Modeling Transport Phenomena of Microparticles
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Lecture-04
Exact Solutions of Navier-Stokes Equations in Particular Cases

Hello welcome so in the last lecture we discussed how to write down Navier-stokes equations in simple Cartesian components and then in case of a unidirectional flow how the equation gets simplified and then we discussed the general boundary conditions, that are to be implemented and the fluid comes in contact with an impermeable boundary okay .

So now we make use of these tools and then try to solve some problems typically this is called exact solution of Navier-stokes equations in books, but I mean you have to take it with a pinch of salt, exact solutions for limited configurations because the Navier stokes equations are highly non-linear because of the inertial term you got Grad u, so we would not be able to get exact solutions, considering the term either you kill that term a priori with an assumption that inertial terms of neglected compared to viscous terms okay, viscous forces.

Then you get some linearized equations alternately you can shuffle no source equations, but bring in a particular geometry and then the assumptions by virtue of the geometry you plug in into the system then automatically the corresponding system gets simplified. So these are the two approaches okay. So maybe we will stick to the second approach that I mentioned just to give you a feeling that we are really considering Navier-stokes but the end of the day you will see you would not see the complete non linear equations.

Because we are going ahead with the assumption that we are really getting exact solutions of a Navier Stokes, so the simplest configuration which comes to our mind is already, if you see for a unidirectional we have almost got the solution because we got u as a linear function of the co-ordinate involving two arbitrary coefficients. So we are almost through with that except that we require two boundary conditions right so anyway not just that alone, we will add little more physics and then try to solve the particular problems.

So the first problem that we are going to consider is very popular in literature any elementary book on fluid mechanics would talk about this okay. So that is flow between two parallel plates right, so you have a two parallel plates and then flow driven by pressure gradient or one of the plate is moving okay so that is the configuration.

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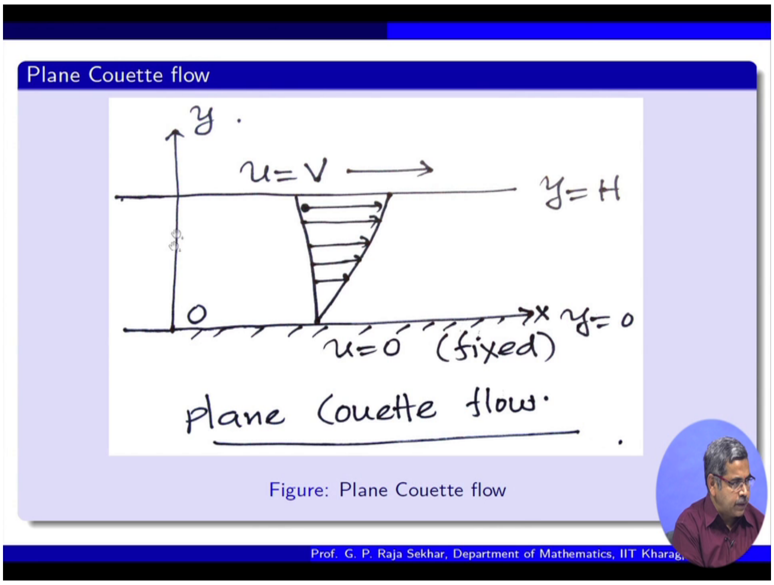

So let us have a look at it, so this is called plain Couette flow, so what is the configuration you have seen x axis y axis and we are assuming that two plates are there at y = 0 and y = H, and the bottom plate is fixed okay, and upper plate is moving since bottom plate is fixed the velocity is 0, upper plate is moving the velocity is given the term, velocity of the plate okay, so in literature this is Couette flow, plane Couette flow okay.

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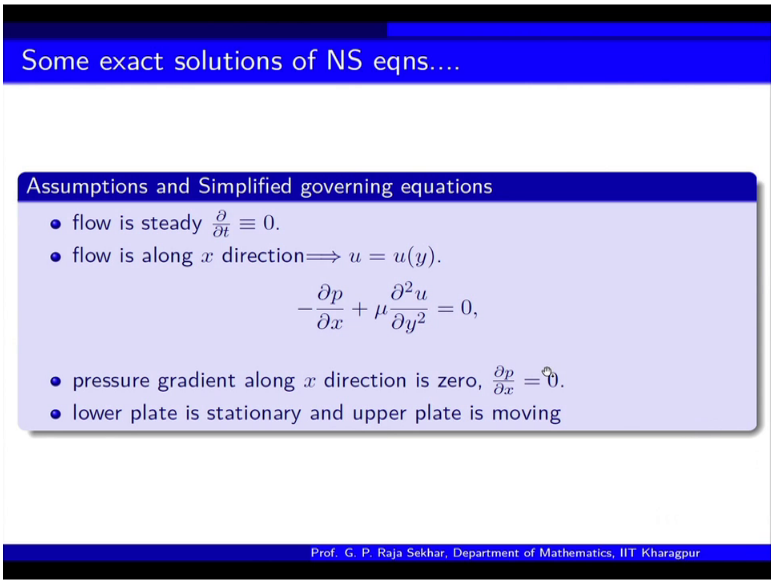

So what are the assumptions of the simplified equations flow is steady, so that is the first assumption then flow is along x direction, which is nothing but unidirectional therefore using equation of continuity, we have seen that u will be function of y okay. Then y momentum indicates that the pressure gradient along y is 0 okay. So that is a mistake so this is y 0 so the x momentum indicates that term this is the equation okay, so okay, so that is an assumption okay.

This this part is known we are enforcing additional assumption the pressure gradient is 0 and lower plate is a stationary and the upper plate is moving okay. As such from the y momentum we are getting Dow p Dow y is 0, so Dow p Dow x is non zero at that stage we have got each equal to constant right, but we are bringing an additional assumption that the pressure gradient is 0, So then the question comes is if the pressure gradient is 0 who is driving the flow right.

So you have two parallel plates and you have only one momentum equation, which involves pressure gradient and velocity and we are killing the pressure gradient then who drives the flow, so one of the plates is driving the flow right, so that is the assumption lower plate is stationary and upper plate is moving so the flow is driven by the moment of the plate okay, so that is a purposefull.

