Constrained and Unconstrained Optimization
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Lecture – 20
Dual Simplex Method

Today in this lecture we will go through the dual simplex method. In the last class what we have done that is if you have a primal problem. How to convert the primal problem into the corresponding dual problem? That we have discussed in the last class. The reason for conversion of primal problem into the corresponding dual problem, is that to reduce the computational efficiency. Computationally effect I can reduce the computational work for that reason we use the dual of a primal variable, as you have observed that if I have more number of constraints then to find the basic feasible solution the computational time it takes much more.

So therefore, if I convert it into the dual then the number of constraints will be reduced because if the number of variables in the objective function are less, in that case number of constraints in the dual also will be less. And in that case what I have to do it my computational time will also be less. So, at first we will start with the dual simplex method, what is that one?

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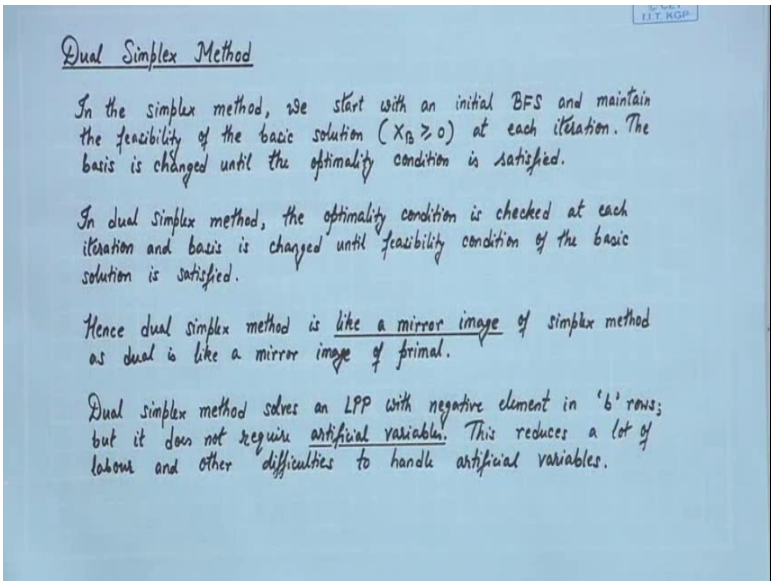

In the simplex method we start with the initial basic feasible solution. And maintain the feasibility of the basic solution, that is X B greater than equals 0 at each iteration. If you note we start with the initial basic feasible solution, and always the basic solution is feasible; that means, all the decision variables are X B greater than equals 0. And the basis is changed until the optimality condition is satisfied.

If you remember the optimal optimality condition is zj minus cj is greater than equals 0 and some vector may be departing some new vector may enter in the basis. In this process the basis will change until zj minus cj is greater than equals 0 or the optimality condition is satisfied. In the dual simplex method what happens? The optimality condition is checked at each iteration, and basis is changed until the feasibility condition of the basis is satisfied. Or in other sense here we are doing the opposite one for general LPP what we were doing? We were first confirming that basic solution is always feasible then we trying we are trying to find out the optimality condition.

But in this case the we are our basis is changing whenever we are checking optimality condition is satisfied or not, and until the feasibility condition of the basic solution is satisfied. That is until X B greater than equals 0. So, we are doing just the opposite one for this reason we call dual simplex method is like a mirror image of the simplex method, as dual is like a mirror image of the primal. Dual simplex method solves an linear programming problem with negative elements in b rows please note this one, with negative elements in b rows, but it does not require any artificial variable for that we will see through this. And this reduces basically a lot of labour and other difficulties to handle the artificial variables.

So, please note this one in dual simplex we use the negative element in b row and for that we never use the artificial variables. And the so many other difficulties it always reduces.

