Probability and Statistics
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Lecture – 08
Properties of Probability Function- II

Probability function we have assuming to be countably additive. But countably additive axiom implies that if we have a disjoint sequence then the probability of union is equal to the sum of the probabilities. What if we do not have disjoint sequence? For example, if I have two sets say A and B then we have probability of A union B is equal to probability A plus probability B minus probability of A intersection B.

That means, if I remove probability of A intersection B from there then we get probability of A union B less than or equal to probability A plus probability of B. This is called Subadditivity. So, if in general if we consider any sequence of sets then the probability of union will be less than or equal to the sum of the probabilities.

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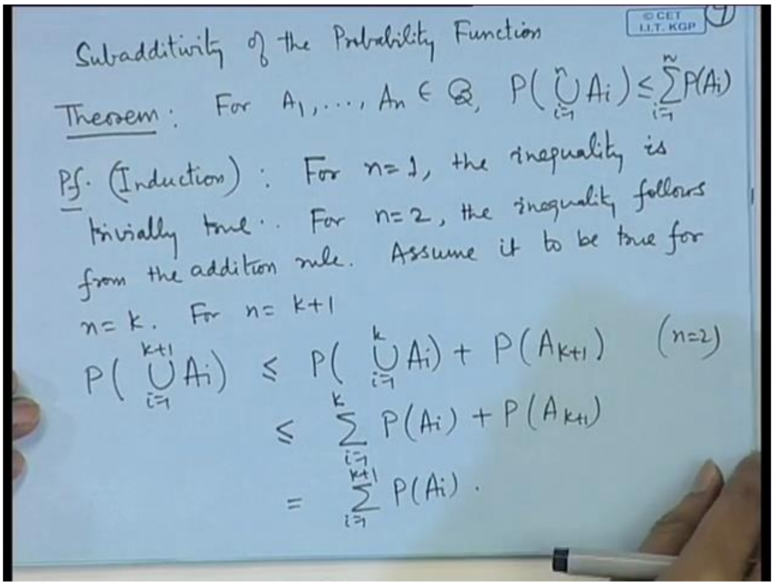

So, we have subadditivity of the probability function. Now we can state in the form of a theorem for A 1, A 2, A n belonging to B probability of union Ai i is equal to 1 to n is less than or equal to sigma probability of Ai i is equal to 1 to n. So, one can prove this by induction because for n is equal to 1 the result is true. And if we look at for n is equal to 2 it is already shown to be true. So, for n is equal to 1 the inequality is trivially true. For n is equal to 2 which we will require for extension from k to k plus 1 case, so for n is equal to 2 the inequality follows from the addition rule. So, assume it to be true for say n is equal to k.

Now, for n is equal to k plus 1 we can write probability of union Ai i is equal to 1 to k plus 1 as less than or equal to probability of union Ai i is equal to 1 to k plus probability of A k plus 1, by using the result for n is equal to 2. So now, on this we can make the use of assumption that up to n is equal to k it is true. So, it becomes less than or equal to probability of Ai i is equal to 1 to k plus probability of A k plus 1, which is nothing but the sum of the probabilities i is equal to 1 to k plus 1. Therefore, by induction the result is true for all of them.

Now, if we want to prove the result for a countable number of these then we can consider the decomposition.

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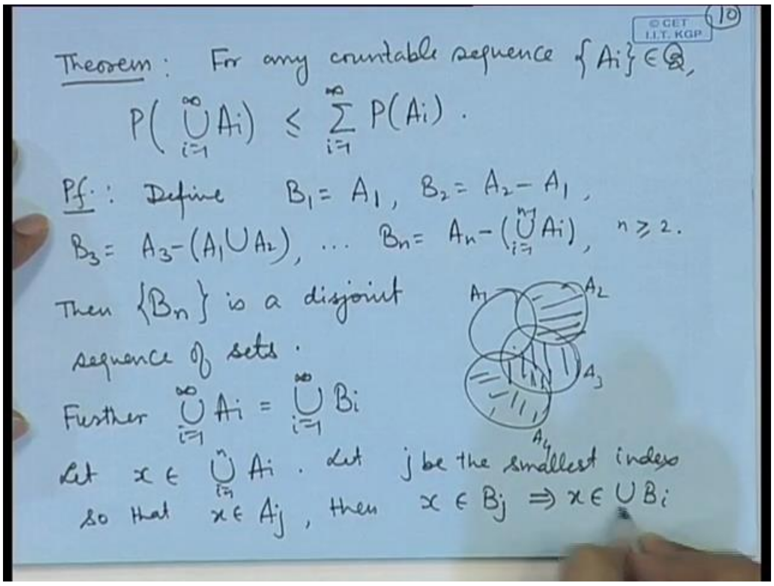

