Probability and Statistics

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Lecture – 07
Properties of Probability Function – I

In the last lecture I have introduced the Axiomatic Definition of Probability.

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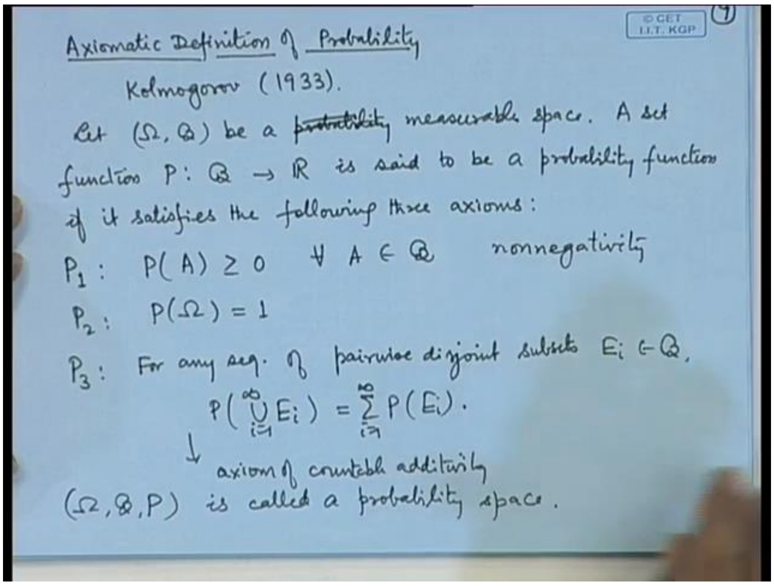

This takes care of the deficiencies or drawbacks left by the classical definition or the relative frequency definition of probability. So, in this definition we give a general framework under which a probability function is defined. This does not tell you how to calculate a probability, but a probability function must satisfy these axioms in order to be a proper probability function.

So, in particular if we have a sample space and a sigma fluid of subsets of that sample space let us call it a script B then a set function P from B to R is said to be a probability function if it satisfies the given three axioms which we name P 1, P 2, P 3; the first is the non negativity axiom that is the probability of A is greater than or equal to 0 for all A belonging to B, so this is the axiom of non negativity. Then probability of the full sample space is 1, basically it makes the P 2 to be a finite function. And the third axiom is the axiom of countable additivity that is for a given pair wise disjoint sequence of sets probability of the union is equal to the some of the individual probabilities.

Thus, this omega B and P is called a probability space.

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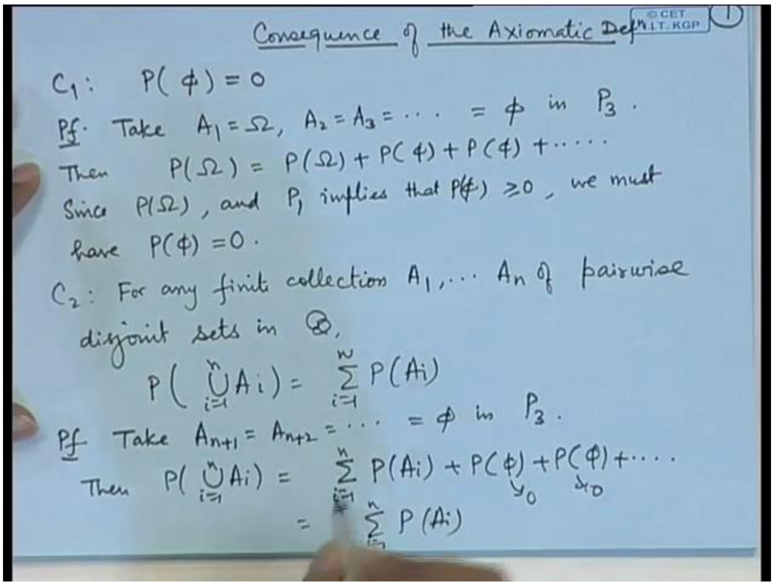

Now, some of the consequences of the axiomatic definition are as follows: the first consequence is let me call it C 1 that probability of the impossible event must be 0. To prove this statement let us take A 1 is equal to say omega and A 2, A 3 etcetera to be phi in axiom P 3. Then we will get probability of omega is equal to probability of omega plus probability of phi plus P of phi plus P of phi etcetera. Since, P omega is 1 and P 1 implies that P phi is greater than or equal to 0, we must have P phi equal to 0.

