Probability and Statistics

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Lecture – 03
Ring, Field (Algebra)

Welcome to this second lecture on algebra of sets. Today I will introduce some algebraic structures which are fundamental to our definition of Axiomatic definition of probability.

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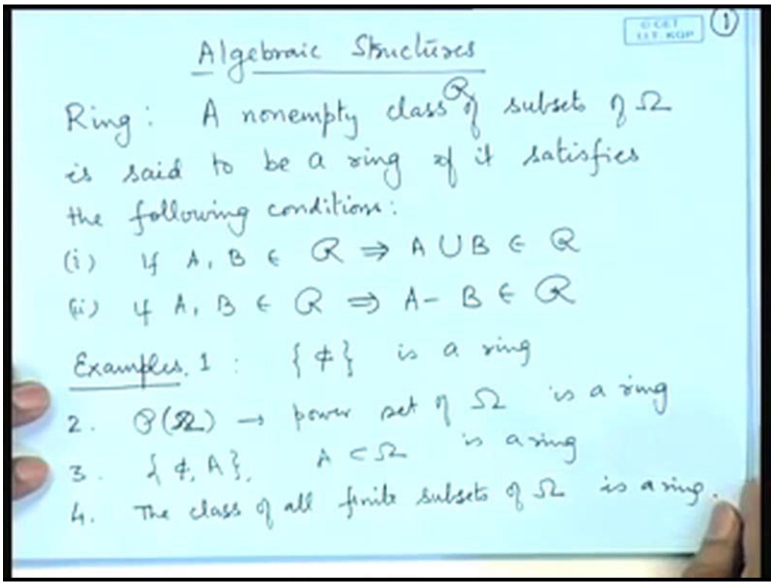

So, let us start with first structure which I call as a Ring. This is a nonempty class of sub sets of; so we continue with our previous definition of the universal said that is omega, so whatever sets we are considering there will be subsets of omega; so nonempty class of subsets of omega is said to be a ring provided it satisfies the following two conditions. The conditions are that should be closed under the operation of unions and differences; that is if A and B belong to say script R. So, let me use the notation the nonempty class to be script R. So, if A and B belongs R, then A union B belongs to R. And if A and B belong to R then A minus B also belongs to R.

That means the ring way structure which is closed under the operation of unions and differences. Let us consider some simple examples to illustrate what is the ring. For example, if I consider a class consisting of simply the null set phi then this is a ring, because if I consider phi union phi it is phi and phi minus phi is also a phi. So, this is a ring. If I consider say the class of the set of all subsets of omega; that is the power set of omega then this is also a ring, because all the sets under consideration will be subsets of omega only. If I consider say a class phi and A, where A is a subset of omega. Then this is also a ring because if I consider phi union A there it is equal to A, if I consider A minus phi it is A, if I consider phi minus A then it is phi. So, this is also a ring.

So, if I consider say the class of all finite subsets of omega. Now suppose I will consider two subsets say A and B which are finite then A union B and A minus B both are finite. Therefore, this is also a ring. Thus we can see that a ring contains certain subsets of omega which certain property, and therefore it may be useful to consider such a structure. Let me look at some of the properties of ring.

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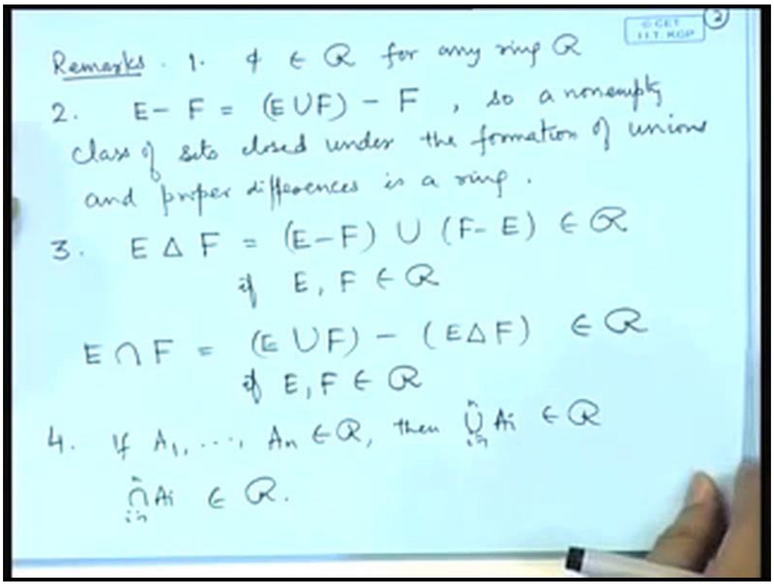

