Constrained and Unconstrained Optimization
Prof. Debjani Chakraborty
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 49
Constrained Optimization

Now, I will.

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In continuation to my previous class, I will talk today on more on KKT condition.

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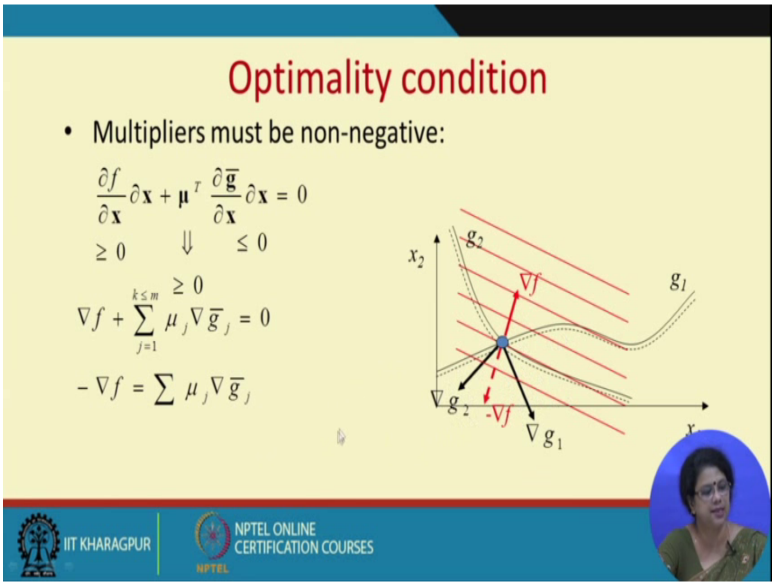

If we consider that, f is a objective function and g 1 is the set of active constraints. Then we know that from the first order condition, we can write that del f by del x plus mu as the KKT multipliers into del g by del x. And if we just consider a small perturbation over x, then we can multiply both side with del x and x is del x in the feasible direction only. Then this is always will be positive and this is will be negative. And certainly to maintain this one the whole equation this the Lagrange multipliers, but mathematically, but we say it as a KKT multipliers this must be positive. And that is why we can say that if we have k number of active constraints then grad f plus summation of mu j grad gj this will be is equal to 0.

Alright and in other way we can say this. Now if we just show it graphically, then these are all the grade grad of g 1, grad of g 2, grad of delta f, then from here this is minus grad f. Then we can say this is if you just consider this direction, this is the direction of the descent direction of the objective function and we can say that this descent direction lies in the cone spanned by positive constraints in gradients. That is why if we consider the half space over grad g 1 here, grad g 1 greater than equal to 0 here, grad g 2 greater than equal to 0 here and if we consider the cone that is spanned by with the vector grad g 1 and grad g 2, then this cone will be say it can be interpreted as the descent direction of the objective function.

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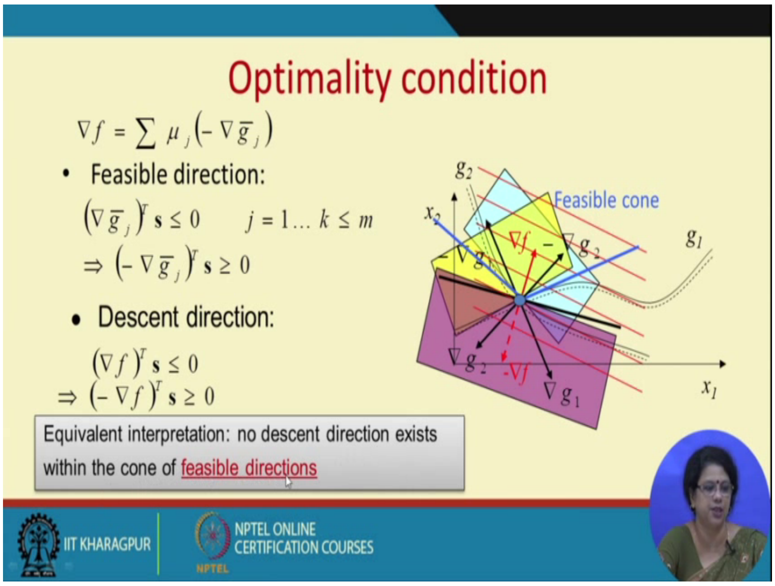

Now similarly to define the feasible direction we know in the feasible direction we have if the direction is the s is the feasible direction, then grad g into s must be less than 0.

That is why minus grad g bar must dot s must be greater than equal to 0. We are only concerned about the active constraints; we are not really concerned about the inactive constraints. That is why we have clubbed all key active constraints here that is why instead of gi we are talking it as gi bar, that is why minus gi bar dot is greater than 0 for every constraint we can make one half space in the negative direction of grad g 1. This is the half space, over minus grad g 1. Similarly we can have another half space over minus grad g 2. And if we consider thus cone that is spanned by minus grad g 1 and minus grad g 2 and in this direction we can say that this is the feasible cone. That is why that was the descent cone, now it is the feasible cone. Thus, we and this is the, this is my descent direction. Already I have explained to you.

That is why we can interpret that no descent direction exists within the cone of the feasible direction, for the minimization problem alright. We are dealing with the minimization problem.