The second consequence is that for any finite collection A 1, A 2, A n of pair wise disjoint sets in B probability of union Ai i is equal to 1 to n is equal to sigma probability of Ai i is equal to 1 to n. Let me explain this that why do we need this finite additivity consequence here to be proved. We have assumed the countable additivity axiom, but that does not necessarily imply the finite additivity.

A proof of this can be given using the fact that in A 3 we can take A n plus 1 A n plus 2 etcetera to be phi in the third axiom. Then we will get probability of union Ai i is equal to 1 to n is equal to sigma probability of Ai i is equal to 1 to n plus P phi plus P phi etcetera. Now, if you use consequence one here then these terms are 0 and we get sigma probability of Ai i is equal to 1 to n.

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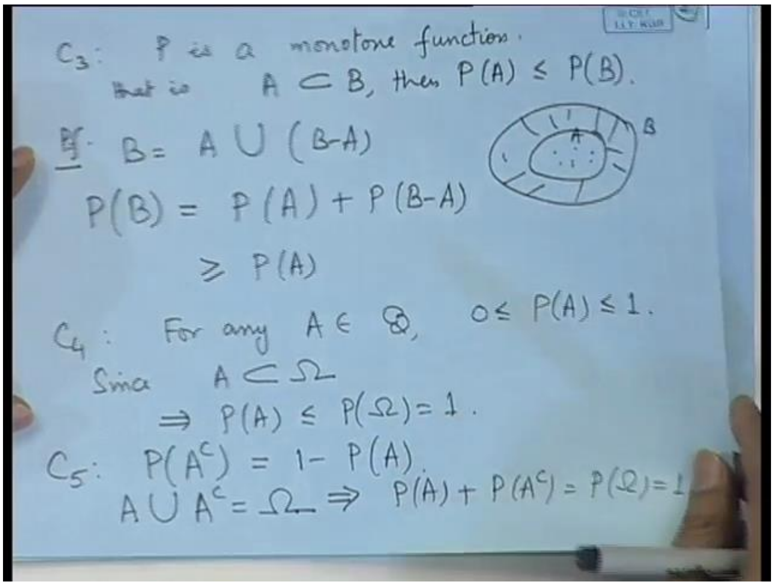

A third consequence is the P is a monotone function, that is if I take say A to be a subset of B then probability of A will be less than or equal to probability of B. Let us look at the proof of this.

Consider say a set A and a set B then I can write B as A union B minus A this is B minus A and this is A. And these two are disjoint, so if I make use of the finite additivity consequence then probability of B is equal to probability of A plus probability of B minus A. Naturally this is greater than or equal to probability of A since probability of B minus A is always greater than or equal to 0.

As a further consequence we have that for any event A probability of A lies between 0 and 1. Now the first part of this is always true because of the P 1 axiom. Now A is a subset of omega for every omega for every A, this implies that probability of A is less than or equal to probability of omega that is equal to 1. If I consider probability of A compliment then it is equal to 1 minus probability of A. This follows because I can write A union A compliment as omega, and therefore probability of A plus probability of A compliment is probability of omega that is equal to 1.

Now, we look at certain further consequences or the definition the first of them is the addition rule.

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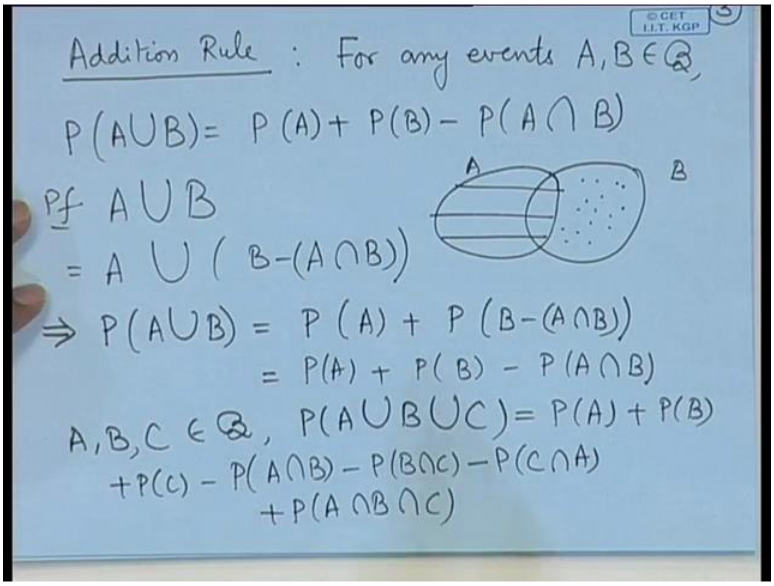