So, if we have for any countable sequence say Ai in B probability of union Ai i is equal to 1 to infinity is less than or equal to sigma probability of Ai i is equal to 1 to infinity. In order to prove this one, one may consider the decomposition of union Ai into a disjoint decomposition in the following way. Let us define say B 1 is equal to A 1, B 2 is equal to A 2 minus A 1, B 3 is equal to A 3 minus A 1 union A 2 and so on. In general Bn is equal to An minus union of Ai i is equal to 1 to n minus 1.

If we consider a Venn diagram then it will be clear that what sets we are defining. Suppose these sets are A 1, A 2, A 3, A 4 etcetera. Then A 1 and then A 2 minus A 1 is this set, then A 3 minus A 1 union A 2 becomes this set, A 4 minus A 1 union A 2 union A 3 becomes this set. Naturally you can see here that we are considering the union as a disjoint union. So, then Bn is a disjoint sequence of sets.

Further union of Ai i is equal to 1 to infinity is equal to union of Bi i is equal to 1 to infinity. To prove this let us observe that union of Bi is already a subset of union Bi, because each of the B is a subset of the corresponding Ai's. Now any point of Ai, let us consider say x belonging to union of Ai. Let j be the smallest index so that x belongs to a j, then x will belong to Bj consequently x will belong to union of Bi's.

As a result since union of Bi is already a subset of union of Ai we are now getting union of Ai is a subset of union of Bi. Therefore, we must have union of Ai is equal to union of Bi's.

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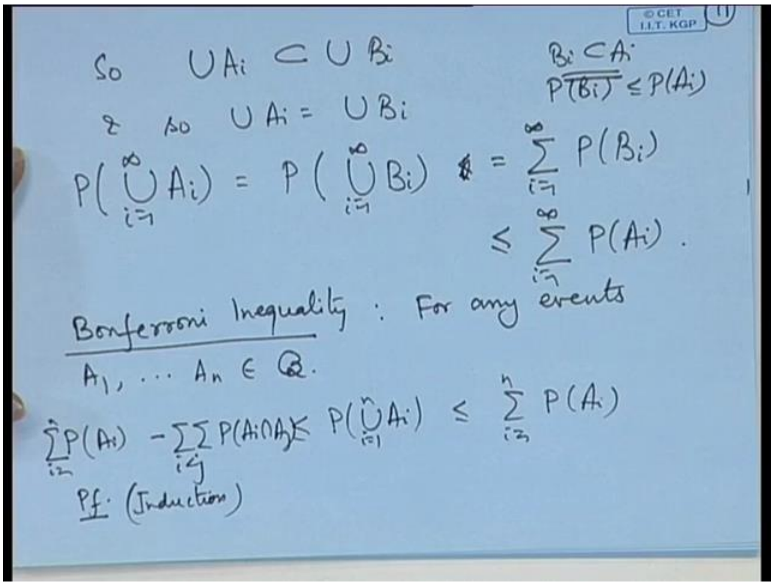

So now, if we consider probability of union of Ai i is equal to 1 to infinity it is probability of union of Bi i is equal to 1 to infinity which is less than or equal to, which is actually equal to sum of the probability of Bi's because Bi's are now disjoint and we can use the axiom of countable additivity. Now, each Bi is a subset of Ai, therefore probability of each Bi is less than or equal to probability of Ai, therefore this becomes less than or equal to sigma probability of Ai i is equal to 1 to infinity. This proves the countably additivity of the probability function.

We also have something called Bonferroni inequalities which basically give that the probability of the unions are bounded between two bounds. So, for any events A 1, A 2, A n in B probability of the union which is already less than or equal to sum of the probabilities, it is however greater than or equal to probability of minus. Let me not prove it here the proof will be by induction.