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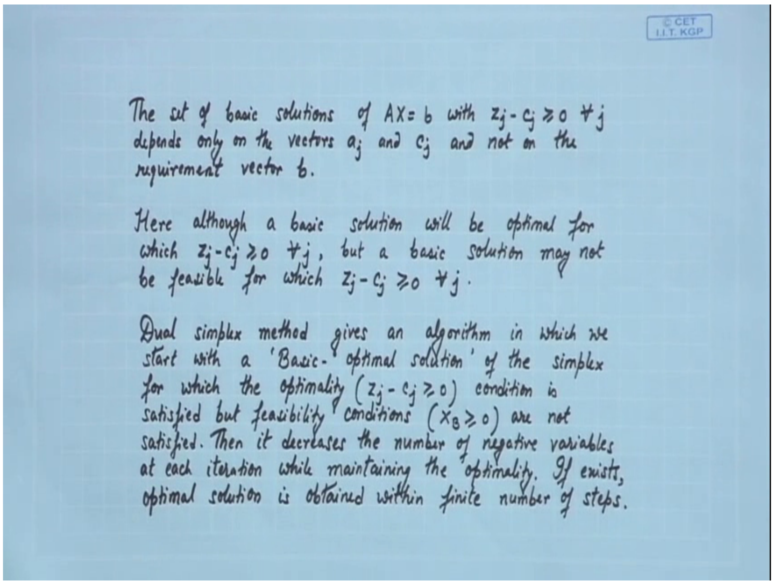

The set of basic solutions of ax equals b with zj minus cj greater than equal to 0 depends only on the vectors aj and cj, not on the requirement vector b. Please note this one through the set of basic solutions a of ax minus b with zj minus cj greater than equals 0; obviously, will depend on the vectors only aj and cj, at the involvement of the requirement vector b is not there.

So, although a basic solution will be optimal for which zj minus cj greater than equals j, greater than equals 0 for all j, but a basic solution may not be feasible for which zj minus cj greater than equals 0. So, basic solution may not be equal to may not be feasible means X B greater than equals 0 this condition may not be satisfied and for this reason we are saying that that a basic solution may not be feasible, but zj minus cj may be greater than equals 0 for all j whereas, if you remember for normal simplex case always basic solution is feasible that is X B is always greater than equals 0, that we are ensuring and we are checking the optimality condition there.

So, in dual simplex it is the opposite one. So, dual simplex method gives an algorithm in which starts with a basic optimal solution of the simplex method for which the optimality condition is satisfied. Optimality condition means zj minus cj greater than equals 0. So, whenever we will try to find out a basic optimal solution that ensures that the optimality condition that is zj minus cj is will always be satisfied, but feasibility condition that is non negativity on the decision variable or in other sense X B greater than equals 0 may or may not be satisfied. Then it decreases the number of negative variables at each iteration while it maintaining the optimality. That is it at each iteration it maintains the optimality and then tries to reduce the number of negative variables in the basis.

If the optimal solution exists then optimal solution will be obtained within the finite number of steps.

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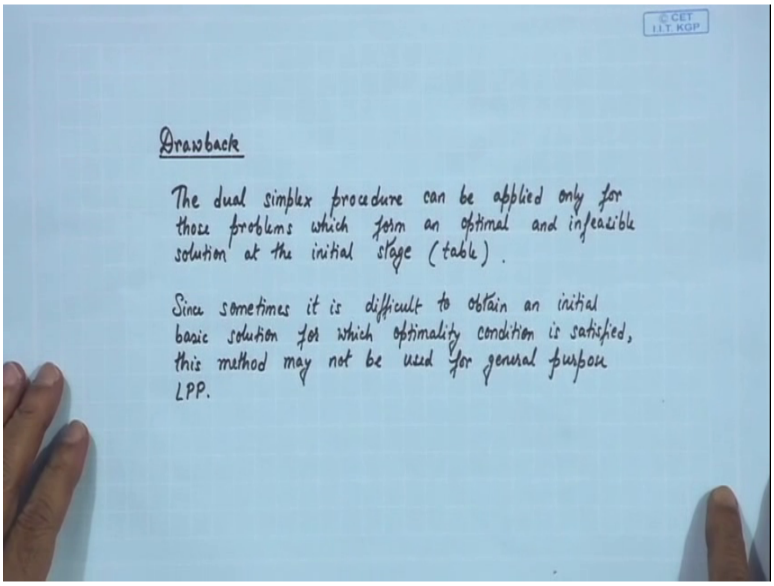

So, this is the basic idea of the dual simplex method. So, what is the drawback of this dual simplex method. The drawback of dual simplex procedure is it can be applied only for those problems whose which form an optimal and infeasible solution at the initial stage. Please note this one which form an optimal and infeasible solution at initial stage or initial table. Otherwise we cannot use this dual simplex method.