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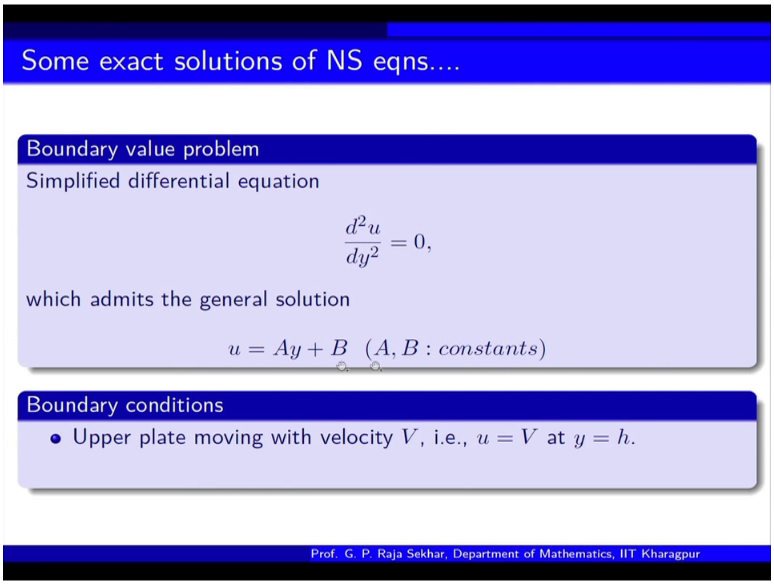

So very simplified equation okay, and it admits a solution linear, then upper plate is moving with velocity V so therefore you enforce this condition and lower plate is at rest so therefore you enforce this condition then determine A and B, so if we take this condition B will be 0 simple and if enforce this condition then A will be V by H okay.

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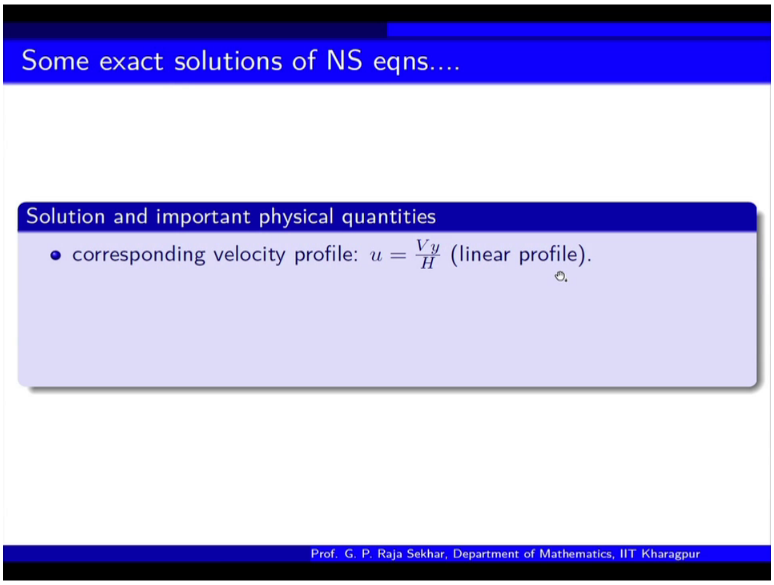

So that is what we get the corresponding velocity profile is linear, so which means you have two plates okay.

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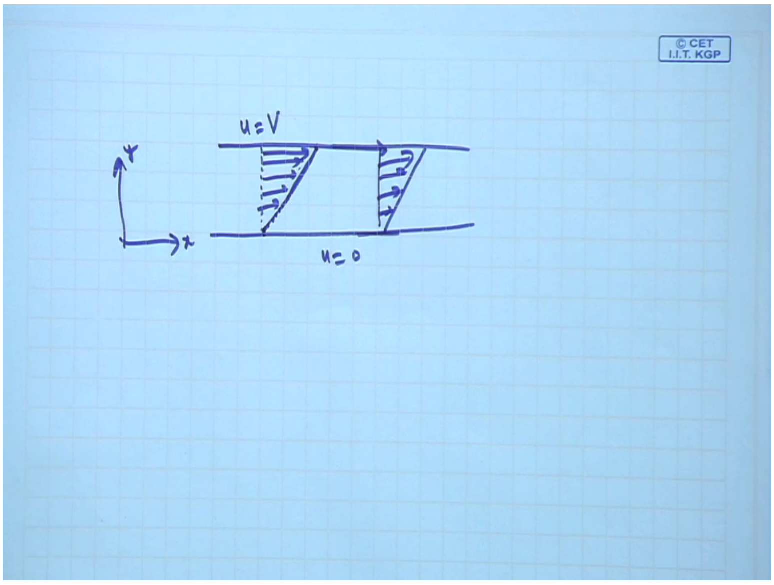

So this is x and y this is a trust, therefore u is 0. So this is moving suppose these are the fluid particles, since the plate is moving with velocity V, what will happen? The velocity structure will be like this, why we are realizing this, this plate is stationary, this plate is moving okay. Since this is stationary viscous effects are more here, the fluid a auguring to this they are at rest, then the plate which is moving that will drag the because again here no slip this is nothing but no slip, this is also no slip.

The particles are into this surface they move along with the velocity of the plate so therefore velocity is taken by these layers but due to the viscous effects they will drag slowly. So as you are coming down the velocity is expected to be reduced then ultimately it should reach the no-slip stationary, so therefore beyond a certain distance you will expect such velocity profile okay so if you take this such velocity profile is expected, so it is a kind of a shearing happening okay so that is the thing. So this is a linear profile.