For example; if I consider any ring then phi will belong to that ring. That means, if I consider any set trivially for any ring R. Since, if I take any sets a E belonging to our since it is a nonempty set so at least one set belongs there, and if I take E minus E then it is phi. So, every ring will certainly consist of the null set. If I write say E minus F as E union F minus F, then here you can see that E union F will be certainly including F and if E is any nonempty set then E union F is certainly going to be larger than F; that means, F will be a proper subset of E union F. So, it follows that a nonempty class of sets closed under the formation of unions and proper differences is a ring.

We can also see some other properties, for example; if I look at say symmetric difference then symmetric difference is defined as E minus F union F minus E. Now, if E and F are in R then E minus F and F minus E both are in R and therefore its union is also in R. That means, the symmetric difference of two sets E and F will belong to the ring if E and F belong to ring. That means, a ring is also closed under the operation of symmetric differences.

Similarly if I consider say E intersection F then this I can represent as E union F minus E delta F. Now just on the previous statement we approved that if E and F belong then E delta F also belongs to R. E union F is already there, and therefore the difference is also in R. Therefore, this will also belong to R if E and F belong to R. That means a ring is also closed under the operation of symmetric differences and intersections.

Since, we have taken that if there are any two given sets A and B then A union B belongs to R. Then by mathematical induction we can prove that if we have sets A 1, A 2, A n belonging to R then union of Ai is i is equal to 1 to n will also belong to R. That means, the ring is closed under the operation of taking finite unions. In a similar way we can also look at intersection Ai i is equal to 1 to n.

Since we have already prove that for given two sets the intersection is in R, therefore by induction we can prove that intersection Ai i is equal to 1 to n also will belong to R. That means a ring is closed under the operation of taking finite unions and finite intersections.

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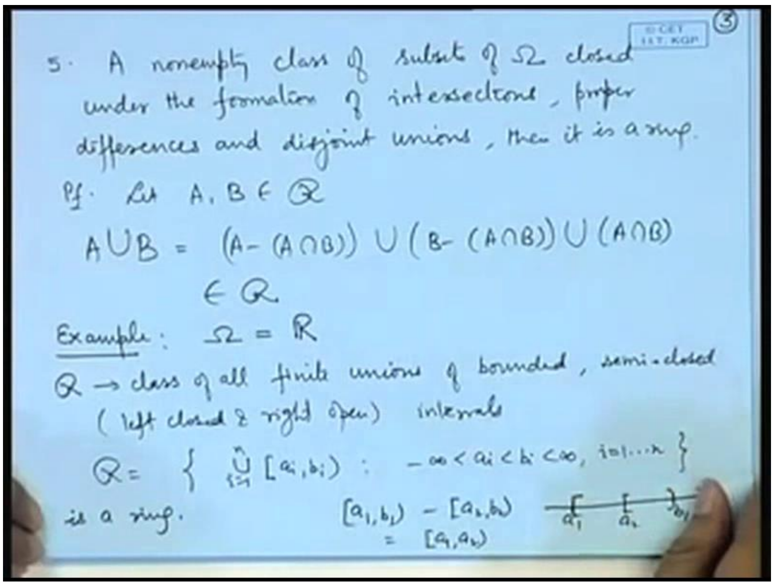

We can consider an alternative definition of the ring in the form that if we consider a nonempty class of subsets of omega which is closed under the formation of intersections, proper differences and disjoint unions then it is a ring.