So, the initial table which we will obtain that must form an optimal solution that is optimality condition has to be satisfied. But the solution will not be feasible that is X B greater than equals 0 for all j will not be condition may not be satisfied. Since sometimes it may become difficult to obtain an initial basic solution for which optimality condition is satisfied this method may not be used for general purpose with LPP. So, basically dual simplex method is not used always not for all general purpose entity for certain type of problems only which satisfy the earlier one as I told that initial table must be optimal and infeasible solution then only we can apply the dual simplex method.

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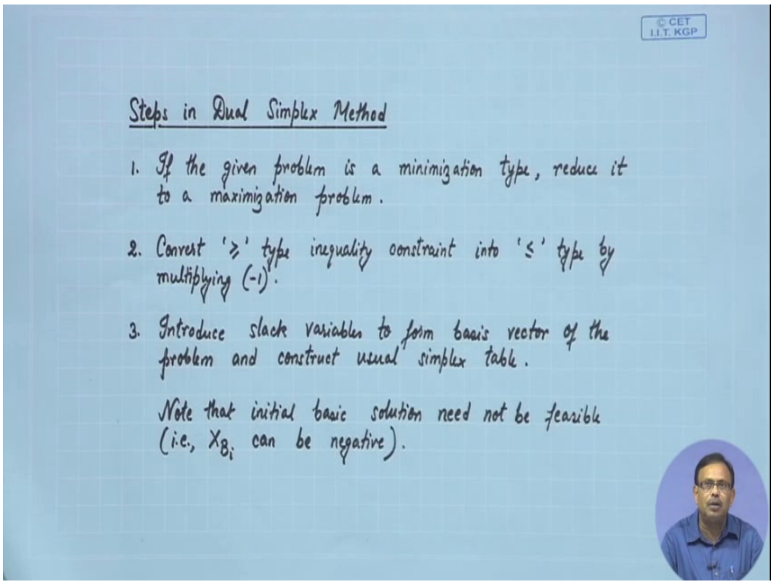

Now let us see what are the steps they are in dual simplex method. If the given problem is a minimization type then reduce it to a maximization problem. As we have seen whenever you are writing any LPP dual LPP first we are writing in the standard form. The standard from was maximize z equals cx subject to ax less than equals b. Here also point number one if the given problem is a minimization type then reduce it to a maximization problem. Number 2 convert the greater than equals type inequality constant into less than equals type, and you know we can do it by multiplying minus 1 on both side of the given inequality.

So, if you have greater than equals type inequality by multiplying minus 1 on both side convert it to the less than equals inequality. Number 3 introduce slack variables in the less than equals type inequality constant to form the basis vector of the problem. And then construct the usual initial simplex table. So, you introduce the slack variables for a less than equals type constraints inequality constraints to form the basis vector of the initial simplex table. Note that initial basic solution need not be feasible. That is whenever you have the initial basic feasible solution their X B i can be negative also.