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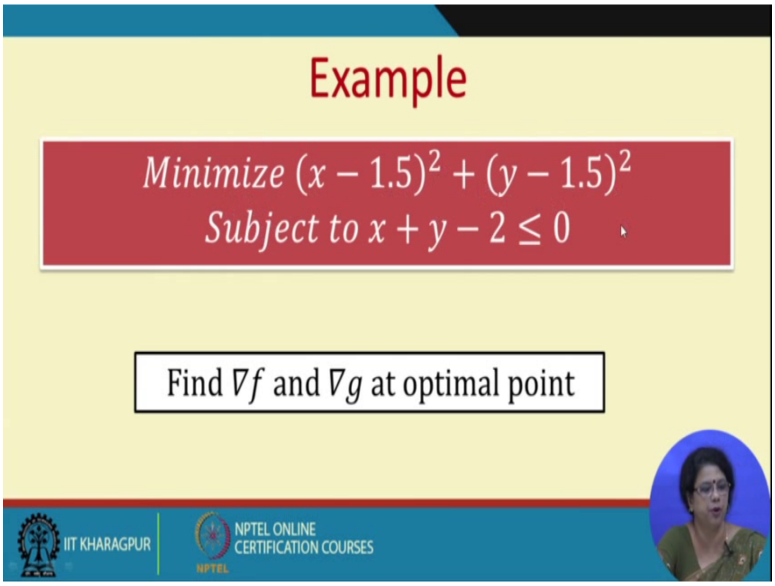

Now with this idea, let us go through the series of examples one by one, I will solve it solve few for you and few I will give you for your assignment. Now there is one non-linear programming problem has been given minimization of x minus 0.5 square, y minus 1.5 square subject to x plus y minus 2 less than is equal to 0. We have to find out the grad f direction and grad g direction at the optimal point.

Now, let us start working on it. Now if you formulate the Lagrange multiplier for this problem then L can be is equal to x minus 1.5 square plus y minus 1.5 square plus lambda into we have the constraint x minus y plus y minus 2 less than is equal to 0.

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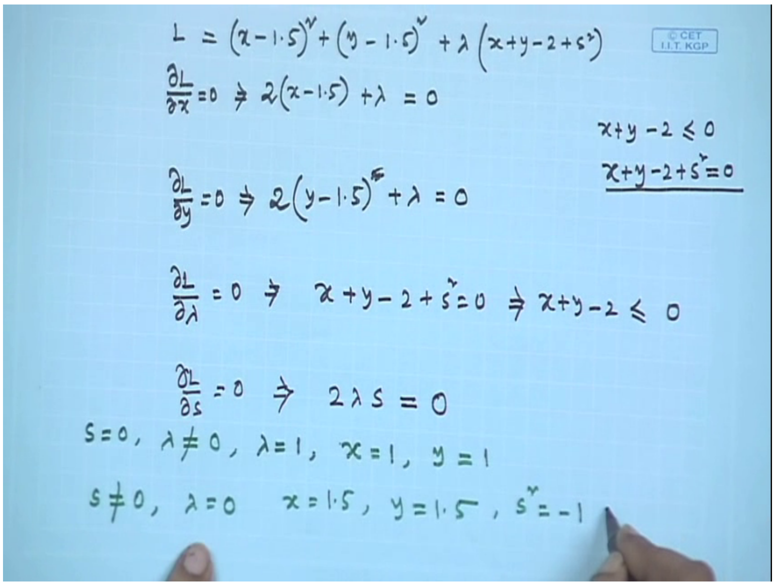

That is, why we can introduce the positive slag variable, since we are enforcing positive, that is why the variable s can be written as s square here. Thus this equality constraint if we just include here then this is my Lagrange function minus 2 plus s square this one. Now from here if you consider the optimality condition that is del L by del x must be equal to 0, then we will get 2 of x minus 1.5 plus lambda must be is equal to 0. The other optimality condition del L by del y equal to 0.

We can get from here 2 into y minus 0.5 whole squares plus lambda into y is equal to 0, alright. Now if you consider the with respect to the Lagrange multiplier del by del lambda equal to 0, then we will get x plus y minus 2 plus s square equal to 0. In other way we can say that x plus y minus 2 must be less than is equal to 0. What else we will get? We will get del L by del s is equal to 0 from here we can get 2 lambda s is equal to 0. These are all the conditions given to us alright. Now if we just try to consider the values of this if we consider s is equal to 0, then lambda is not is equal to 0, then the value of x would be from here if we just adjust if we consider lambda in terms of x and lambda in terms of y if we substitute this one here we will get that this is equal to and s is equal to 0, if we just put it here; we will get one equation that is with lambda only and from here we will get lambda is equal to 1. And if lambda is equal to 1, we will get x is equal to 1 and we will get y is equal to 1 alright and that is why my solution would be x equal to 1, y equal to 1, lambda equal to 1, s is equal to 0.

That is why what we can see? That, let me consider the other thing, that s not is equal to 0, but lambda equal to 0. If we consider lambda equal to 0 here, on the first then what we will get we will get x is equal to 1.5 and y is equal to 1.5 alright, but you see what else we will get. We will get the value of s square is equal to from here if we just substitute 3 minus 2 s square is equal to minus 1. Now from here we can say that this is not a feasible point, because s square must be positive as we know then the only the optimal solution is this one, x equal to 1 and y equal to 1. What is our question? Our question is that find out grad f and grad g and grad g at optimal point that is 1, 1 this is our question.

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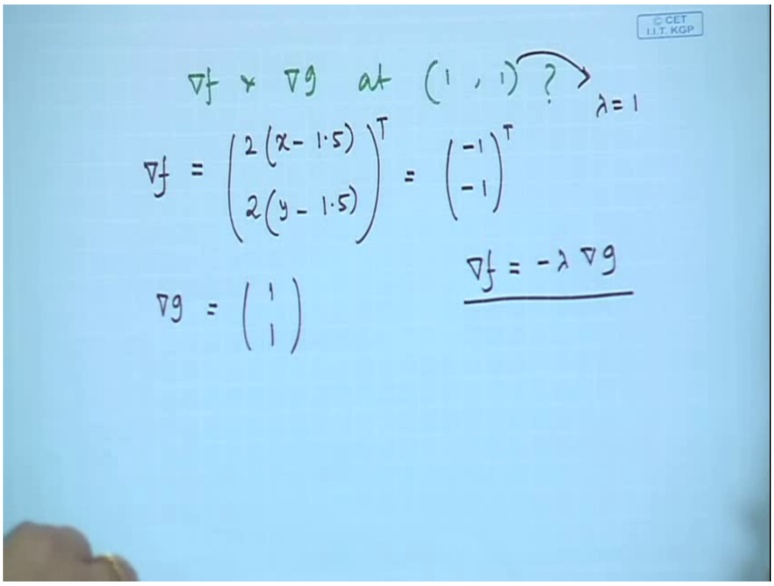

Let us try to calculate the value for this. What is my grad f? grad f is equal to 2 into x point x minus 1.5 plus oh this one only grad f. And grad what we get if we this is substitute 1 here this is 0.5. 0.5 into 1 that is why we can get that grad f is equal to minus 1 minus 1. And if we consider grad g my g was x plus y minus 2, that is why grad g is equal to always 1, 1. Now this is not dependent on the value of this, but we got at this point lambda is equal to 1.