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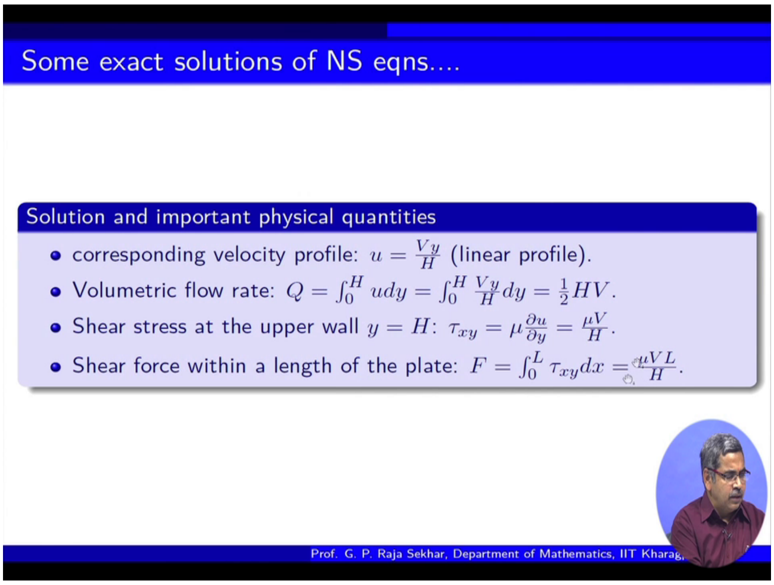

Then volumetric flow so what is the volumetric flow rate, so you have you take a small element aerial element okay, so then whatever flow with whatever velocity is coming out so that should be trapped within that area okay, so if you keep on changing the velocity naturally the total amount okay, which is coming out of that varies right, so therefore you integrate the velocity across that element, so that is what we are doing so 0 to H okay, means at a cross section here okay.

So this is distance between the plates 0 to H so that will be behaving like this, this is the total volume flux okay. Then shear stress at the upper wall you see the expression of shear stress already it is written here, so Dow xy is Mu Dow u Dow y that is the shear stress right, so it is a symmetric tau xy is yx. So here we are on a surface having no normal along y direction, there is y equal to x so normal y is a long way direction then x component of it.

So that is nothing but, this so we have u and the Dow u Dow y is nothing but V by H and therefore this is the shear stress, so you would not appreciate much what is happening here and all that because it is very simple channel flow it is a linear profile. So there is not much of happening except that the fluid velocity is taking the particles attached to the upper plate so they are dragged so you see a linear profile okay.

Otherwise so the volume flow rate and the shearing effect they are expected to be dependent on what are the other remaining variables involved what are they you increase the velocity so then what happens more sharing happens then you increase the distance between the plates okay, so correspondingly the more volume will be trapped okay, right you increase the velocity or you increase the distance between the plates the more volume flux is supposed to be accumulated, so that is what is in agreement okay.

So you can plot and then visualize similarly, if you increase the distance then the shearing effects or reduced that is expected right because the impact of the movement of the plate will not reach the other plate. So naturally if you keep on increasing, there is no shearing effect so that is in agreement, so for a particular length within a length you can add the corresponding measurement x we are talking about so it is like this.

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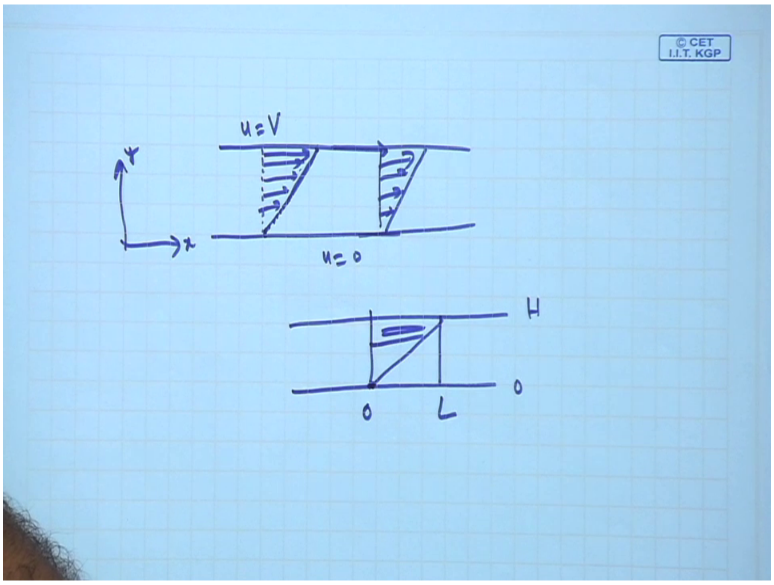

So far we are talking about only integration between 0 to H, but now within this length of sharing how much sharing is happening okay. That quantifies quantification this is shearing happening that quantification is given by this, so naturally sharing is proportional to viscosity. So that is what if you get it okay, so these are some important physical insights even with such a elementary problem okay. So now let us a slightly complicate okay, not complicating we are adding another feature you can say like that okay.

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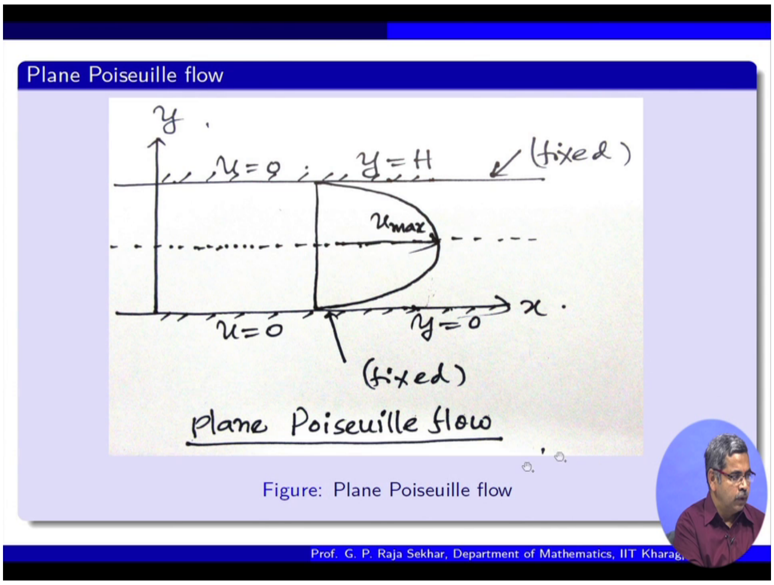

So what is the additional feature so we are now both plates are fixed then the question comes who is driving the flow if you recall we have killed the pressure gradient in the earlier case, right so now we have fixed the place, therefore somebody has to drive the flow that means we are now anticipating pressure is driving the flow okay. I would not talk about this profile right away okay, we will see this later on.