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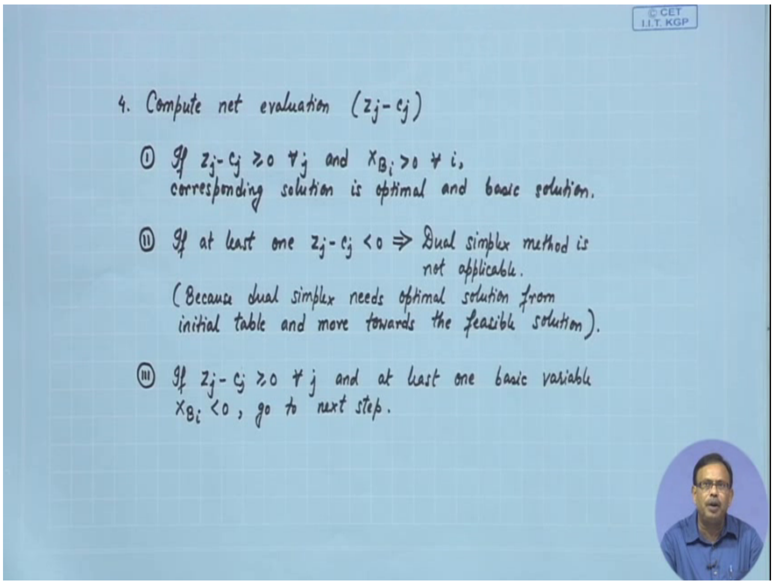

Step 4 compute the net evaluation zj minus cj which we are calculating always for simplex tables at each iteration. Here basically 3 different types may occur. If zj minus cj is greater than equals 0 for all j, x B i that is all the variables in the basis are greater than 0 which we are saying x B i greater than 0 for all i. Then the corresponding solution is optimal basic solution this is the optimal solution. So, if zj greater than equals 0 and x B i greater than 0, in that case whatever solution you have obtained that is the optimal solution and which are satisfied the feasibility condition.

Number 2 at least one zj minus cj is less than 0. If at least one zj minus cj is less than 0, then dual simplex method is not applicable. Please note this one, if at least one zj minus cj is less than 0 then dual simplex method is not applicable. Because the reason is as we have told the initial dual simplex table needs the optimal solution from the initial table and move towards the feasible solution. That is it always satisfies the optimality condition then tries to obtain the feasibility condition. Number 3 if zj minus cj greater than equals 0 for all j, but at least one of the basic variable x B i is less than 0; that means, the infeasibility has occurred that is all the decision variables are not satisfying the feasibility condition for that reason we are saying X B is less than 0, then follow the next step whatever we are doing.

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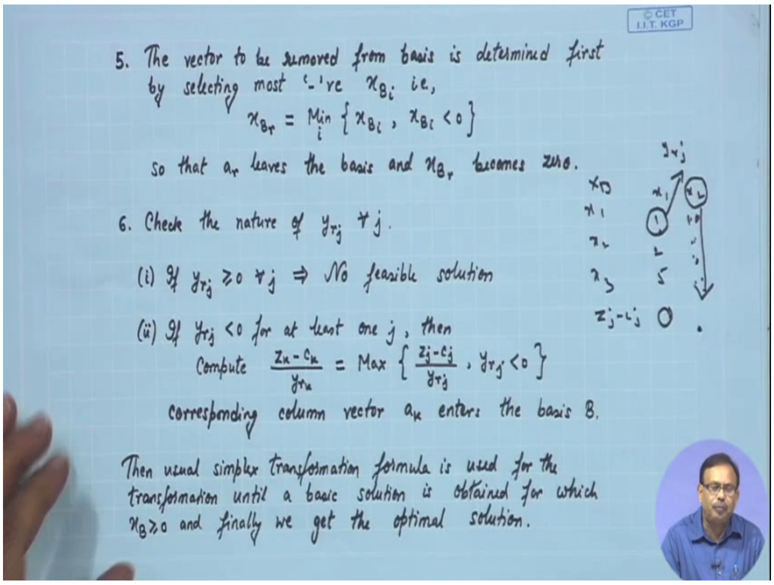

Here you see the vector which has become negative, that vector instead 5 can be removed from the basis which can be removed from the basis which is determined first by selecting the negative x B i. That is x B r equals minimum of x B i. So, x B r is minimum of x B i; that means, in the basis whatever vectors are there. The minimum value negative value whatever the x B i is taking that one will be the going out if r th element is r th vector is going out which I am denoting as x B r minimum of I x B i where x B i less than 0. Then ar will leave the basis and x B r becomes 0. So, please note this one that what will be the departing vector that we are deciding first compared to the normal simplex method where we were deciding what will be the entering vector first by checking the zj minus cj value. So, here what will be the departing vector that we are checking by comparing the values of the vectors of the basis that is x B i. And minimum of these x B i which is equals to x B r then ar will be the departing vector.