That is why can we not check that grad f is equal to minus lambda grad g, that is coming very easily. Now, this is the question, answer of this question. Now coming to the next example where we are considering two inequalities together.

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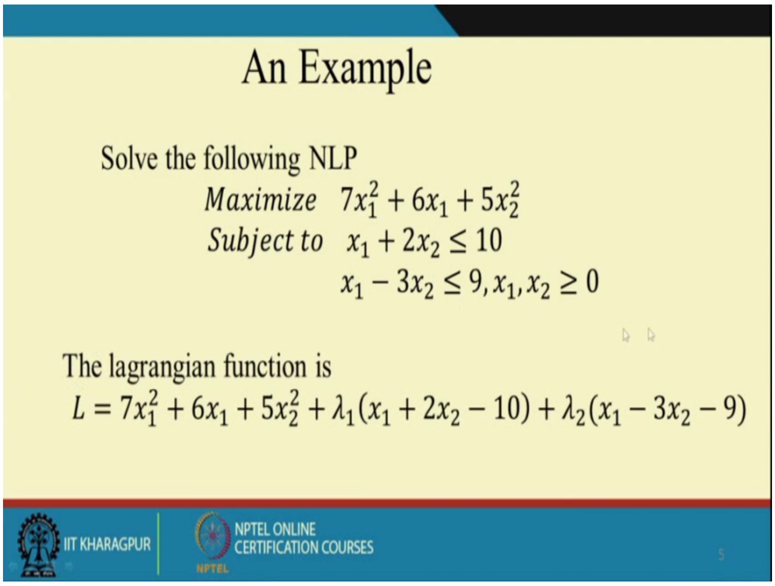

Now maximization of 7 x 1 square plus 6 x 1 plus 5 x 2 square subject to x 1 plus 2 x 2 less than is equal to 10 and x 1 minus 3 x 2 less than equal to 9. Now with this one we have to find out the optimal solution, using the KKT conditions. What is the first task? First task would be to formulate the Lagrange function. Now in the Lagrange function you must have been seen that we did not include the non-negativity constraints here, but you we may include it if this is. So, then objective function plus lambda 1 into first constraint, now here the this one is missing it must be s 1 square and here must be s 2 square will be there, because this is the less than equal to type that is why we have the slack variables here.

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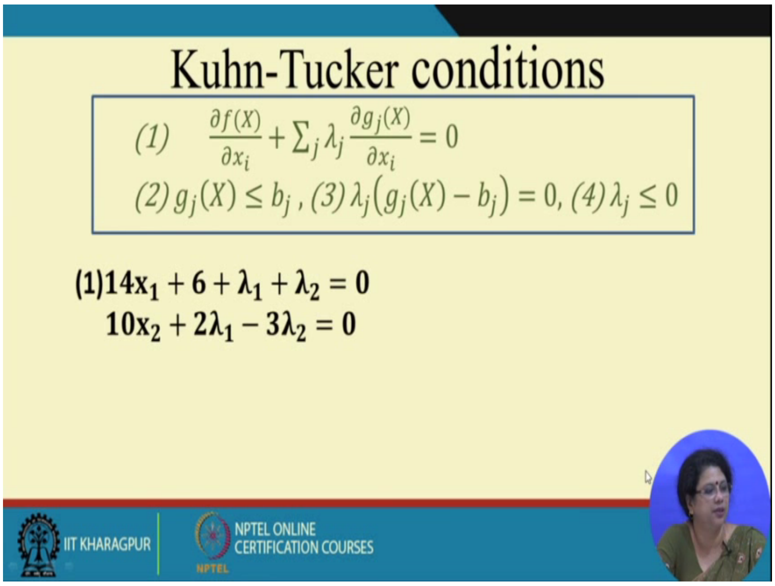

Now, is it that maximization problem you just see, that is why let us see what would be the value for lambda 1 and lambda 2.

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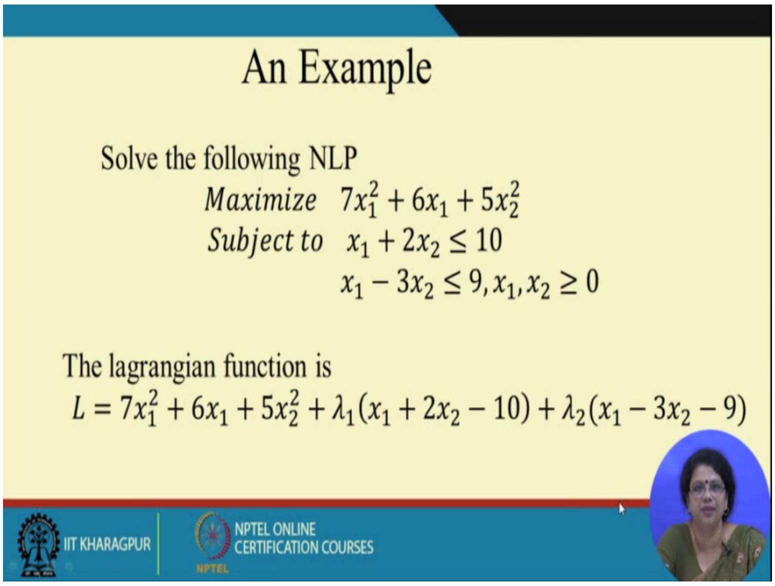

What you have learnt that if for the maximization problem the constraints are of less than equal to type then always the lambdas must be negative, but in the maximization if the constraints are of greater than equal to type, then lambda must be positive that is that is why in this case lambda must be negative.