So this plate is fixed this plate is fixed located at x = 0, located y = H, therefore we have stationary plates corresponds to no slip there u = 0 and u = 0 okay.

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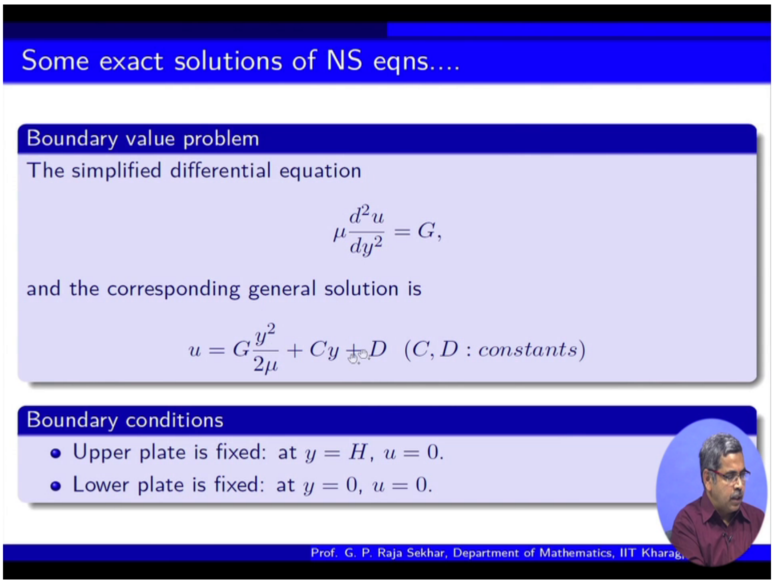

Again same assumptions instead of the earlier assumption of pressure gradient is 0 we are taking pressure gradient is constant okay. So once pressure gradient is constant this is a constant therefore, this has to be constant okay. And both upper and lower plates are stationary that is the assumption, then we get a simplified equation which admits the corresponding solution, still we have two arbitrary constants right.

So go for the corresponding no slip boundary conditions eliminate the constants, only additional contribution is this time you have a dominating quadratic term which is not 0. You see when I take y = 0, u 0, this constant goes off but when you take u = 0 at y = H, this constant survives and this is very much, there so the dominating part of this profile is quadratic that means you expected quadratic profile okay.

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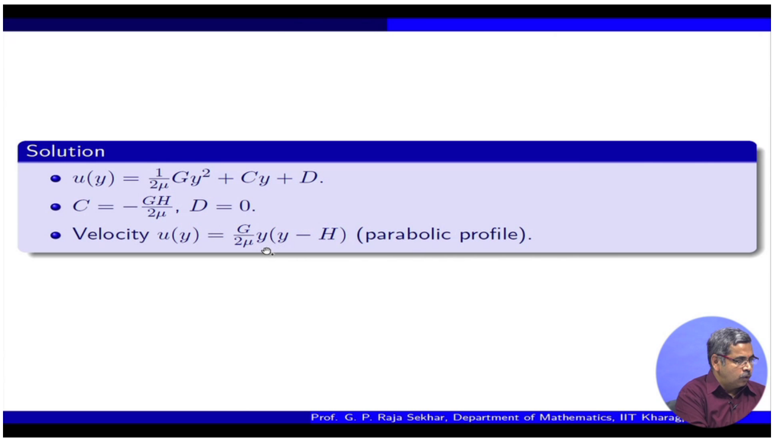

So this is the velocity and coefficients are determined, and this is the parabolic profile okay, so what is happening here you have stationary plates.

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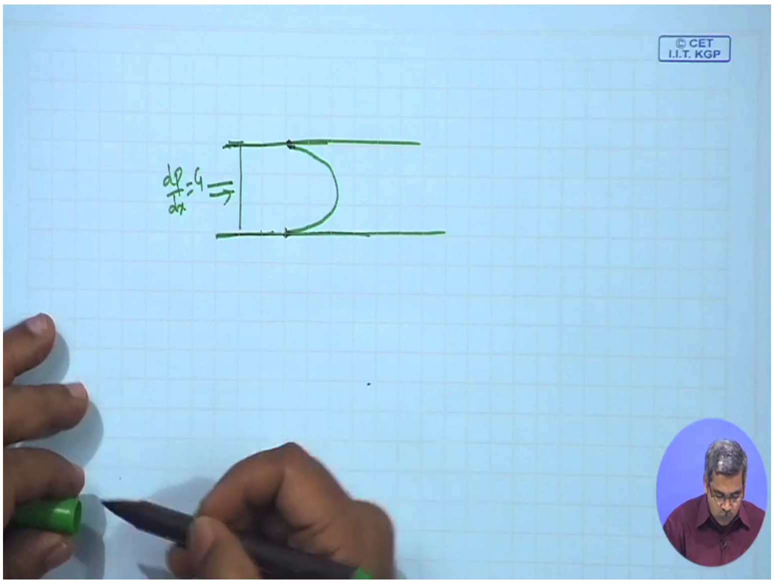

But the flow is driven buying constant pressure gradient okay. So then, when this is stationary viscous effects are more prevalent at these boundaries. Therefore due to no slip we expect the some pressure is pushing from this side, but viscous effects are more prevalent since the plates are stationary. We expect the particles other into this should retain the velocity of the plates, but to the pressure is driving the flow, so beyond certain distance we expect this profile, why because it is symmetry right.

Now you see the profile that what is happening and this is very much a satisfying y = 0 this is 0, y = H this is 0, so that is what exactly the flux are in our profile okay. Now magnitude definitely magnitude depends on the magnitude of the pressure gradient, if you are having a large pressure gradient then correspondingly the velocity is expected to be large or otherwise correspondingly low and the viscous effects you have a water in the channel okay.