So, next you have to choose which one would be the entering vector. To check the what would be the nature of corresponding Yrj. Yrj means here I want to say whenever you are writing x 1 x 2 x 3, this is your xbc in that case here also x 1 x 2 some values of the one 2 5 or 10 like this way some values are there. So, these values we are calling as Yrj this we are calling as here Yrj. If Yrj greater than equals 0 for all j, that is corresponding to that departing variable if Yrj all that values are greater than equals 0 then we say that no feasible solution.

So, please note that if corresponding to this parting vector ar whatever values are there which we are denoting in this Yrj. If all Yrj greater than equals 0 in that case there will be no feasible solution. Whereas, if Yrj is less than 0 for at least one j then you have to compute zk minus ck by yrk, which is equals what maximum of zj minus cj by y rj. So, zj minus cj value divided by the corresponding element value your zj minus cj is here. So, zj minus cj value this one this divided by this Yrj wherever it is pointing where a Yrj must be negative Yrj should be negative. Corresponding to which one? Corresponding to column vector ak enters in to the basis b.

So, corresponding to the column vector ak; that means, if a 2 is the departing vector then I will compare here the y values with zj minus c values, cj values. And I will not check compare the ratio for all y 1 values I will take only those y values which are negative here. Then usual simplex transportation problem is used for transformation until a basic solution is obtained for which X B greater than equals 0 and finally, we get the optimal solution. So, I am repeating the process once more in this case your if you see you are computing first the value of zj minus cj.

If zj minus cj greater than equals 0 and x B i greater than 0 then the solution is optimal solution. If at least one zj minus cj is less than 0, then I cannot apply dual simplex method. Because the basic criteria is not satisfied that the solution is not optimal. And if zj minus cj greater than equals 0, but at least one variable basic variable x B i less than 0 in that case what you are doing you are calculating the first you have to check what is the departing variable. Here departing variable we are calculating by this one x B r equals minimum of y x B i, then if ar leaves the basis in that case I will check corresponding to ar in columns what is the values of Yrj values. If all Yrj are greater than equals 0 then the problem will have no solution.

Whereas if Yrj less than 0 then we are computing zk minus ck by yrk equals maximum of which one? In that case corresponding column vector ak will enter into the basis and I will repeat the steps until I obtain the optimal solution, and it satisfies the feasibility condition that is X B greater than equals 0.

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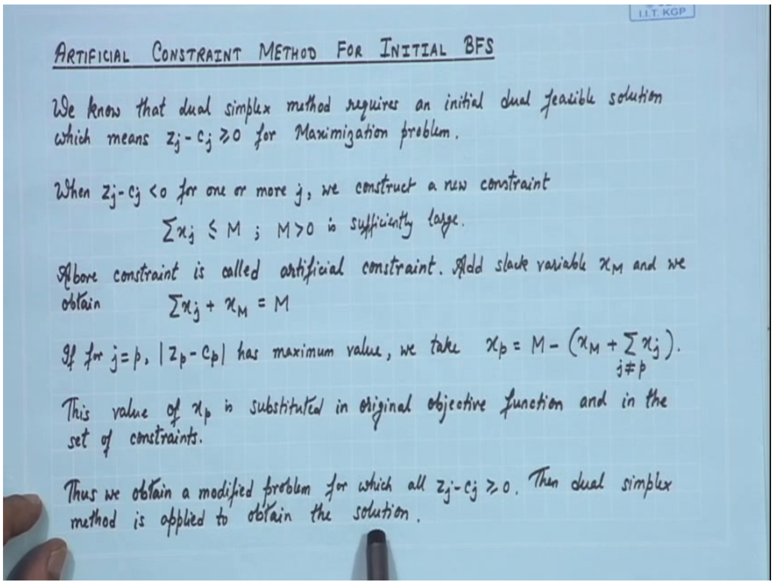

Now let us discuss we will discuss that part also, whenever we are doing the examples artificial constant method for initial basic feasible solution. As you know that dual simplex method requires an initial dual feasible solution, which means zj minus cj should be greater than equals 0 for maximization problem. As we have told if zj minus cj is not greater than equals 0 in that case we cannot use this dual simplex method.