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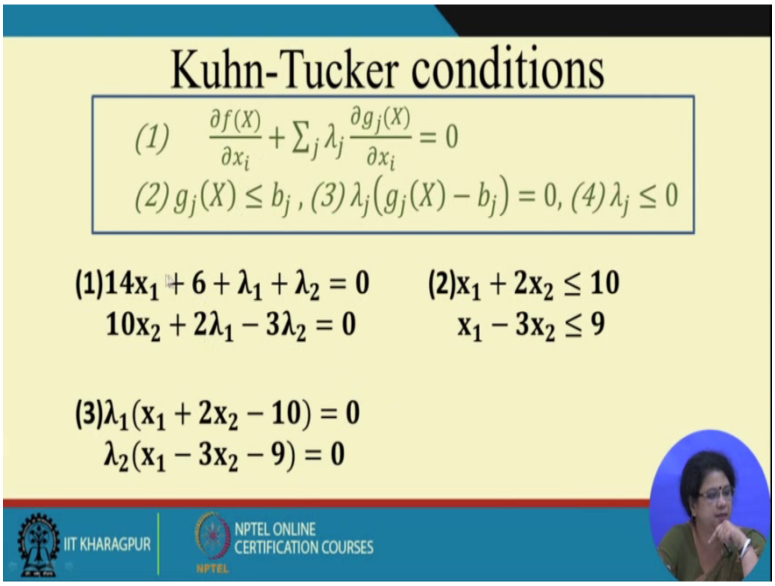

Then these are the KKT condition, because this is the optimal condition, this is the feasibility condition, this is the complementary slackness property and this is the lambda condition.

If we just substitute all these things together, now from here we will get from the first set we will get this equation and how many equations and how many unknowns we are having so, many equations together and we are having 4 unknowns only from here. Now since there are 4 equations and 4 unknown very easily we can find out the values for lambda 1, lambda 2, x 1, x 2, but you see that we have to consider the values in different conditions by considering lambda 1 equal to 0, lambda 1 not equal to 0, lambda 2 equal to 0, lambda 2 not equal to 0, that is why so many options will come from here to get the set of values for the 4 unknown variables.

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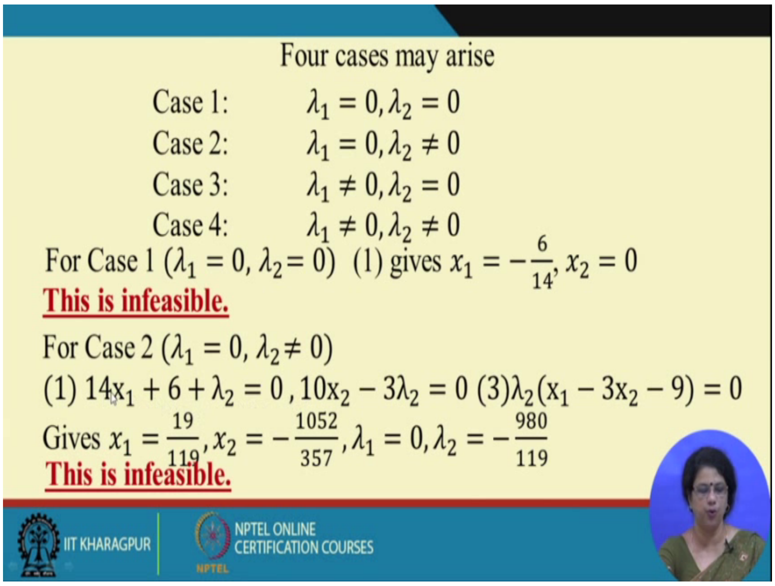

That is why let us consider 4 cases one by one. In the first case we will consider 4 both the lambdas 0, in the second case second and third one of the lambdas not 0 and in the fourth case both the lambdas are not 0. Let us see the values of the set of values of x 1 and x 2 all in all this four cases. Now if you consider the first case lambda 1 lambda 2 is equal to 0.

And if we substitute in the in the 4 equations I showed you, then we will get the value for x 1 this 1 and x 2 is equal to 0, but you see the feasibility one of the feasibility condition is that decision variables must be non-negative that is why that feasibility condition is not being satisfied. Thus for the case when lambda 1 equal to 0 and lambda 2 equal to 0, we are getting infeasible solution. That is why we are not really interested to get this result as an optimal solution. Now let us consider the second case, lambda 2 lambda 1 equal to 0 and lambda 2 not equal to 0 then if we just substitute in 1, 2, 3, 4 in all the equations we are getting 3 equations and 3 unknowns. From there we will get x 1 is equal to this, x 2 equal to this, lambda 1 0 and lambda 2 this.

Then again we could see that, x 2 is the negative value, that is that is violating the set of feasibility conditions that is the again I should say that the decision variables must be non-negative. Non negativity constraint is being violated that this is not the in feasible solution at all. Let us see the third case.