For any events A and B probability of A union B is equal to probability of A plus probability of B minus probability of A intersection B. In order to prove this is statement let us consider any two sets A and B. Then A union B can be expressed as A union B minus A intersection B. So, we can write A union B as A union B minus A intersection B. You can observe here that this is a disjoint union; therefore if I consider probability of A union B it is equal to probability of A plus probability of B minus A intersection B.

Now, at this stage we notice that A intersection B is a subset of B and if we look at the statement that probability of B is equal to probability of A plus probability of B minus A then this implies that probability of B minus A is equal to probability of B minus probability of A. That means, if A is a subset of B then probability of B minus A can be expressed as probability of B minus probability of A. Therefore, here we can write this as probability of B minus probability of A intersection B.

Now, naturally one can think of the generalization of this rule. For example, if I consider say for three events: suppose A, B and C are three events then we must have probability of A union B union C is equal to probability of A plus probability of B plus probability of C minus probability of A intersection B minus probability of B intersection C minus probability of C intersection A plus probability of A intersection B intersection C.

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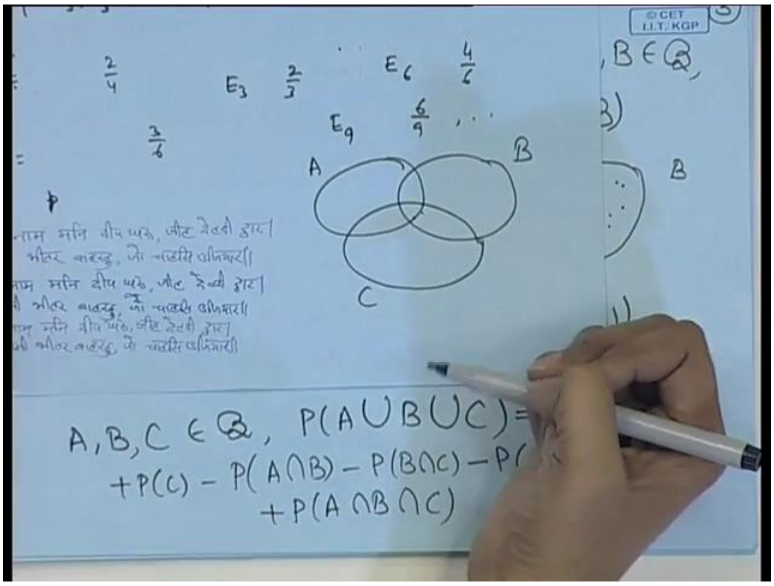

One can look at this statement from the point of view of set theory or Venn diagram. If I consider three events say A, B and C. Then the union can be expressed as A union B union C. However, here we have to remove A intersection B, B intersection C and C intersection A. If we remove that then the set A intersection B intersection C has been removed three times. So, we have to add it once to get this portion here. So, A intersection B intersection C has to be added here.

So, this gives us a rule for considering a general addition rule and we have the following result.

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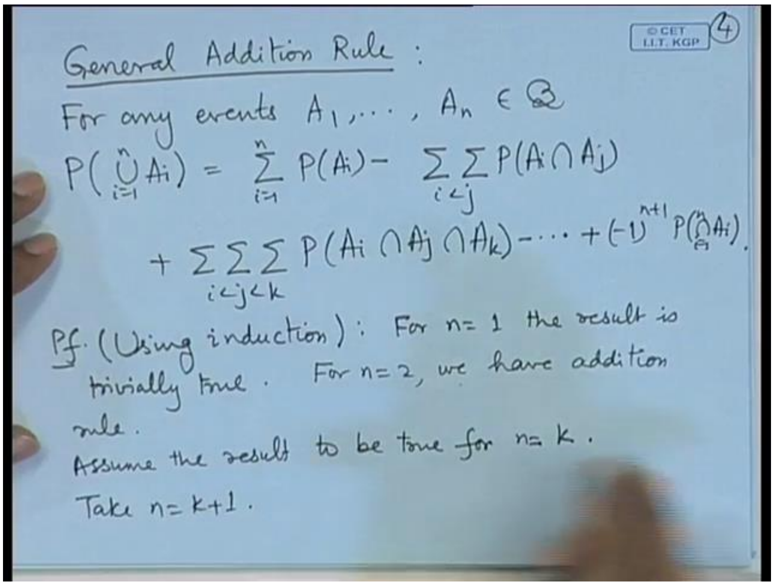

