratic equation which has real values the coefficients are real, but the roots of complex which is possible.

Abel Ruffini theorem tells that, it is possible to analytically find roots of 2 polynomial of degree 4. Beyond that, we need computer methods and this is importance. So, eigenvalues needs a computer methods in most of the cases are through (Refer Time: 24:53) or through approximation we need to find out the roots of the polynomial equation.

So, and if you look into mat lab or Blass or in pack or all the linear algebra, library is available, course available for or functions available for finding out eigenvalues of a large matrix and they are little complicated computer programs. Because, they are trying to solve the equation the polynomial equation in an iterative method through (Refer Time: 25:19) and the analytically, it is not possible to find out this solutions if the degree of polynomial is more than 4.

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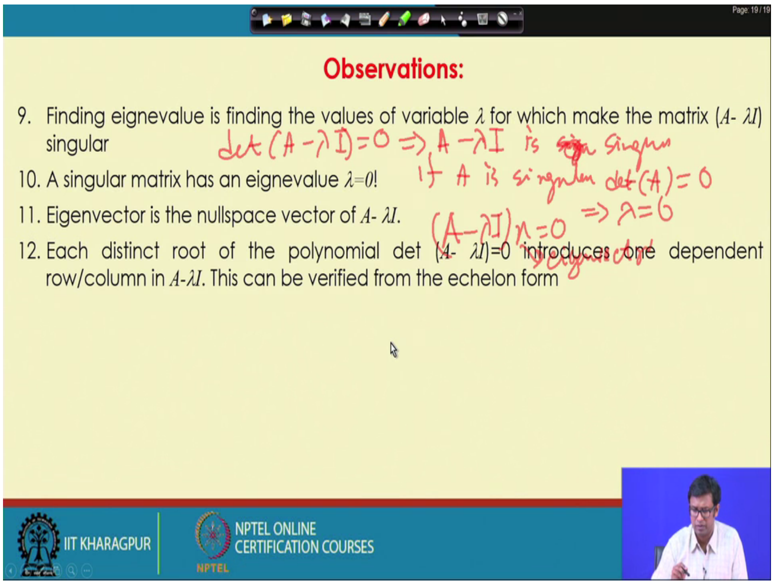

Finding eigenvalue is finding the values of variable lambda which will make a matrix A minus lambda is equal to singular true. Because, if A minus lambda is not singular, then eigenvectors will be 0 trivial and the this will be a trivial solution and the rate equation will also have trivial solution 0 is equal to 0 in both sides. So, it is important to find lambda which will make the matrix norms, matrix singular a singular matrix has an eigenvalue lambda is equal to 0. So, this is, I think this will more discussion.

For example, we wrote determinant of A minus lambda I is equal to 0 which gives us A minus lambda I is singular. Now, if A is singular determinant of is equal to 0 that gives us lambda is equal to 0. So, determinant of A minus lambda A which is determinant of A is equal to 0. At least, there is one lambda which is 0 which will give determinant of A to be 0.

So, a singular matrix will always have an eigenvalue which is 0. Eigenvector is the null space vector of A minus lambda and we have seen it in the last example. Because, A minus because we are solving, I should wire because, we are solving A minus lambda I x is equal to 0 and this x is eigenvector. So, it is null space vector of A minus lambda I.

Each distinct root of polynomial which is determinant A minus lambda is equal to 0 introduces one dependent row or column in A minus lambda I. So, what happens? For example, again, we will write the equation in the AKLN forms.

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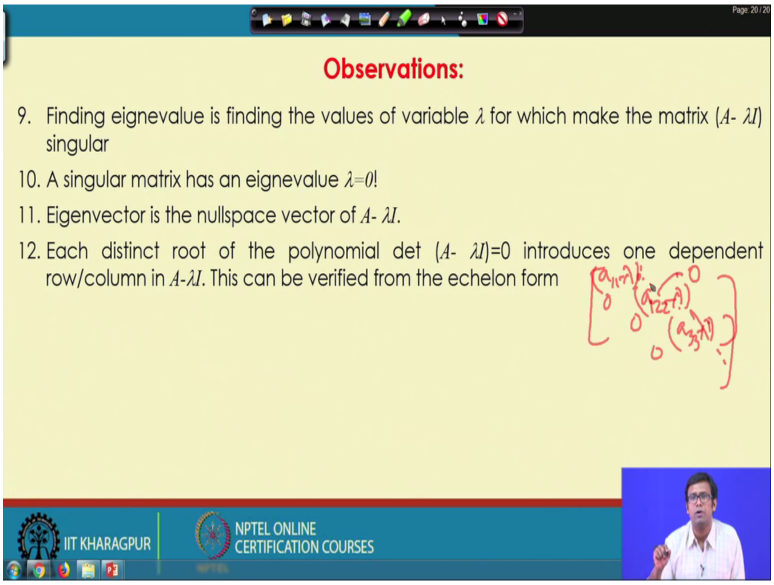