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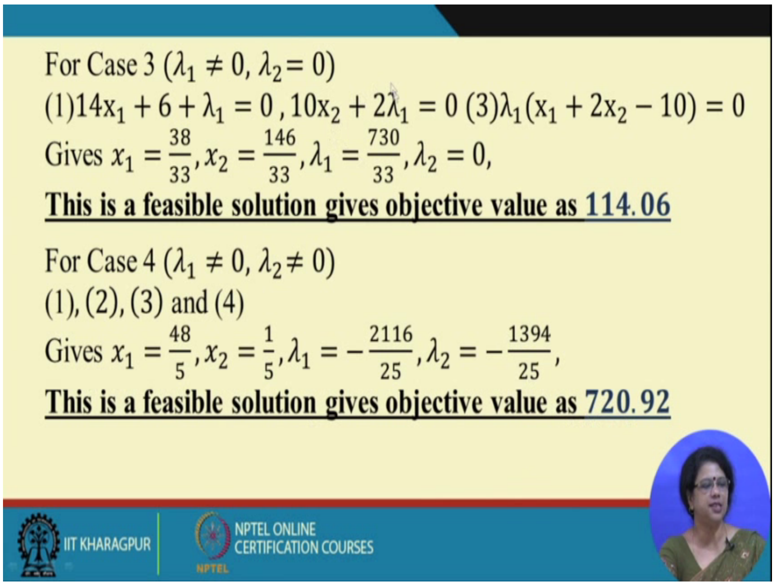

Lambda 1 not is equal to 0, lambda 2 equal to 0. If we just substitute the values in 1, 2, 3, 4, we could see that x 1 is coming this and from this set we could do very nicely we can check the non-negativity constraint is satisfying. Thus we can say that this is a feasible solution alright; that is why this is the solution. And in the second case we are getting the same lambda 1 not is equal to 0, lambda 2 not equal to 0 and we are getting this.

Now if we just compare both the solutions together, and if we just try to satisfy the these are the feasible points that there is no doubt about it, but this the first is not the KKT point, because this is not a KKT point what is the reason for that, second one is the KKT point. The reason is that, here is you can see that KKT multiplier is coming as a positive, but for maximization less than equal to type KKT multiplier cannot be positive it must be always negative. Thus we conclude that this is not the optimal solution, in the next case this is the optimal solution. Not only that it is the only reason the other reason is that you see though this is the feasible point, we are getting better objective functional value with this combination.

That is why we can declare that this is the KKT point, that is the necessary condition for us, but thus this is the optimal solution for the problem. Now the question comes we are dealing with the first order necessary condition in the KKT Karush-Kuhn-Tucker theorem, but we never said about the sufficiency. Have you said anything about the, whether this is the sufficient condition or not? That is still remaining, that is why we have to address that whether this is at all the optimal solution at the end or not. There are certain conditions, when KKT solution not only the necessary this is also sufficient, that part I will discuss later on. Let me consider another example.

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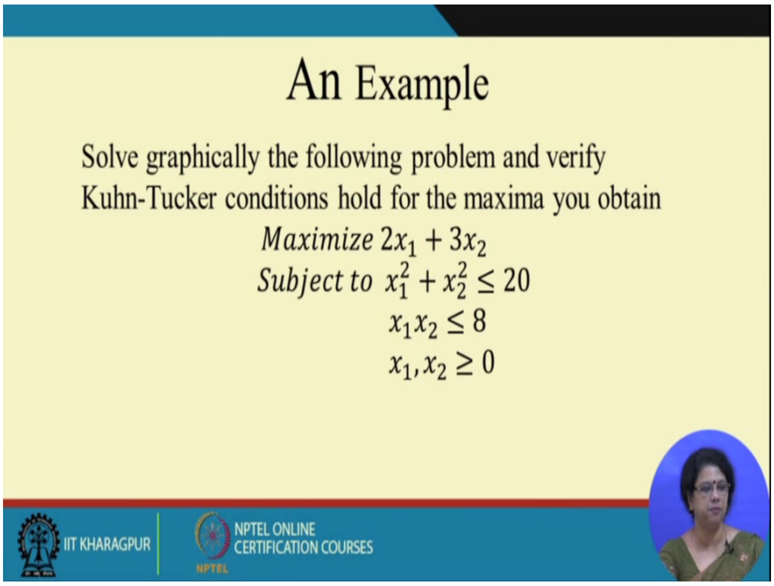

Here you see we are considering, the example where the objective function is linear and there are both the constraints, these are the non-linear in nature. Now we will try to find out the solution for this problem graphically first, then we will solve it with our technique.

Now, if you just draw the graph of the constraint set x 1 square plus x 2 square is equal to 20 and x 1, x 2 equal to 8. These are the two constraints for us.

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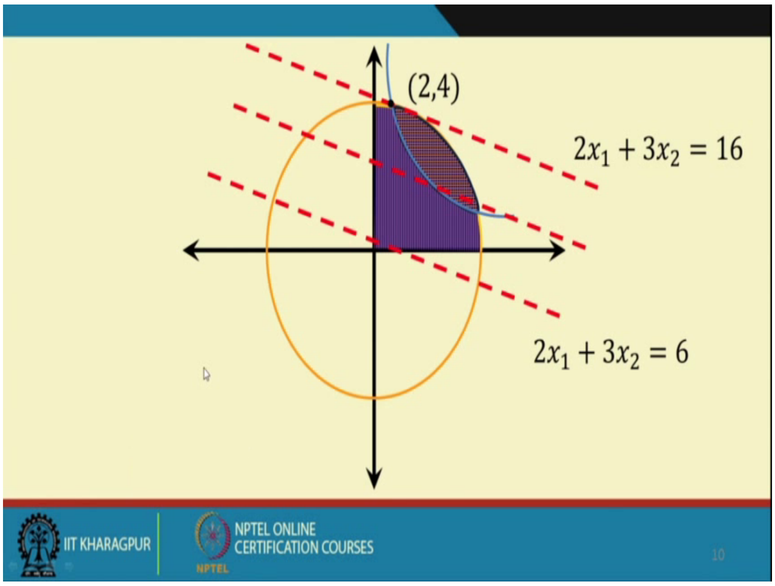

Then, this is the first constraint, this is the second constraint. Then we can say that the overlapping region is my feasible space. Now there is one objective function, objective function is 2 x 1 plus 3 x 2. Now this objective function is moving. This is one of the level set of the objective function with the value 6, but there is every provision to improve the value for this. Are we dealing with the maximization problem or we are dealing with the minimization problem. We are dealing with the maximization problem that is why always we will try to get the better value of this objective function. That is why objective functional value will move this way and since this is the crossed region is my the feasible space, because this is the region that is the region we are considering together.