General Addition Rule: so if we have events A 1, A 2, A n then probability of union Ai i is equal to 1 to n can be expressed as sigma probability of Ai i is equal to 1 to n minus double summation probability of Ai intersection Aj i is less than j; plus triple summation probability of Ai intersection Aj intersection Ak i less than j less than k minus and so on. Finally, you will have minus 1 to the power n plus 1 probability of intersection Ai i is equal to 1 to n.

One can prove this result using induction; for example if we take n is equal to 1 the result is trivially true. For extension from k to k plus 1 we will need the result for n is equal to 1 which has already been proved. For n is equal to 2 we have addition rule. Now, assume that the result to be true for n is equal to k. Now take n is equal to k plus 1.

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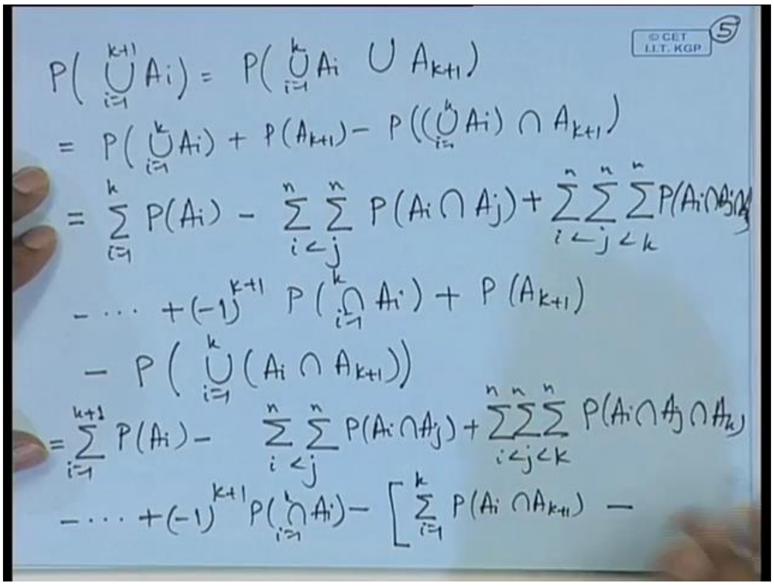

So, we need to consider probability of union Ai i is equal to 1 to k plus 1. And we can consider it as probability of union Ai i is equal to 1 to k union Ak plus 1. So now, I can apply the result for the union of A and B two events so this becomes probability of union Ai i is equal to 1 to k plus probability of Ak plus 1 minus probability of union Ai i is equal to 1 to k intersection Ak plus 1.

Now the first part of this can be expended, because we have already assumed that this rule is true for n is equal to k. So, this becomes sigma probability of Ai i is equal to 1 to k minus double summation probability of Ai intersection Aj i less than j. Now these sums are up to n. Triple summation probability of Ai intersection Aj intersection Ak i less than j less than k the sums are up to n and so on plus up to minus 1 to the power k plus 1 probability of intersection Ai i is equal to 1 to k.

Then we have probability of Ak plus 1, and here we apply the distributive property of the unions and intersections. So, this becomes minus probability of union Ai intersection Ak plus 1, i is equal to 1 to k.

Now, if you look at this last term it is again union of k events and since we have assumed the probability of union result to be true for n is equal to k we can apply that formula. So, using that we will get summation of probability Ai intersection Ak plus 1 for i is equal to 1 to k and that term can be adjusted with this. So, let me write it here. Firstly, sigma probability of Ai i is equal to 1 to k minus double summation i less than j up to n probability of Ai intersection Aj plus triple summation i less than j less than k probability of Ai intersection Aj intersection Ak minus and so on plus minus 1 to the power k plus 1 probability of intersection Ai i is equal to 1 to k.

Now this probability of Ak plus 1 can be added to the first term, so the first term becomes probability of Ai i is equal to 1 to k plus 1. Now, let me expand the last union by using the formula for n is equal to k. So, this becomes sigma probability of Ai intersection Ak plus 1 i is equal to 1 to k minus.