So now we are want to talk about if zj minus cj is less than 0 for one or more j then how to construct a new constraint. So, that your zj minus cj will be greater than equals 0. So, when zj minus cj is greater less than 0 for one or more j we construct a new constant constraint summation over xj. So, it is less than equals capital M. Where m is sufficiently large, that is some of those variables I am making less than equals some quantity which is sufficiently large. That above constraint whatever we have discussed these constraint we call it as the artificial constraint, this constraint is known as artificial constraint.

So, since this new constraint is less than equals time to make it equality we have to add one slack variable say x M, and we obtain summation over xj plus x M equals capital M. So, basically we are adding a new constraint at first for which zj minus cj is less than 0 for one or more j. So, summation over xj less than equals capital M. Then using slack variable we are making this less than equals type into equality type by doing this summation xj plus x M equals M. Now if for j equals p say mod of zp minus cp has maximum value in that case we take xp equals capital M minus x M by this. That is from here the value of j for which zj minus cj has maximum value corresponding to that value of j the variable xj which we are denoting here by xp. We are we will replace that particular variable by this quantity capital M minus x M plus summation over j not equals to p xj.

So, the value of xp will be substituted in the original objective function as well as on the objective function. That is you are effectively added the slack variable, but you are reducing the variable which is actually the negative. So, that you are replacing by the other variables. Thus you will obtain a modified problem for which your zj minus cj will be greater than equals 0. Then as usual dual simplex method can be applied to obtain the solution. So now, let us take one or 2 examples by which let us explain the problem.

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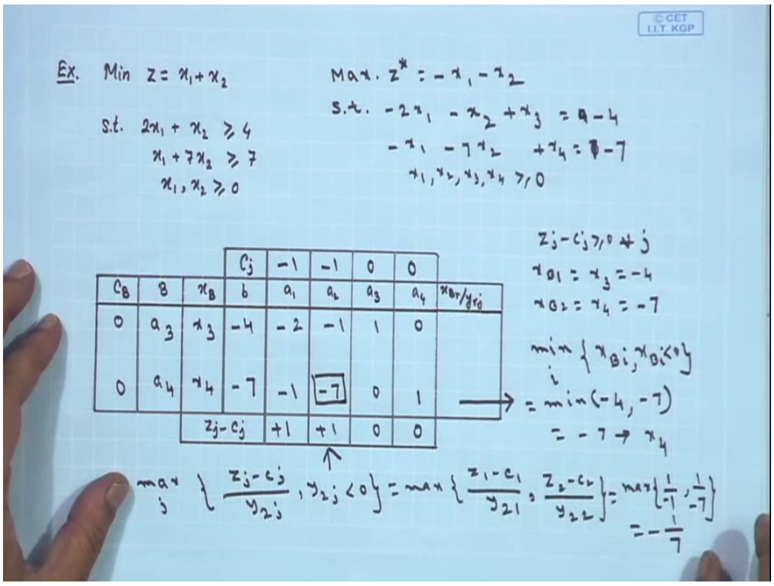

Let us take the problem minimize z equals x 1 plus x 2, subject to 2 x 1 plus x 2 greater than equals 4 and this one x 1 plus 2 x 2 less than equals greater than equals 7.

So, the problem is not in standard form, you can write it in the standard form by making it maximization problem, subject to you have to change this greater than equals into less than equal type by multiplying minus 1 for both the constraints and after that you have to add one slack variable to make those less than equals type into equality type. So, at a time I am writing it in the standard form and I am making these constraints as equality type. So, since you have done it already. So, the problem will be maximize z star equals since it was a minimization problem therefore, I have to multiply it to get it minus x 1 minus x 2, subject to minus 2 x 1 minus x 2 plus x 3 this is equals 4.