For example; if I consider say two sets A and B in the ring R. And if I look at since already I have taken the differences so A minus B is equal to A union B minus B, so proper differences are already there. If I take a union then this union I can represent as A minus A intersection B union B minus A intersection B and A intersection B. Now here you see these three sets are disjoint: the first two are proper differences and this whole union is a disjoint union, therefore this will belong to R. So, this can be considered as an alternative definition of the structure ring.

Let us consider some below example in the sense that it will consist of many more sets. Let me take omega to be the set of real numbers and R is the class of all finite unions of bounded then semi closed. So, we can consider, basically say left closed and right open R reverse. Basically, I am saying R is the collection of the sets of the form union i is equal to 1 to n ai, bi where minus infinity less than ai less than bi less than infinity for i is equal to 1 to n. If we consider such collection then this is a ring. You can look at the proof of this statement. Suppose I consider two sets finite unions then there union will again be a finite union. If I take say difference: let me explain the difference suppose I take only a 1, b 1 minus say a 2, b 2. You draw it on a line suppose this is a 1, this is b 1, this is close this is open and this side a 2, b 2. Suppose this is a situation then if I consider a 1 b 1 minus a 2 b 2 b 2 then it is simply equal to a 1 a 2 which is again an interval of the same form.

So now, if I consider unions and then I take their differences then it will be unions of the intervals of the same form, and therefore this is a ring. Let us consider say the difference that here unions are and intersections are treated in the definition of a ring. Here I said at if a ring is closed under intersections then a class of sets closed under the formation of intersection and differences is not necessarily hearing.

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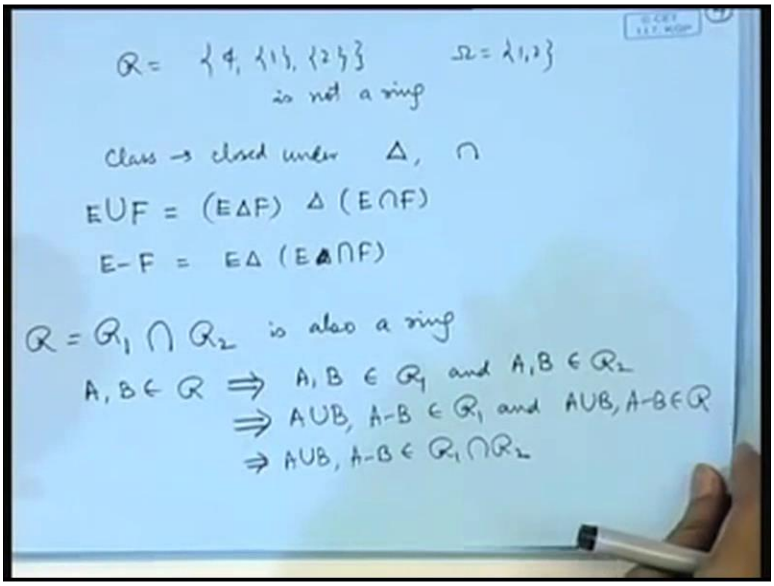

For example; if I consider R is equal to say phi 1 2, where omega is equal to 1 2, then this is not a ring. Here it is closed under the operation of taking intersections, but it is not a ring. So, the definition of ring is not symmetric in the treatment of unions and intersections.

If I consider say a class which is closed under the formation of say symmetric differences and intersections then it will be a ring. For example; if I consider say E union F then it can be represented as E delta F delta E intersection F. Similarly if I consider E minus F then this I can write as E delta of E intersection F. So, you can see that the class of sets which are closed under the formation of symmetric differences and intersections they are ring. On the other hand if you consider it is closed under the operation of symmetric differences and unions, then I will show we get a ring. Therefore, the definition of ring becomes symmetric a unions and intersections if you replace difference by a symmetric difference.