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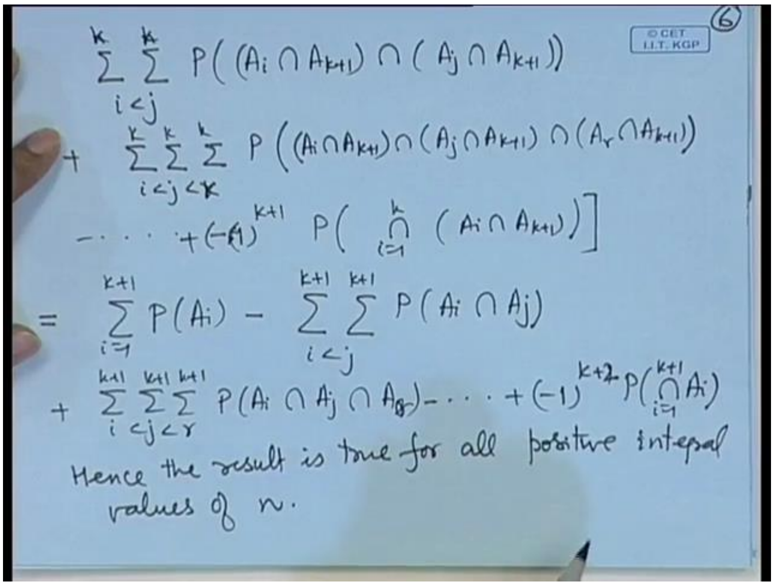

Now you will have double summation probability of Ai intersection Ak plus 1 intersection with Aj intersection Ak plus 1, where i is less than j and the sum goes up to n only goes up to k. So, I think I have made some small mistakes here; these sums are up to k. Then you will have triple summation i less than j less than k probability of Ai intersection Ak plus 1. So, you may put it as r intersection Aj intersection Ak plus 1 intersection Ar intersection Ak plus 1 and so on minus 1 to the power k plus 1 probability of intersection Ai intersection Ak plus 1.

Now, let us look at the terms. The term sigma probability of Ai intersection Ak plus 1 can be combined with this term here with a minus minus getting adjusted, and therefore if you see here. Now we already had all the intersections up to k. Now we have A 1 intersection Ak plus 1 A 2 intersection Ak plus 1 and Ak intersection Ak plus 1, so this gets adjusted here and you will get a term. So, the first term remains as such probability of Ai i is equal to 1 to k plus 1, in the second term you will get i less than j, and now the summation is up to k plus 1 probability of Ai intersection Aj.

Now, let us look at this term; this term is Ai intersection Aj intersection Ak plus 1. Where i's and j's are varying from 1 to k. And if we look at the third term in the previous expression here we had all the intersections of three sets up to k. So, this term gets adjusted in this one and you will get plus triple summation probability of Ai intersection Aj intersection Ar i less than j less than r up to k plus 1.

In a similar way if I look at this term here it will be intersection of the four terms, and the last that is Ak plus 1 that means it is taking care of all the terms of the intersection taken four sets at a time. So, in this way all of the terms are combined. If you look at this term this is actually intersection of all of the Ai's from i is equal to 1 to k plus 1. And since there is a minus sign outside of the square bracket this becomes minus 1 to the power k plus 2. So, you will get minus 1 to the power k plus 2 probability of intersection Ai i is equal to 1 to k plus 1. Hence, the result is true for all positive integral values of n.

Let us look at the some applications of this one. Now before giving the application let me also consider the limit of the probabilities or probability of the limit. As I mentioned that we have defined monotonic sequences and for the monotonic sequences of the sets the limit always exist.