Second one is minus x 1 minus 7 x 2 plus x 4 equals 7 sorry, you know since I multiplied both by negative. So, it will be minus 4 this will be minus 7. And your x 1 x 2 x 3 and x 4 are greater than equals 0, where your x 3 and the x 4 are the slack variables. So, the basic basis will be basis vectors will be x 3 and x 4, x 3 is minus 4 x 4 is minus 7. So, I am writing here x 3 x 4 this is a 3 a 4, if you see now I have changed this one in place of x 1 x 2 x 3 as we were writing earlier.

Now, we are writing these as a one a 2 a 3 a 4 which corresponds to this one. Your cj value is minus 1 minus 1 0 and 0. So, that your cb will be corresponding to x 3 x 4 it is 0. And 0 your b values are minus 4 and minus 7, now write the rows that is minus 2 minus 1 1 0 and minus 1 minus 7 0 and 1. So, you are getting zj minus cj means you are obtaining plus 1, here it is plus 1, here it is 0 and here it is 0.

If you see here your zj minus cj is greater than equals 0 for all j. So, your X B 1 is what your X B 1 I am just explaining elaborately this one X B 1 equals x 3 that is equals minus 4, X B 2 is equals to x 4 that is equals minus 7. So, your solution is optimum they are satisfying these optimality condition, but infeasible since that value of the decision variables are negative over here. So, you have to choose which from basis which vector will depart. How I will choose it? I will choose it by these minimum of i x B i where x B i is less than 0.

So, this is equals minimum of minus 4 X B 1 is minus 4 X B 2 is minus 7. So, this is minus 7. This is happening for which variable for x 4. So, your departing variable is x 4 here your departing variable will be x 4. So, like this way we calculate the departing variable. So, x 4 will leave the solution. Now which one will enter for that you have to calculate maximum over j zj minus cj by y 2 j, such that your y 2 j should be less than 0. So, x 4 is the departing variable. So, if this I can write down maximum of z 1 minus c one divided by this will be y 2 1, because this is corresponding to this y 2 1 and z 2 minus c 2 by y 2 2.

So, the values are maximum of this is maximum of 1 by this is 1 by minus 1, your negative is coming on this 1 by minus 1 and 1 by minus 7. So, minimum is equals to minus 1 by 7 which is occurring for this y 2 2. So, your entering vector will be this one.

So, your a 2 will be the entering vector in the basis and this is the pivot element. So, like this way you are selecting which one will be the departing vector, and which one will be the entering vector, first we are selecting the departing vector.

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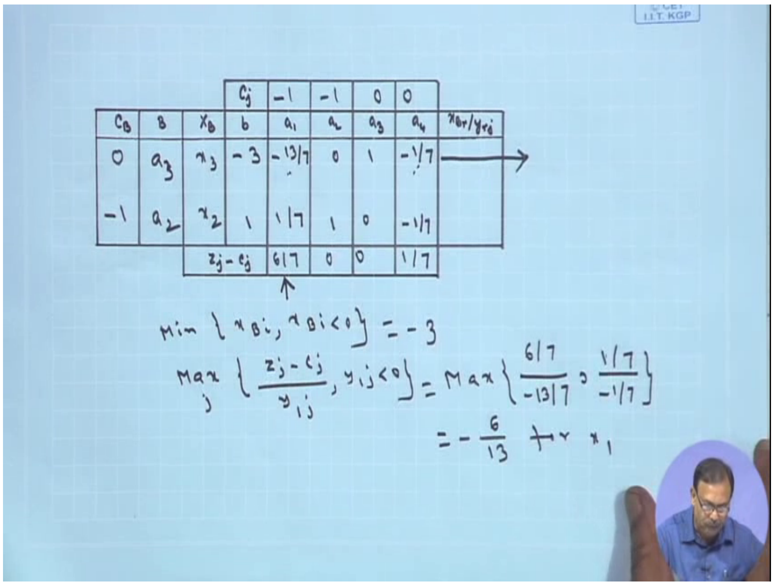

So, from here we are calculating this one we are next your x 3 will be coming here and x 2 in will be replaced by x 4 sorry x 4 will be replaced by x 2.