Then further thing is that suppose I say R 1 is a ring and R 2 is a ring. Then if I take the intersection of this let me call it R then this is also a ring. For example, if I take say the sets a B belonging to R, then this implies that A and B both belong to R 1 and both belong to R 2 as well. Now, since R 1 is a ring and R 2 is a ring that means that A union B and A minus B will belong to both R 1 and R 2. And this will be in that A union B and

A minus B belong to R 1 intersection R 2. Therefore, the intersection of two rings is again a ring.

Ring is one of the primary structures in the study of algebraic structures. Now we proceed to define a slightly larger a structures.

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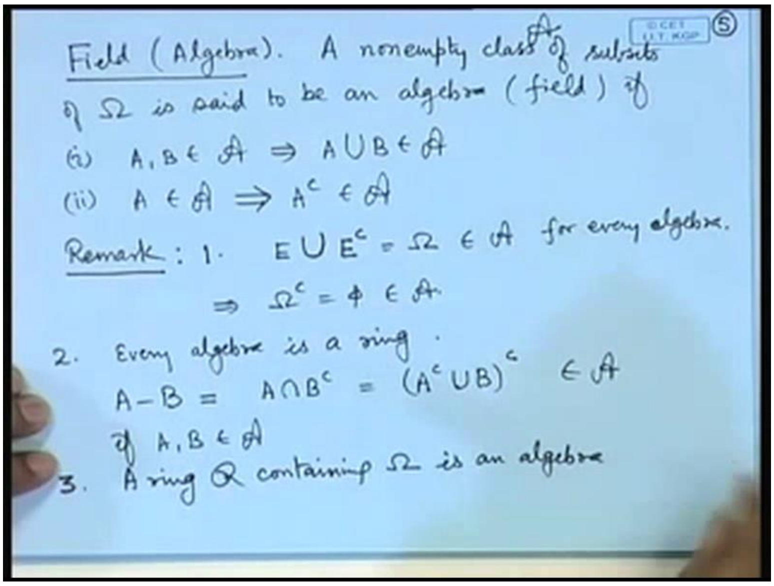

The first of the generalisation is in the form of a field are an algebra; it is also called algebra. So, if I consider a nonempty class of subsets of omega it is said to be an algebra; a nonempty class of subsets of omega is said to be an algebra are a field if it satisfies a following two properties: that is if A and B belong to the class so let me denote this class as script a then this implies that A union B belongs to A. And the second is that A belongs to a script implies that A compliment belongs to script A.

Now if you look at this definition, this definition is a little modification of the definition of the ring. A ring was closed under the operation of forming unions and differences. Here a field or an algebra is closed under the operation of unions, and in place of differences the property has been replaced by it is closed under the operation of taking complementations. So now, let us look at the consequences of this definition. How does it derive in the sense or you can say with respect to a ring. So, the first is that given any set E, its compliment will be there and therefore its union E union complimentary will be there, now this is the full set that means this will belong to algebra. So, you have seen that a ring always consisted of the empty set. Now the full set consists of always included in an algebra. And now you take omega compliment that is equal to phi then this will also belong to an algebra.

Our second remark is that every algebra is a ring. Now to prove this a statement we have to prove that the difference between two sets will belong to the algebra. Now if I take A and B has two sets in algebra then A minus B I can express as A intersection B compliment which is equal to A compliment union B whole compliment. Now that definition of algebra if A belongs to algebra then A compliment belongs to an algebra. Now A compliment union B will belong to algebra and therefore its complementation will belong to an algebra. Therefore, if A and B belong to the algebra then A minus B will also belongs, and therefore every algebra is a ring. So, in some sense now we can say that algebra is an extension of the definition of the ring.

Alternatively, we can look at it; that if I consider a ring and I include the set omega here then this becomes an algebra, because if algebra put omega there then for any given set its complementation can be obtain by taking omega minus that given set. And therefore, that in will become automatically an algebra.