Now, if this is the case then we will get the optimal solution here, and the point is coming as 2, 4. Let us check I will set the similar kind of example in the assignment as well and there we will check whether the graphically and algebraically we will get the same result or not. Here I am not doing anything again I am constructing the Lagrange function.

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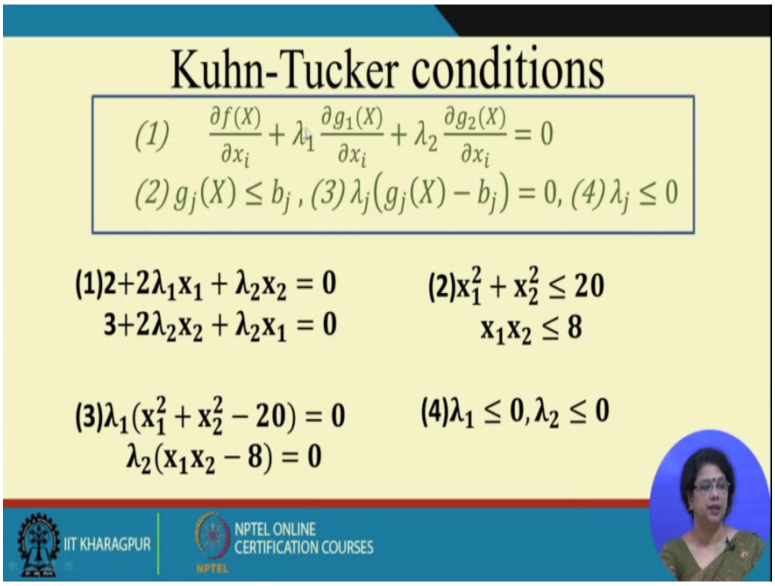

When first order first order necessary condition, this is the set of feasibility condition, I am sorry optimality condition, feasibility condition, complementary slackness property and this is a sign enforcement on lambda, this is the maximization problem. And if we consider this we do not know which is the active constraint which is the in active constraint initially because we do not know the value for lambda 1 and lambda 2, but you have idea from the graph which one is active and which one is the in active constraint. Similarly here also, let us formulate the for x 1 and x 2, let us get 4 equations and 4 unknowns from here, alright; and again for different condition for lambda 1 and lambda 2. Let us try to solve the problem of it.

You just formulate for different values of lambda. Let us see different cases again. Whether lambda 1 0 lambda 2 0 both are 0 both are non zero.

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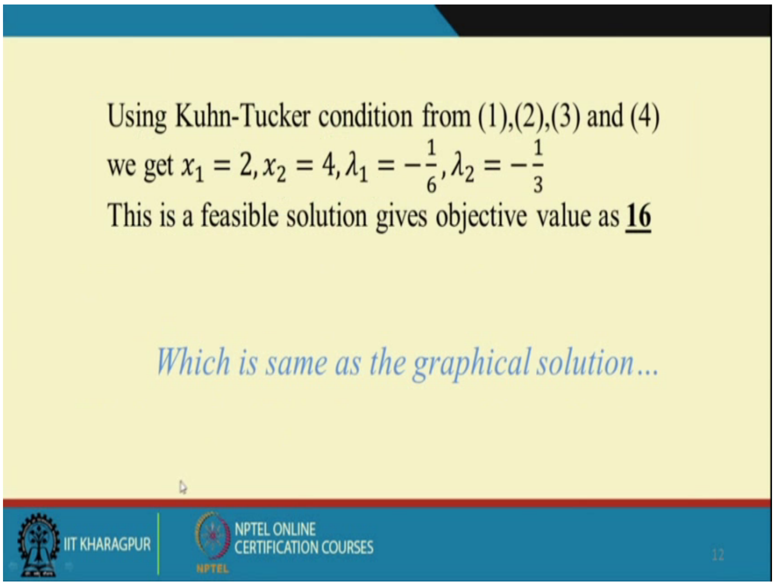

Here, also we will you can get the same set of cases and you can say that the through the Kuhn-Tucker condition we will get the solution as 16. And this is also the point 2 and 4 together alright this two check it.

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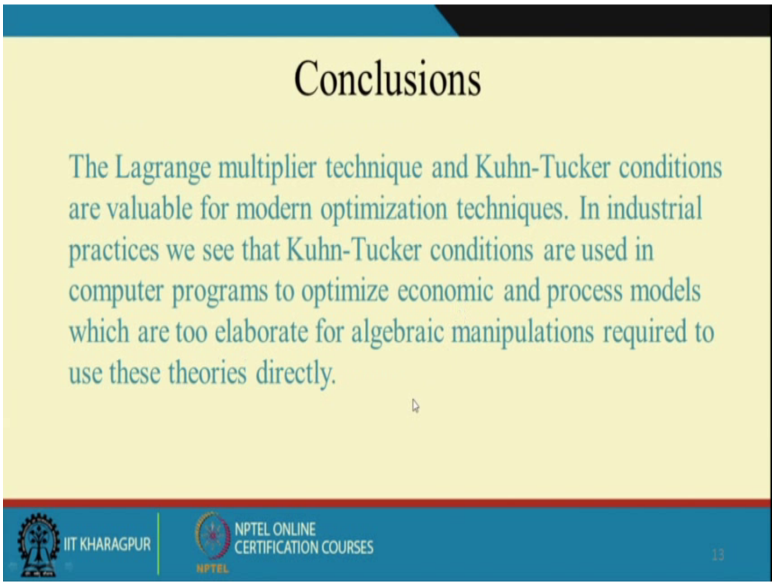

That is why we can conclude that that Lagrange multiplier technique and Kuhn-Tucker conditions these are the very important for us.