Then you take some oil, you maintain the distance same, you maintain the pressure gradient the same, it is natural that the velocities for oil will be much less than water so that is reflected in this solution, so these are very elementary problems but we can really nicely capture the corresponding physics okay.

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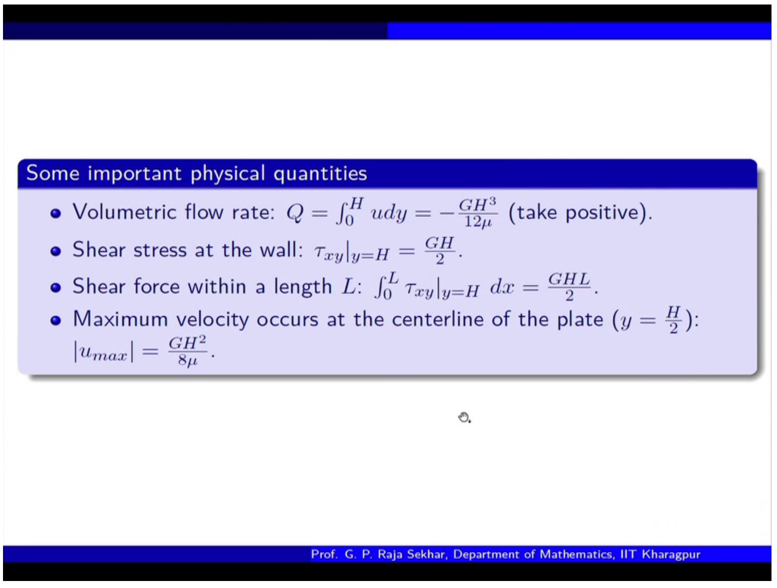

Now we can compute the volumetric flow rate again using the similar arguments, as we have given shear stress at the wall and the shear force within the length okay. So this point I said that, we will discuss little later, the maximum velocity occurs and the centre line of the plate from velocity, you can compute the maximum, maximum occurs at the centre line why so this is what is happening.

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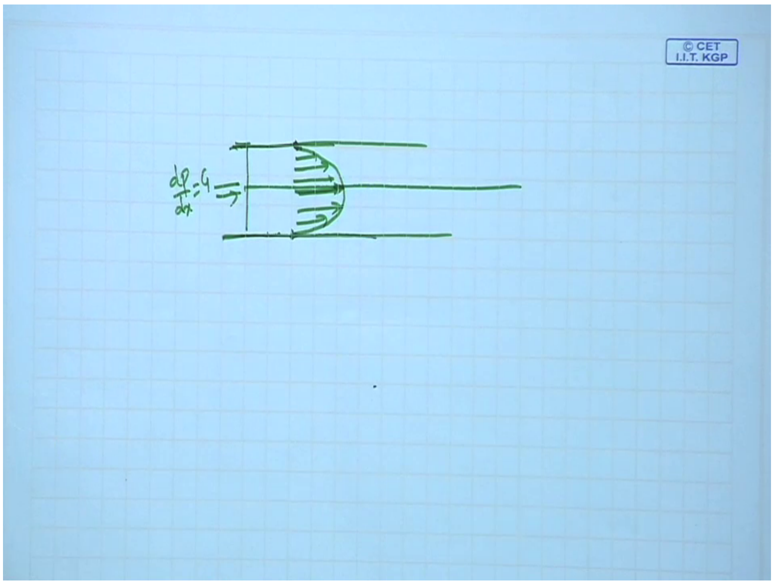

It is symmetric and both the plates are rest or so maximum at the centre line okay. So the maximum velocity of the centre line can be computed it is varying over square of the distance between the plates and it is always proportional to the pressure gradient okay. So one can normalize so some people can define G by 8 as a Mu G, that means we are normalized in the pressure gradient and simply call it, it is some GH square by Mu so there is no pre factor so that is also these are some adjustments okay .

So far we discussed very simple cases so we did not really appreciate how the Navier stokes equation gets reduced into such simple form because it is unidirectional so left hand side is 0 and then the y momentum is almost gone etc., etc., okay. Now let us move on to a Curvilinear coordinate system, say very simple cylindrical system then see how the Navier Stokes equations really reduces under particular assumption.

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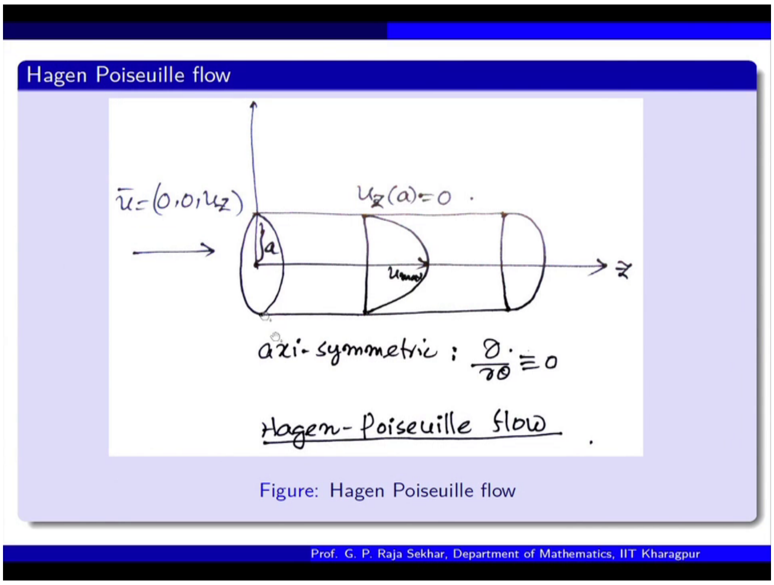

So what is the next case that we are going to discuss the flow between channel is called a plane Poiseuille and flow inside a pipe is called Hagen Poiseuille flow okay. The assumptions are listed here, it is again unidirectional but along z direction and this is another assumption. What do you mean by axis-symmetric? Okay.