So, you will have here a 3 and a 2 cj minus 1 minus 1 0 0 this will be 0 and minus 1 as I was saying. I am directly writing here minus 13 by 7 0 1 minus 1 by 7 1, 1 by 7 1 0 minus 1 by 7 here zj minus cj is 6 by 7 0 0 this will be 1 by 7. So, here again you see zj minus cj is greater than equals 0, but this variable value is minus 3. That is all X B not greater than equals 0.

So, I have to there must first calculate what is the departing vector. Your departing vector is minimum of x B i that is bi x B i where x B i is less than 0. So, that is only one that is minus 3 therefore, your x 3 will be the departing vector. Now x 3 will leave the basis which one will enter into the basis, for that you have to calculate zj minus cj by y 1 j corresponding to this. So, y 1 j means this one. So therefore, where y 1 j could be less than 0. So, this is maximum of again I am writing 6 by 7 divided by minus 13 by 7 this one. 6 by 7 minus 13 by 7 and then 1 by 7 by here it is minus 1 by 7, because on this you are having negative only on these 2.

So, 1 by 7 minus 1 by 7 and the minimum will be minus 6 by 13 that is for x 1. So therefore, your x 1 will enter into the basis. So, once you are obtaining this one then let us go to the next table where x 2 will be replaced by x 1. So, sorry x 3 will be replaced by x 1.

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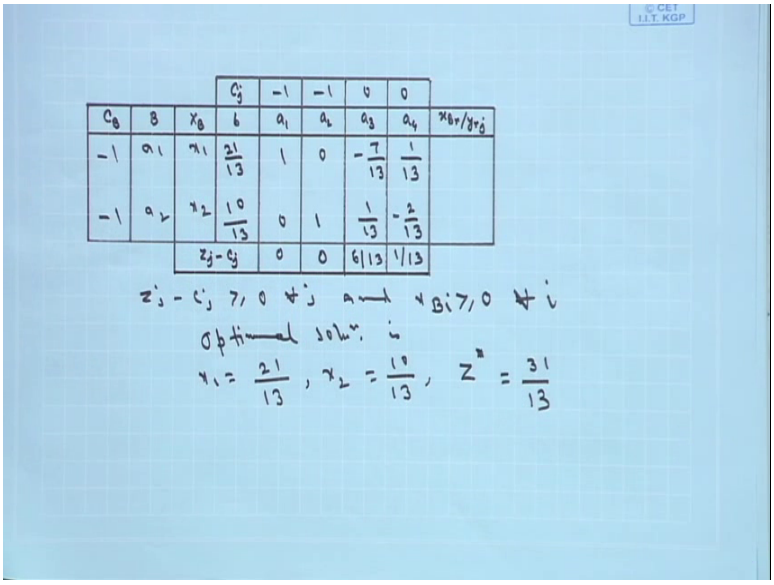

So, you will have x 1 x 2 here a one a 2 here minus 1 minus 1 0 0 and these 2 values are minus 1 and minus 1. So, here again your pivot element is this 1 1 by 7. So, I will make this one as 1, I am writing again 21 by 13 1 0 minus 7 by 13 and a 4 is 1 by 13, then 10 by 13 0 1 1 by 13 minus 2 by 13.

So, if I calculate zj minus cj. This will be 0 this will be 0 this will be 6 by 13 and last one will be 1 by 13. So, you see your zj minus cj greater than equals 0 for all j. And your X B is greater than equals 0, or x B i greater than equals 0 for all i since x 1 is positive x 2 is positive. Therefore, your optimal solution if I have to calculate your optimal solution is x 1 equals 21 by 13 x 2 equals 10 by 13 and z star if I calculate this will be 31 by 13 because this into this plus this into this. So, z star is 31 by 13. So, I hope your basic idea is how to enter a how to select which vector will depart and then how to select which vector will go. We will enter into the basis that is the basic idea of this dual simplex method. Otherwise it is similar to your LPP normal LPP iterations whatever we have done earlier.