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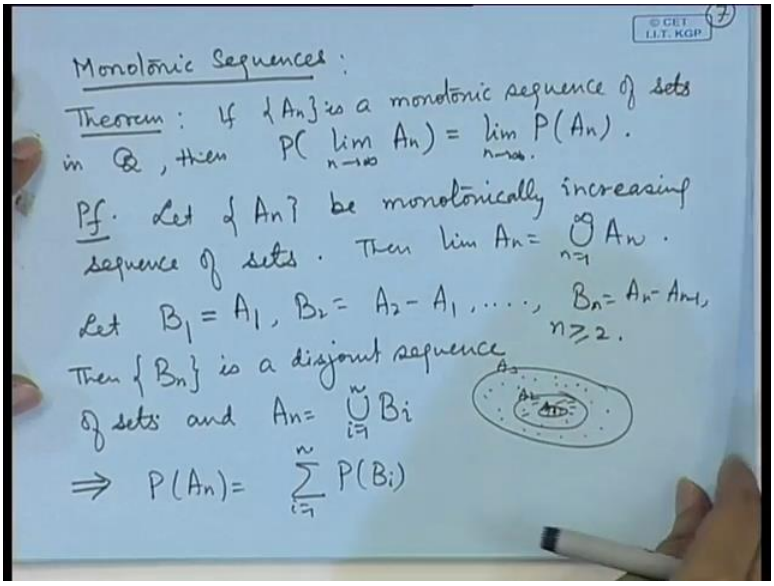

So we have the following result for monotonic sequences of the sets. We have the following theorem: if An is a monotonic sequence of sets in B, then probability of limit of An is equal to limit of probability of An. To prove there is result let us consider An to be say monotonically increasing sequence; let An be monotonically increasing sequence. If that is so then limit of the sequence An will become union of An n is equal to 1 to infinity.

In order to prove that we have to look at probability of limit means probability of the union; now what we do we decompose this union by defining a new sequence of sets by saying say B 1 is equal to A 1 B 2 is equal to A 2 minus A 1 Bn is equal to An minus An minus 1 for n greater than or equal to 2. If we look at this one, basically what we have done the sequence of sets is like this; say A 1 this is A 2 this is A 3 and so on. So, if I look at the union of Ai's we are decomposing it as a disjoint union.

This A 2 minus A 1 will be this portion then A 3 minus A 2 will be this portion. So, we will have that Bn is a disjoint sequence of sets and An is equal to union of Bi from 1 to n. Naturally this implies that probability of An is equal to probability of Bi sigma i is equal to 1 to 1.

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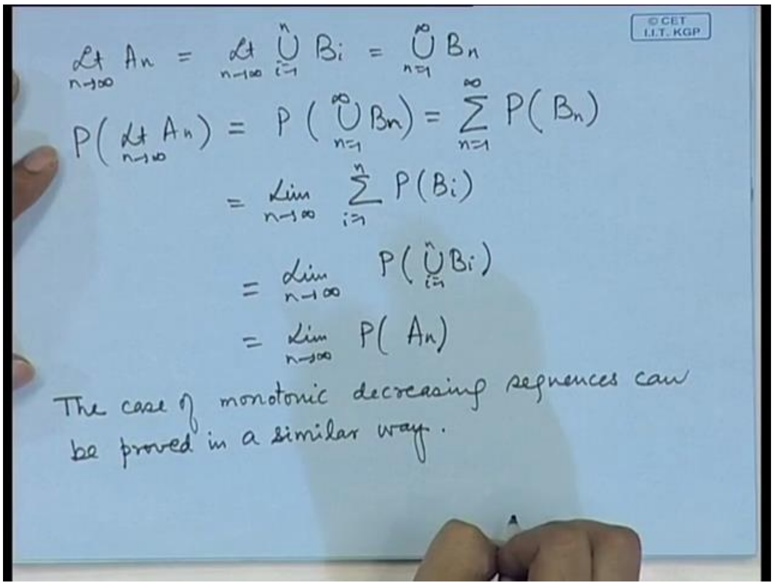

Now, if we look at limit of the sequence An as n tends to infinity then it is equal to limit of union Bi i is equal to 1 to n, n tending to infinity which is equal to union of Bn n is equal to 1 to infinity, because union Bi is a monotonic increasing sequence and the limit will be the ultimate union of these sets. So, if I look at probability of limit of An as n tends to infinity then it is equal to probability of union Bn n is equal to 1 to infinity.

Now, Bn is a disjoint sequence of sets then by the axiom of the countable additivity this becomes probability of sigma probability of Bn n is equal to 1 to infinity. Now this we can write as limit as n tends to infinity sigma i is equal to 1 to n probability of Bi, which we can write as probability of union of Bi i is equal to 1 to n which is equal to limit as n tends to infinity probability of An.

Thus, we have proved that probability of a limit of a sequence of monotonic sequence of sets is equal to limit of the probability of the sequence of the sets. We may also consider the case of monotonically decreasing. Now that can be obtained by taking the complementations here or you can define a reverse way.

Thank you.