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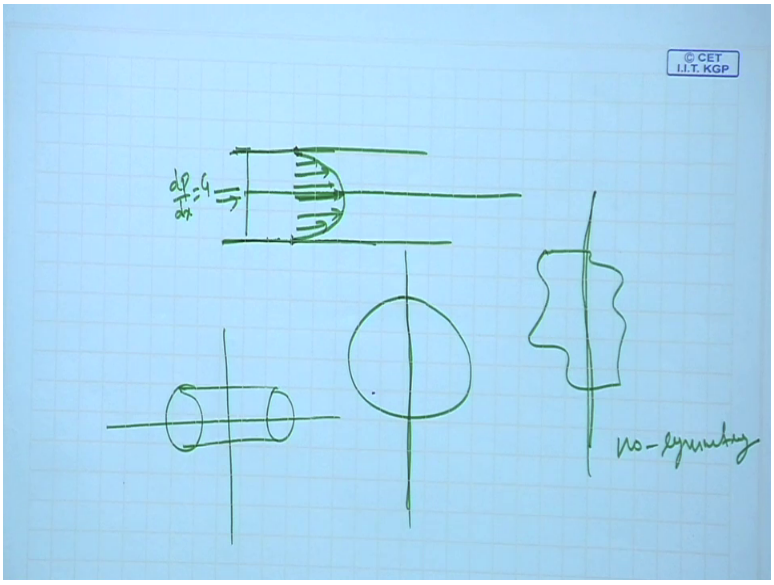

So you take some faced some object like this, then go for an axis so then can you call it this is acts of symmetry unfortunately no symmetry okay. Whereas you take like this assuming I have drawn a nice picture so this is symmetric, similarly you take a cylinder you have this, this is also symmetry, and this is also symmetry axis okay. So now when we define flow quantities which are independent of Theta in cylindrical coordinate systems, so that means you have the corresponding rotations happening okay, and they are invariant.

So the flow quantities are independent of Theta and whereas in spherical coordinates the flow quantities are invariant with respect to the azimuthal angle 5, so then the corresponding flow is called axisymmetric okay, so we assume flow is excessive metric in this case okay.

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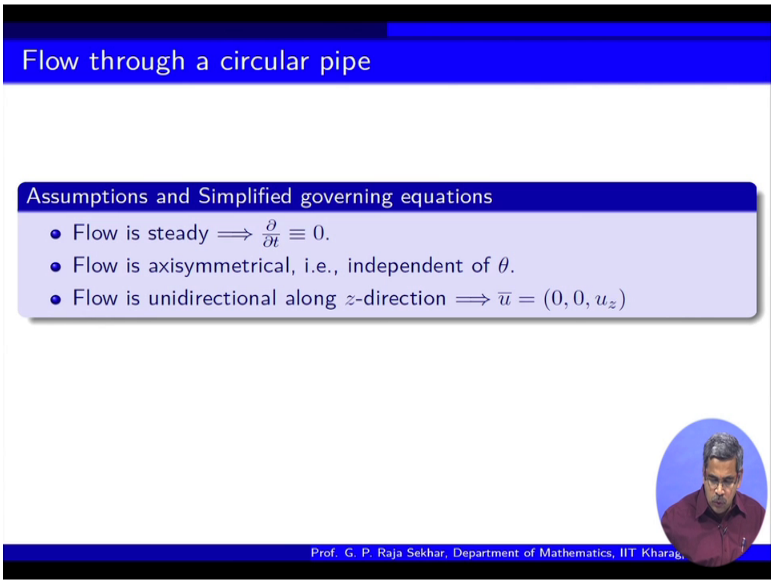

Now very straightforward assumption every time we are doing flow is steady okay, then axisymmetric that is independent of theta then flow is unidirectional along z direction okay. So here I would like to show you how the corresponding Navior stokes reduces slightly okay so that you get some idea.

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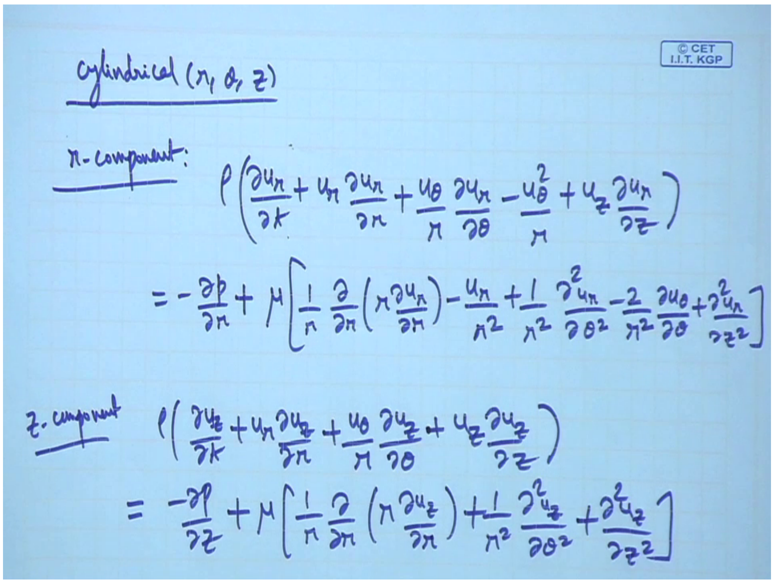

So what I am trying to do is we are considering cylindrical (r, Theta, z) coordinate system, then we are writing r component okay, so maybe we discuss maybe one or two components not all, so this is full Navier-Stokes I am writing in component form just to get an idea, so the suffix indicates the component not the partial derivatives, so this is a local derivative and this is the convective derivative and the corresponding pressure gradient then the viscous term.

So here you can appreciate the role of curvilinear co-ordinate system compared to the Cartesian, in the Cartesian we simply could take the Laplacian on component u and component V, here we cannot do that because the unit vectors they are functionally depending on the co-ordinate system. So therefore we cannot do that, so one has to operate the Laplacian on the corresponding unit vectors okay.

So flow is along z direction therefore we must appreciate by writing the corresponding z component. So Theta we are not so much interested in particular the current case, our assumption is flow is axisymmetric, so therefore we are not so much focusing, once we use these assumptions, you will realize immediately what will be the corresponding equation for a Theta component.

So this is the left-hand side okay, so we have considered two components and the please pay attention to our assumptions steady anyway slow is axisymmetric independent of Theta and unidirectional so we are considering along z direction okay.