We can see; the right hand side has already been proved. To prove the left hand side if we take n is equal to 1 then it is trivially true. For n is equal to 2, there is equality by the addition rule. So, assuming for n is equal to k if we write for n is equal to k plus 1 then we can split it into two terms that is union of Ai i is equal to 1 to k union A k plus 1, on that we apply the addition rule and then apply the assumption for k. That will prove the general Bonferroni inequality.

In a similar way we have what is known as Boole's inequality.

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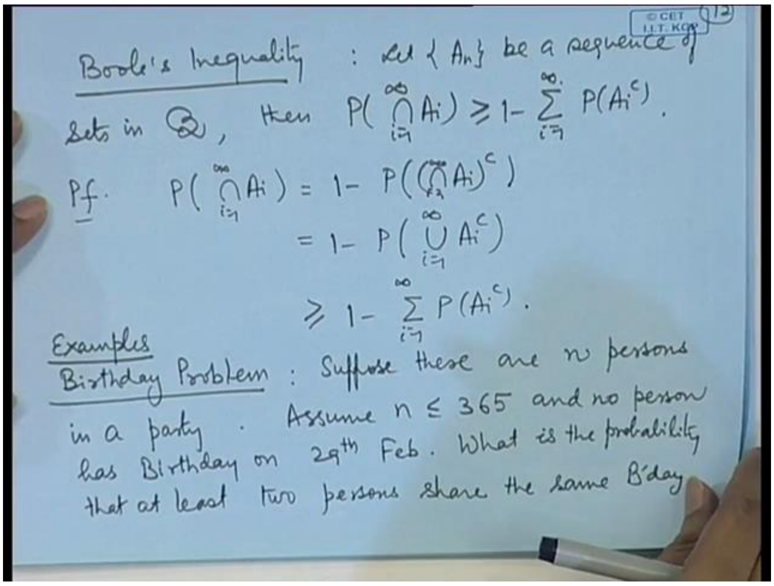

The Boole's inequality gives a relation between the intersections likewise. For example, if I have An is any sequence of sets in B, then probability of intersection Ai i is equal to 1 to infinity is greater than or equal to 1 minus sigma probability of Ai complement i is equal to 1 to infinity. To prove this we simply use the subadditivity, because we can write probability of intersection Ai i is equal to 1 to infinity as 1 minus probability of intersection Ai complement. Now this can be written as 1 minus probability of union Ai complement by using De Morgan’s laws.

At this is stage we can use the countable subadditivity. So, this will become greater than or equal to 1 minus sigma probability of Ai complement. Let me give some examples of applications of basic rules of probability. Let me start from a Birthday Problem. Suppose, there are n persons in a party; assuming that the number of persons is less than or equal to 365 and no person has birthday on 29th February. What is the probability that at least two persons share the same birthday?

In order to analyze this problem let us consider the set theoretic description.

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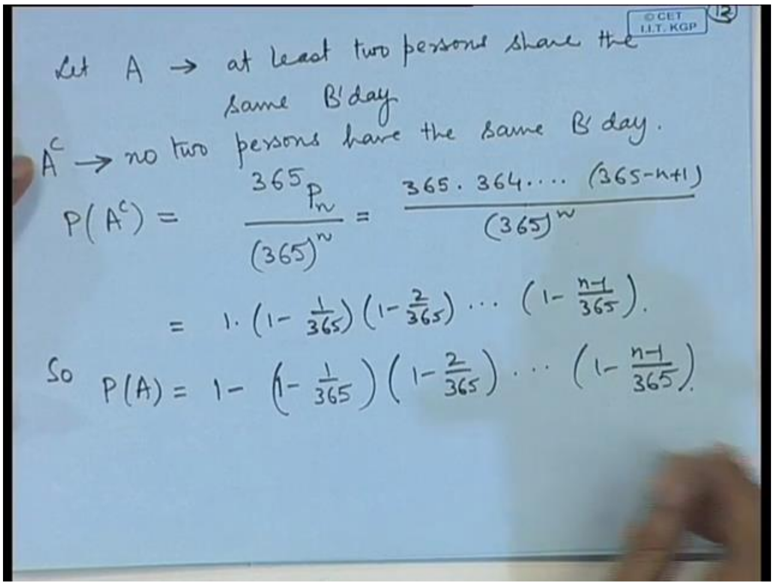

Let us consider A to be the event that at least two persons share the same birthday. Then if you look at this event it is slightly complicated event in the sense that two persons may share three persons may share and so on. And finding out the probabilities of each them may be a little bit complicated. Because if we say two persons share then which of the dates and all others must be on some other dates and they should not be the same. Suppose we say three persons share then which one of the 365 days, and all other persons must be on distinct days which distinct days. So, this is a complicated way to analyze.