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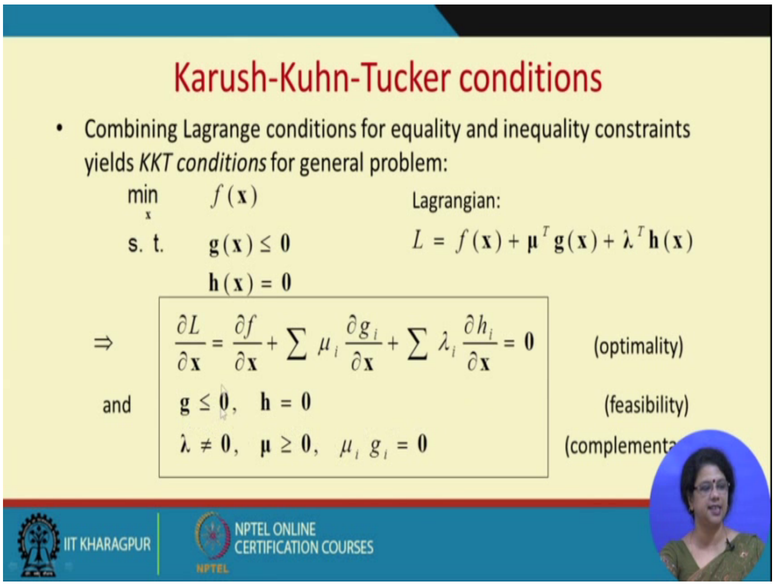

And we are applying this cases for any economic problem or any industrial problem, where we are dealing with the non-linear programming with in equality constraints, but the question comes, are we only getting the non-linear programming problem with inequality constraints separately and equality constraints separately, or we get both the conditions together, that is the question for us. That is why we have to we have to generalize the concept. We have to consider the case the non-linear programming problem, where we will deal with the less than equal to type, as well as the equal to type of constraints.

Now for the equality type of constraints, we know the Lagrange process is there and for the inequality type of constraints we know the KKT process is there, that is why the combination of Lagrange theorem and KKT theorem will give the answer to get the solution of this that also we are naming it as a KKT condition alright. That is why here also the same thing we will formulate the Lagrangian function where L would be is equal to f x plus mu in to g x plus lambda in to h x. Since h x is these are all the equality constraint the corresponding multiplier we call it as a Lagrange multiplier.

And where we are having the inequality constraint, we are calling it as a KKT multiplier, but these are the conventions. In general we can say all are the Lagrange multiplier or we say it as a KKT multiplier all the multipliers as a KKT multipliers. But here also the same question comes, if we consider the minimization problem, what should be the feasibility condition? What should be the optimality condition? What should be the complimentary slackness property not only that what should be the sign of mu and lambda? These are all the answers we have to provide for this set where we are dealing with the non-linear programming problem with inequality and equality constraint both.

That is why this is we can say; this is the first order optimality condition and this is the feasibility condition g less than is equal to 0, rather g x less than is equal to 0, h x is equal to 0, but from the complimentary slackness condition that is mu i g i is equal to 0. We can say for the minimization problem mu is greater than equal to 0 but we will say lambda not is equal to 0, but the interpretation of lambda for equality case all you know and from there we have to apply this one for a set of problems and that is and let us see what can be said about it.

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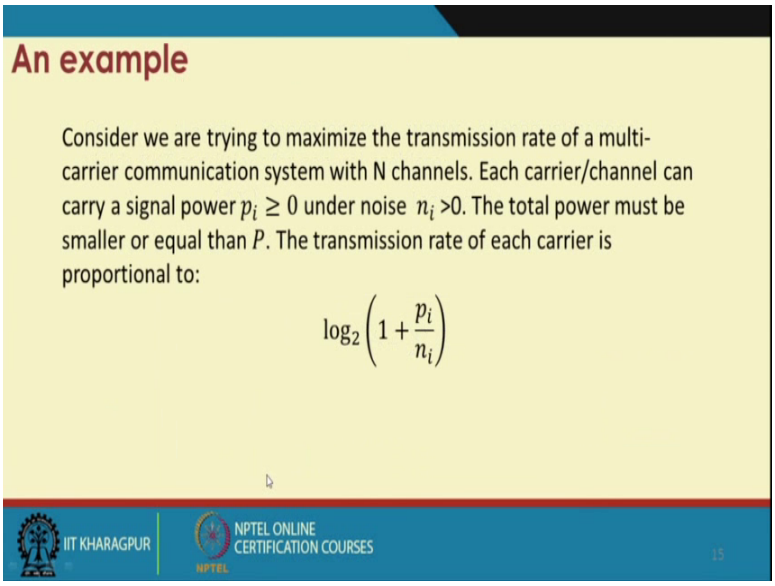

In the next class I will bring I will show you few problems, and where combination of inequality and equality both are there. Not only that in the next class, I will discuss about the sufficiency condition of the optimal solution which we are getting through KKT condition alright. Now today let us I will conclude my class with these example. Now there is a this is a one of the application we can say this is the application is a very simple one it seems, but just you see. Maximize the transmission rate of a multi-carrier communication system where there are N channels. And each carrier or channel can carry a signal pi under the noise ni and the total power must be smaller or equal to P. That is the constraint for us and the transmitter transmission rate for each carrier is proportional to log 1 plus pi by ni, with the base 2, that is given for us.

That is why from this problem you could see that I can vary from 1 to N capital N because there are N channels. And individual transmission rates are there. Now we can have the summation of this as a transmission rate, we have to maximize the transmission rate and there is a noise where ni is the noise included in the transmission rate. Now what else is there total power has been given as P that is why summation of pi from I to N must be less than equal to capital P that is why if we formulate the problem.