So, the matrix in the AKLN form will have a 1 1 1 a, then a non 0 value then a 1 a 2 2, then 2 non 0 values, then a 3 3 and remaining all this terms 0 etcetera. So, the characteristics equation will be minus lambda minus lambda and it will goes 1. So, if 1 lambda is 1 root of this equation, then a 2 2 this minus lambda term, this will be 0 because, the determinant has to be 0.

So, if this is 0, this is 0 pivoted column and that will be a dependent lower column. So, one distinct root will include will enforce one dependent column in A minus lambda your one dependent row in A minus lambda by making the pivot A minus lambda is 0.

So, each distinct root of polynomial A minus lambda, you will introduce one dependent row or column in the matrix.

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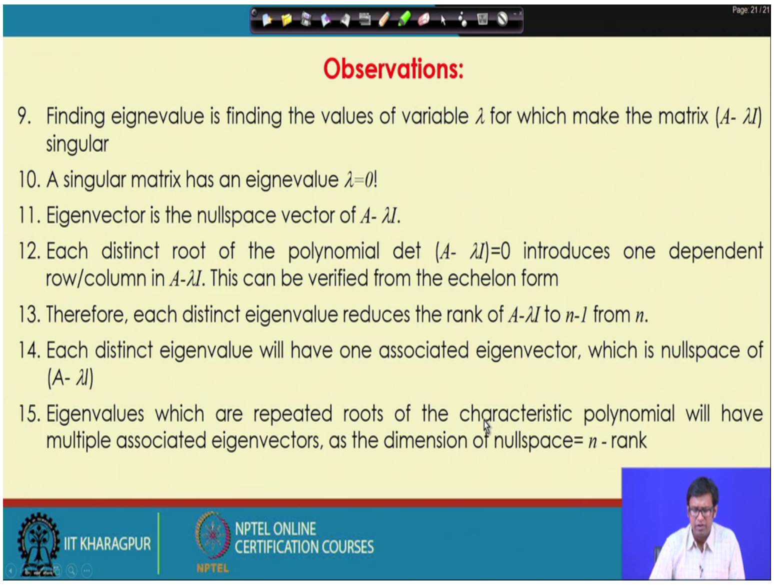

Therefore, each distinct eigenvalue will reduce the rank of A minus lambda I from A into n minus 1. So, rank will be reduce by 1. So, what will be the dimension of the null space here? Rank of the null space here 1 because, there will be only 1 null space vector n minus n minus 1. The 1 null space vector.

Therefore, each distinct eigenvalue will be associated with one Eigen eigenvector which is null space of A minus lambda I eigenvalues which are repeated roots of the characteristic polynomial will have multiple associated eigenvectors as a dimension. Because, they will introduce more pivots, more 0 pivots.

So, the dimension of the rank of the matrix will reduce by more than 1 and the dimension of the null space will be n minus rank which is more than 1. So, there will be multiple associated eigenvectors. And this idea is called geometric multiplicity of eigenvectors that with repeated roots of the polynomial, we get multiple eigenvectors associated with these roots.

And, we will see for a real matrices, the number of Eigen the number of eigenvectors associated with real rather with real symmetric matrices the number of eigenvectors associated with one repeated root is the number by which the root is repeating. For example, lambda 3 is repeated in 3 times or 3 roots of the equation of a polynomial P. P lambda is equal to 0.

So, lambda is equal to 3. We will have 3 associated eigenvectors that is geometric multiplicity of eigenvectors for repeated roots. That is also observable here. So, we will start try to introduce Eigen values and eigenvectors and we will look into more detail of the properties of eigenvalues and eigenvectors in the next few classes.

Thank you.