However, if we use the set theoretic representations we can look at the complementary event; A complement, this means no two persons have the same birthday. Now this becomes somewhat simpler, because if we look at the probability of A complement assuming all the birthdays to be equally likely this number will be simply 365 Pn divided by 365 to the power n. Here the denominator denotes the total number of possibilities for n persons to have birthdays, because each person can have any of the 365 days as a possibly birthday. And therefore, n persons can have possible number of birthdays as 365 to the power n.

If we make the assumption that none of them have the same birthday then it becomes a problem of choosing n numbers out of 365 which are distinct. So it is nothing but the number of permutations taken n at a time from 365; that is equal to 365 into 364 up to 365 minus n plus 1 divided by 365 to the power n, which we may write as a way of representation as 1, 1 minus 1 by 365 1 minus 2 by 365 and so on up to 1 minus n minus 1 by 365. And so probability of A becomes 1 minus the product given by these terms.

An interesting thing would be to look at that how many people are required so that at least two will share a birthday. If we think from a Layman point of view then we may think that the numbers since the number of possible birthdays is 365 to the power n. So, n should be somewhat large in order that this probability is significant. So, let us look at the table of probabilities.

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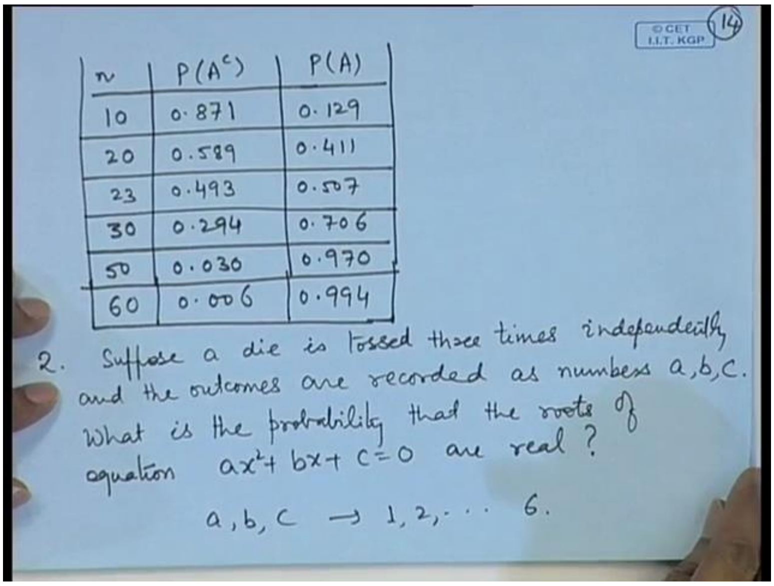

Let us consider say n probability of A complement and probability of A. So, simple calculation table can be prepared if I have n is equal to 10 then the probability of A complement is 0.871 and consequently probability of A becomes 0.129. If we take n is equal to 20 probability of A complement is 0.589 and probability of A becomes 0.411.

If we take n to be 23 then probability of A complement is 0.493 and probability of A becomes 0.507. That means, with as less as only 23 persons the probability that at least two share a common birthday is more than 50 percent. So, it is from a Layman's thinking this is counter intuitive we need very few persons to at least two of them to share a common birthday. If I take n is equal to 30 then this probability becomes 0.706. If we take 50, the probability is 0.97 and for n is equal to 60 the probability is 0.994; it is nearly 1. That means, in a set of 60 people the probability is nearly 1 that at least two of them will share a common birthday.

So, here you can see that the elementary rules of probability have been used for calculation. For example, we have used the property of the complementation to evaluate the actual probability we have used the method of classical probability by assuming all the birthdates to be equally likely for all the persons.