Next is the property of the intersections. Since, the ring always is closed under the intersections, therefore an algebra will also closed under intersections. Since, there inverse closed under symmetric differences and algebra is also closed under symmetric differences.

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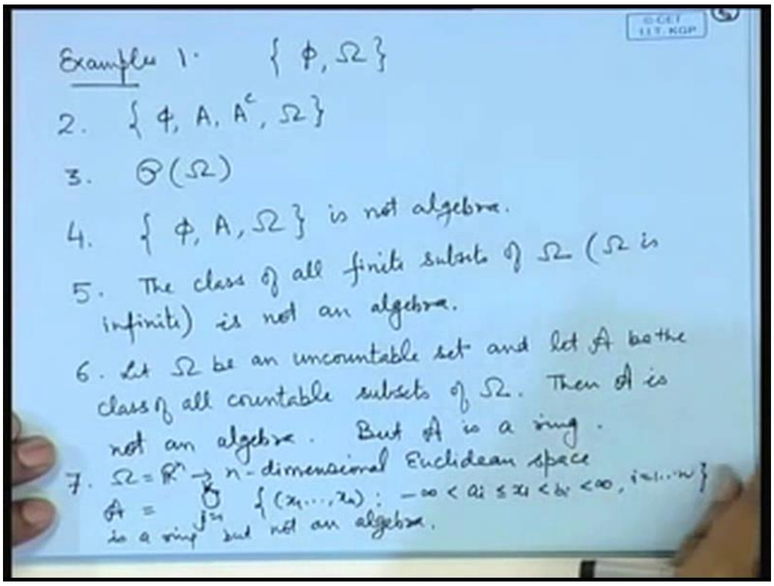

Let us look at certain examples here. So, if I consider say phi omega then this is you can say and a smallest algebra. If I include A and A compliment then you can consider it has a smallest algebra containing the set A. If I consider the power set of omega certainly it is an algebra because all the subsets are already included here.

See if I consider a set like phi A and omega then this is not an algebra, because A compliment is not there. We have considered the class of all finite subsets of omega, where omega may be infinite. Then this is not an algebra this was a ring, but this cannot be an algebra because the complementation of a finite set will become infinite set and that is not there in this class. So, this cannot be an algebra.

Let us take some non trivial example here; let omega be an uncountable set. Let us define the set A to be the class of all countable subsets of omega. Then once again if I consider a countable subset of omega its complimentary become uncountable, and therefore this A cannot be an algebra. However you consider it will be a ring, because if I take any two countable sets its union is again a countable set and if I take difference of the two countable sets it can be a finite or a countable set. Therefore, it will be a ring. From the example it is clear that an algebra are a field is an extension of the definition of a ring. In the sense that it contains some more sets.

Let us consider omega to be the n dimensional Euclidean space. And let us consider A to be the class of all finite unions of the intervals of the form. Say X 1, X 2, X n where ai is less than or equal to xi less than bi less than infinity for; and let me put this as say j is equal to 1 to k and here i is equal to 1 to n. So, basically you can consider them as unions of certain cells then this is a ring, but not an algebra. The reason is that y it is an modern algebra because, if I take the complementation of such a set then that will be say minus infinity to ai and union bi to infinity which is not a set of this form.

Let us construct some classes which may be algebra.