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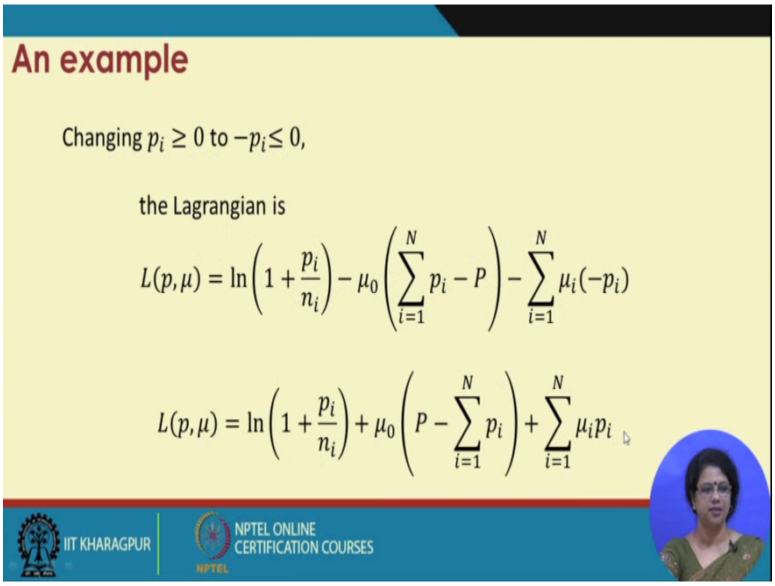

Then we can say that this one. Just you see that instead of log 2 x, we have considered log ln x that is the base e, because we are having a more comfortable with a log values with base 10 and base e that is why instead of taking the base with 2, if we just maximize this one that is sufficient, because both are equal the pattern is equal. Thus we will consider this one as a our non-linear programming problem maximization of this and this type of problem many of you come across in your engineering study. Now how to solve without KKT without KKT condition that is, really difficult to solve this kind of problem that is why we will apply the KKT condition on this. And this is the KKT Lagrange function for us. And from the Lagrange function we can have the stationary conditions rather the feasibility condition; we can have the optimality condition.

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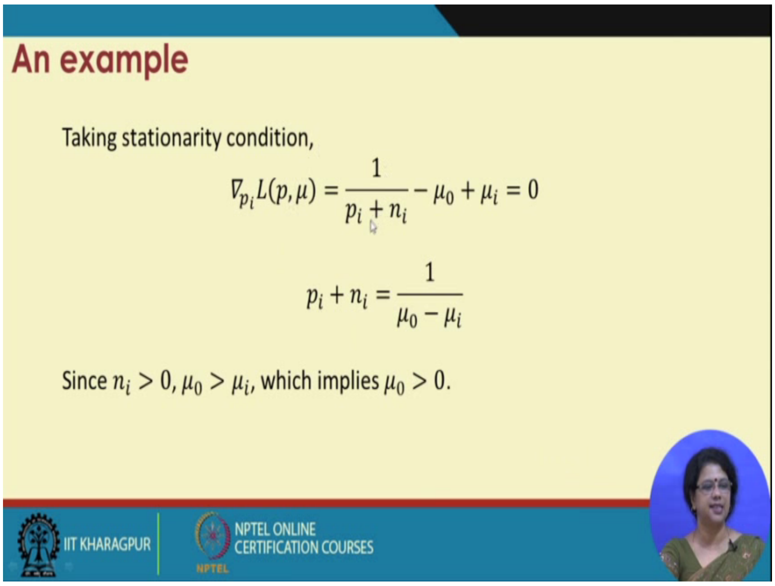

Now if you just do all this things, we will get and the what is suggest to you that, you just derive the feasibility condition, optimality conditions. Complimentary slackness property and the rather the sign of the multipliers, whatever conditions you know to the KKT theorem you just deduce all this things, and try to get the solution of it in the similar manner, because this will be whatever this stitching material will be provided to you, you can take help of it, and I suggest that you do the calculation on your hand and try to get the solution of it and the solution will come as this one that pi is equal to we will get solution as 1 by mu naught minus ni alright. And we can see that the solution satisfies the constraint as well. Now, I suggest that you complete this problem.

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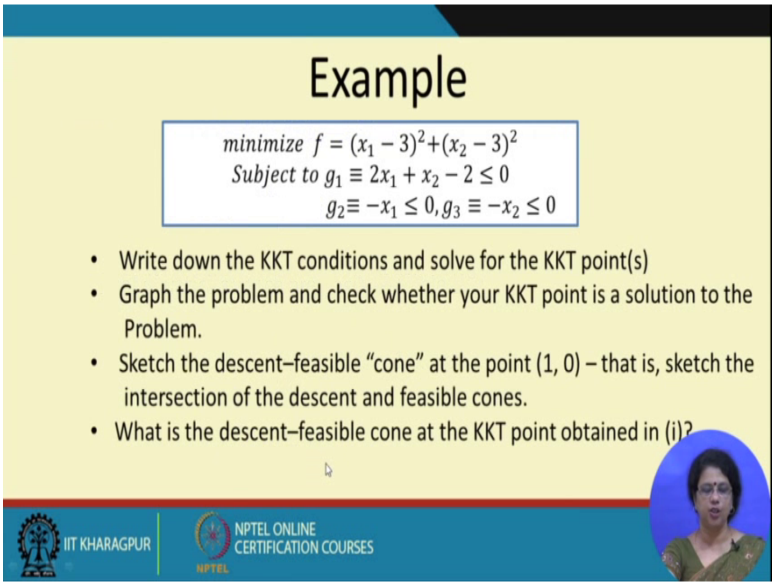

Not only this I am giving you another problem to work with, and you need to find out what is problem this is a minimization problem, you have to write down the KKT condition. Find out the feasible points? And from there you identify which are the KKT points alright and you draw the graph of it, because this is the two dimensional problem. First one is the first one you can consider as a circle and second one simple the line linear line and try to guess that.

Whether the feasible space is the convex space or not? That is another task of yours, to check the feasible space is convex space or not, because the linear line less than 0 can be in. When we have any linear function individually less than equal to 0 can be said as a convex space. And you sketch the graph of it, find out the descent cone? Find out the feasible cone and get the solution optimal solution and see whether the KKT point is obtained in the descent feasible cone or not. All this things please do it in with your hand, and try to solve it on your own.

Thank you very much for today.