Let us look at some other applications of the basic rules of the probability. Suppose a die is tossed three times independently and the outcomes are recorded as numbers a, b, and c. What is the probability that the roots of equation ax square plus bx plus c is equal to 0 are real. Now, if we want to calculate this probability here the outcomes a, b, and c are random; each of the values of a, b and c can be numbers 1, 2 up to 6. Therefore, the quadratic equation ax square plus bx plus c is equal to 0 will have the real roots if b square minus 4 a c is positive. So, we have to look at the number of cases where b square minus 4 a c is greater than or equal to 0.

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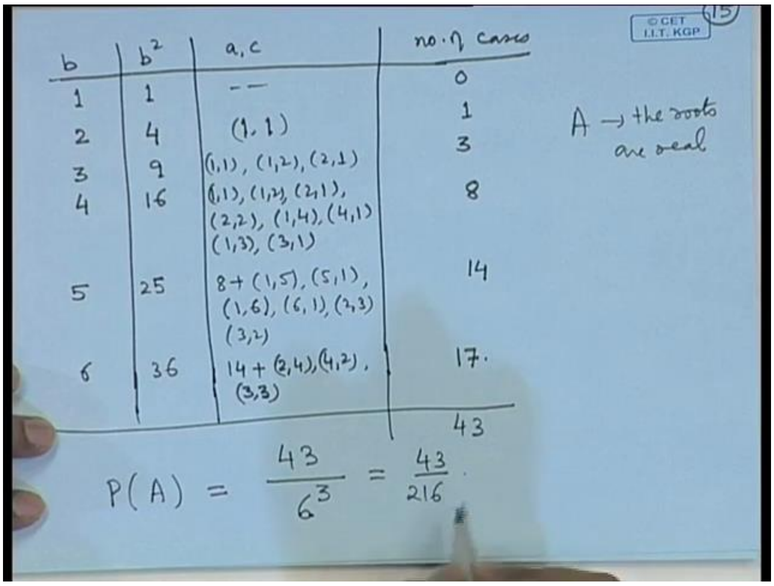

So, this has to be done through an enumeration and we can prepare the table that, what are the possibilities of b, and therefore the corresponding values of b square, what are the possible values of a and c which lead to 4 a c being less than or equal to b square. So, let us take say b is equal to 1, then b square is equal to 1. That means, there is no case which will give me 4 a c to be less than or equal to b square, so there is no possibility here. So, if you look at the number of cases this is 0.

If we take b is equal to 2 then b square is equal to 4. And if we I consider a and c to be 1 1 then 4 a c will become 4. So, there is one case which will give me b square greater than or equal to 4 a c. If we consider B is equal to 3 then b square is equal to 9, now 1 1, 1 2, and 2 1; there are three cases which will give me b square greater than or equal to 4 a c. If we have b is equal to 4 then b square is equal to 16 we will have the cases 1 1, 1 2, 2 1, 2 2 which will correspond to 4. So, 1 4, 4 1, 1 3, 3 1; 1 2 3 4 5 6 7 8 cases are there which will give me b square greater than or equal to 4 a c.

If we have b is equal to 5 then b square is equal to 25. Then all the above cases that is 8 cases plus we will also have 1 5, 5 1 and possibly 1 4, 4 1. So, we will also have 1 6, 6 1, 2 3, 3 2 basically 1 2 3 4 5 6 more cases. So, 14 cases are there. If I have b is equal to 6 then b square becomes 36. And all the 14 cases plus we will also have 2 4, 4 2 then 2 5 is not possible, 3 3 I think 3 3 must have come here itself because no it will not come here 3 3 will come here because this will give me 9. So, there are 17 cases.

So, if you look at the total number of cases it is 39 42 - 43 cases are there, total number of cases is 43. And the total number of possibilities if I define A to be the event that the roots are real then the probability of that will be given by the favorable number of cases divided by the total number of cases which is 6 cube here, because 3 dies each of them have 6 possibilities. So, the total numbers of possibilities are 6 cube, there is 43 by 216.

Likewise in this problem we may also find out the probability of the quadratic equation to have complex roots or the real roots were equal etcetera. We may consider all types of possibilities. So, I will end today's lecture by this.

Thank you.