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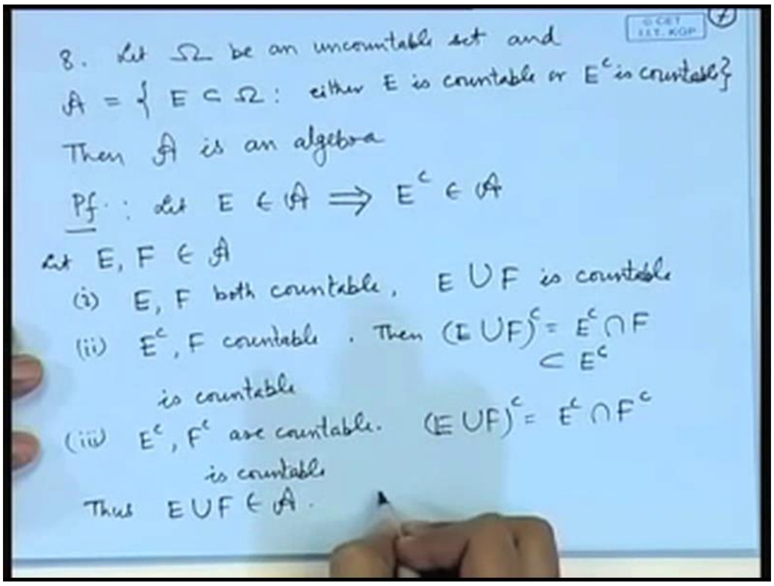

So, let us consider say x to be an uncountable set and A is the class of all sub sets of omega such that either E is countable or E compliment is countable. So, we can see that this is an extension of the example 6, where we are considered only countable subsets. Now here I am considering those sets which may be themselves countable are there compliments may be uncountable. Then the claim is that A is an algebra. To look at a proof of this a statement let us see; suppose I consider E to be in A then by the definition of the class E either E is countable then if I look at E compliment then the compliment of that is countable. If E is uncountable there is E compliment is countable then E will satisfies the same property. Therefore this implies that E compliment belongs to A.

If I consider two sets say E and F belonging to A then there are different cases. See E and F both countable, if both are them are countable then E union F is also countable. If I have say E compliment and F countable then we can write E union F compliment as E compliment intersection F which is a subset of E compliment. So, this is countable. Suppose both E compliment and F compliment are countable, then we can express E union F compliment as E compliment intersection F compliment, and since both E compliment then F compliment are countable then intersection is also countable.

The case where E and F compliment are countable is similar to this because in that case E union F compliment will become a subset of F compliment. So, in all the cases E union

F will belong to A. So, A is an algebra. In fact, given a ring you can always construct a bigger class which will be an algebra. We can do it in the following way.

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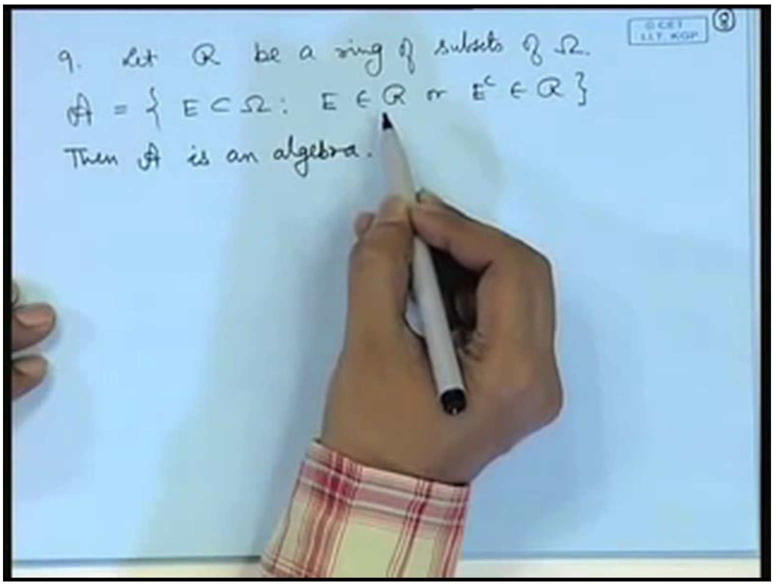

Let us consider R to be a ring and we define the class A to be the class of all subsets of omega such that either E belongs to R or E compliment belongs to R. Then once again is an algebra. The proof of this is statement is almost the same as the previous exercise, because in the previous exercise if you replace the ring to be the class of all countable subsets of omega and you given this definition then the proof will be same.

Thank you.