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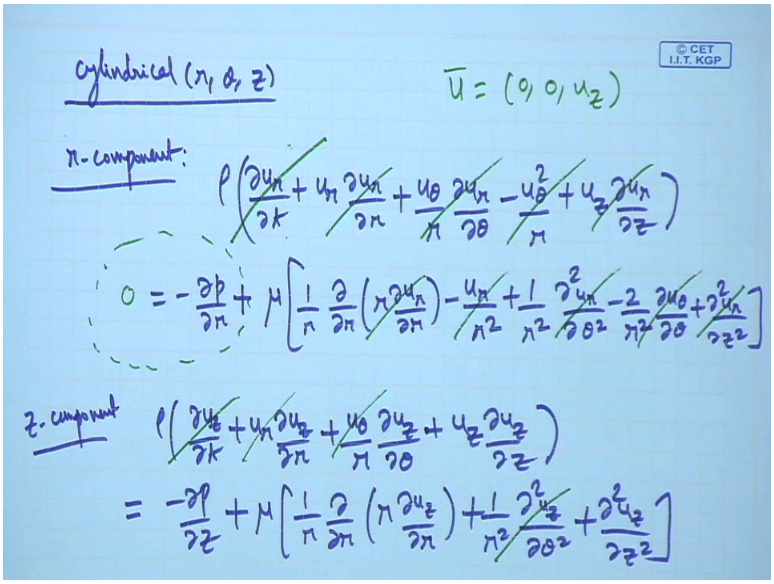

Now please come back, so since it is steady this is gone and the major assumption is u bar is this is our major assumption. So what happens to our momentum, all are gone then this is gone so all that we are getting is, we are getting this okay, then here steady this is gone since uz is nonzero but, there is you are so therefore this term is gone same is case u Theta is gone and this here we keep it for the time being because user is nonzero, then this we keep it axisymmetry, this is one okay.

So we have a reduced equation, so you will see that is the reason we are discussing this case because you will still feel that we are really at non linear equation okay. But one equation we did not use yet that is not the Theta component. If you see Theta component what you really get is complete left hand side is 0, right hand side viscous terms are 0 simply we get Dow p Dow Theta is 0 from Theta component okay nothing much.

What is the other equation that we have not used, which is equation of continuity, so when you have unidirectional what happens to the equation of continuity. For the time being or let us have this we will revisit, so let us go to the equation of continuity.

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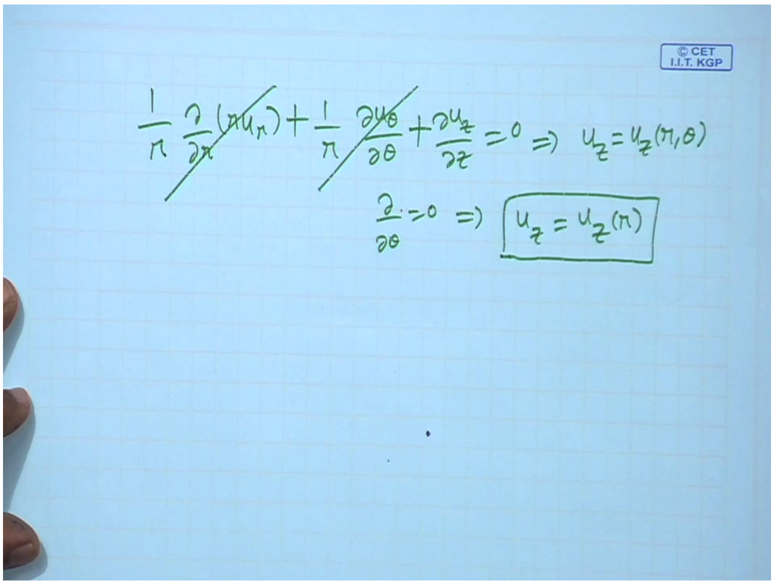

So that will be yeah okay. So now unidirectional so therefore this is gone, this is gone; this implies uz is uz of r and Theta at this stage. But we have an axisymmetry this implies uz is okay, so with this assumption we would like to revisit our z component now.

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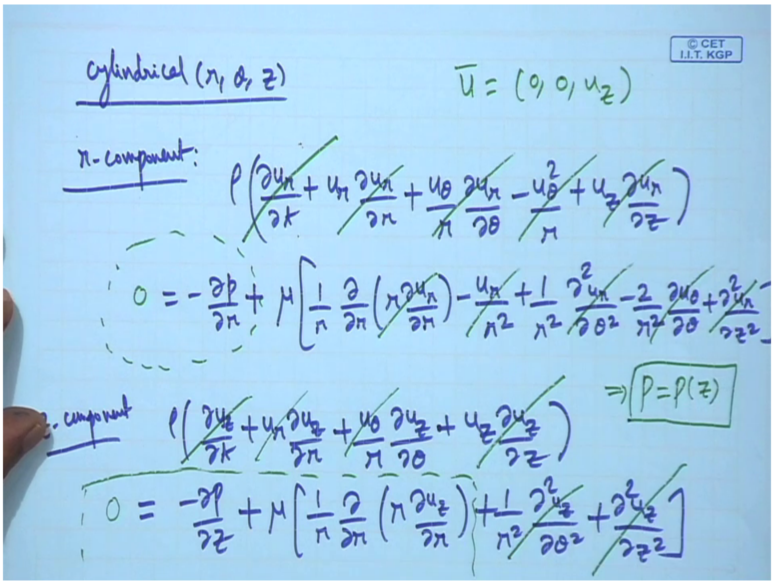

Now you see the non-linear term we have uz is function of r only, therefore this term also goes off, then this term also goes off, what you are getting is so this simplified linear equation, we are getting okay. And we have Dow p Dow r 0 and Dow p Dow Theta 0, so immediate inference is p is function of z okay.

So this is the immediate inference, p is function of z. So now the issue is we have Dow p Dow z, so similar to the case of flow between parallel plates driven by a constant pressure gradient you considered Dow p Dow z is equal to some constant, and then we can solve the corresponding linear ODE, right. So let us see that all that we have discussed uz function of r, I have explained r momentum Theta momentum this is already discussed, z momentum reduces to such a simple form now you assume that pressure is constant.

Then this linear equation can be obtained okay. Now when we are discussing about interior flows this is a case of interior flow right because flow inside a piper so when we discussing about boundary conditions, in particular interior flows. We, we had a remark what was that the question to be asked is are we allowing any source or sink the interior flows, if you are not allowing then you need a boundedness condition okay.

So as you can see the solution is triggering a logarithmic similarity right, so what are the boundary conditions no slip boundary condition on only one boundary that is surface so uz is 0. Regularity condition that is uz should be bounded at r equal to 0, this is required if you are not allowing singularities, if somebody is allowing singularity, so then corresponding source or sink has to be given okay.

So here we assume that there are no singularities inside the pipe therefore uz is bounded immediately to make user bounded, we kill the coefficient A that is 1 and r = A will be used to determine B so that is straightforward. As I indicated boundednesss condition gives you a zero and no slip gives you B and we get this okay.

So the corresponding profile is also quadratic with radial distance so you get the corresponding profile, okay.

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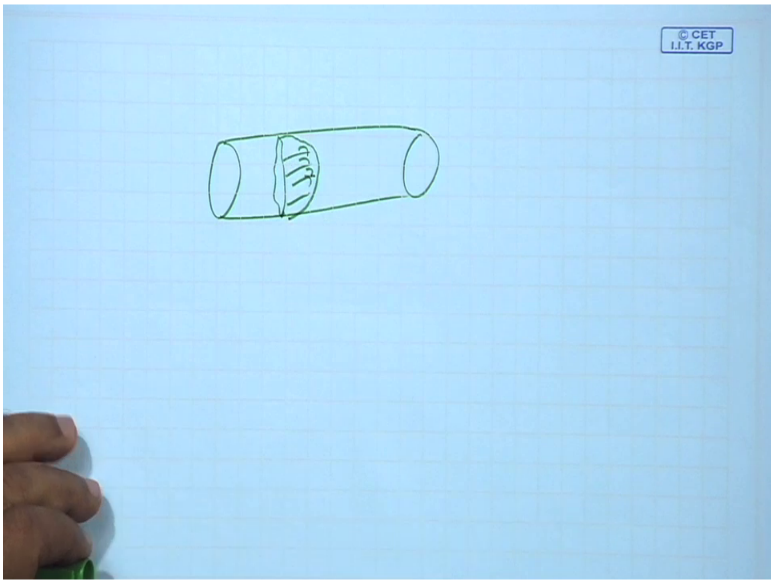

It is like this, again is the profile why because the entire exterior is a stationary okay. So now let us see so this already we have discussed for a parabolic maxima lost occurs at the centre of the tube okay, due to know here at the centre so that can be computed, then average velocity okay.

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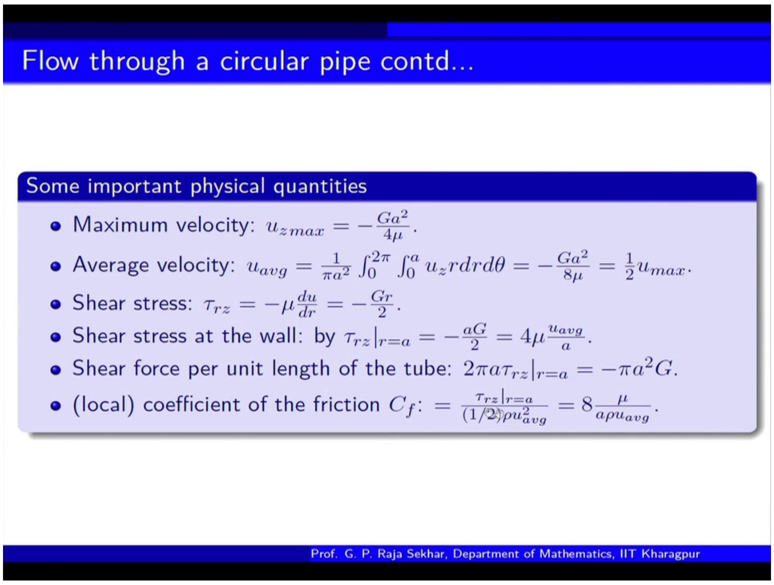

So this is interesting, so what we have taken we have taken the corresponding surface element integrated the velocity and normalized by the area of the corresponding cross section okay. So that is what this is the average velocity okay so that means we have a taken cross section and then we have computed okay.

So there is a relation average velocity is half of a u maximum okay, then shear stress rz, r is this so that is a linear relation with respect to the co-ordinate. Shear stress is having a linear relation with respect to the radial co-ordinate, then shear stress at the wall okay. So you may be wondering why repeatedly we are computing the shear stress at the wall etcetera, so in most of the physical problems this is a lot of applications the interior also you can computer there is no problem.

Once you have a general expression you have everywhere the shear stress, but why repeatedly we are computing on the boundary, so these are the border cases where lot of physics happens, so in most of the applications you have to estimate something called a skin friction, that is the shear stress on the surface so this has a lot of importance. For example if you are talking about flow inside arteries or thin endothelial cocalix, So the shearing will.

If you increase the shearing suppose you inject the drug and that is inducing lot of shearing the tissue gets damaged, so these kind of things can be estimated okay, so that is the reason one has to estimate the corresponding the skin friction okay. So that what we are also computing coefficient of skin friction is the ratio of shear stress on the surface by the corresponding average velocity okay, times the mass okay.

So this enjoys this relation, so those who are interested in analysing, more depending on the skin friction so we pay more attention okay and then analyse. So with this you get an idea about simple geometries and what is the corresponding boundary condition and then how the solutions can be obtained in exact senses, that is reason we are, we stated this exact solution Navier Stokes.

But we have to take it with an assumption that these are for a simple configurations which ultimately are leading to linear ordinary differential equations, for which it theories well-developed therefore exact solution okay. So once you have a non-linear then now we have to go for various alternative techniques okay thank you