Probability and Statistics
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Lecture - 28
Special Continuous Distributions – V

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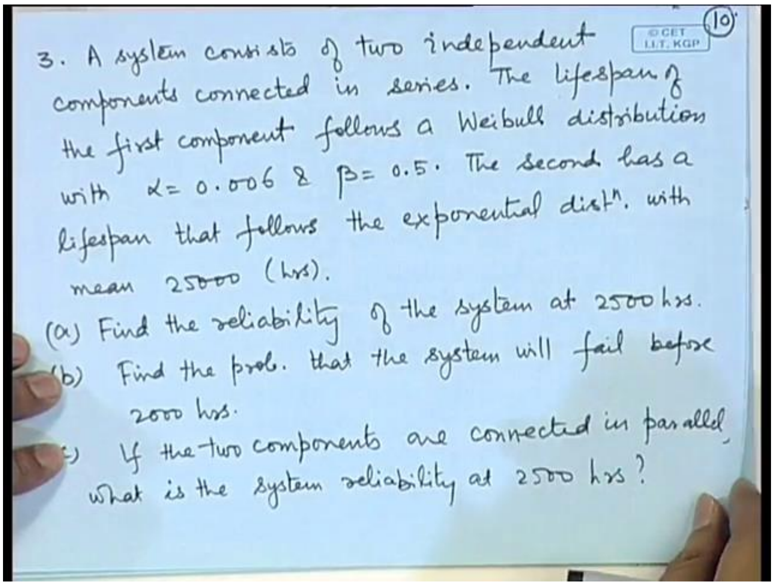

A system consists of two independent components connected in series, the lifespan of the first component follows a Weibull distribution with alpha is equal to 0.006 and beta is equal to 0.5. The second component has a lifespan that follows the exponential distribution with mean 25000 a major eta unit is hours. Find the reliability of the system at 2500 hours, find the probability that the system will fail before 2000 hours, if the 2 components are connected in parallel what is the system reliability at 2500 hours?

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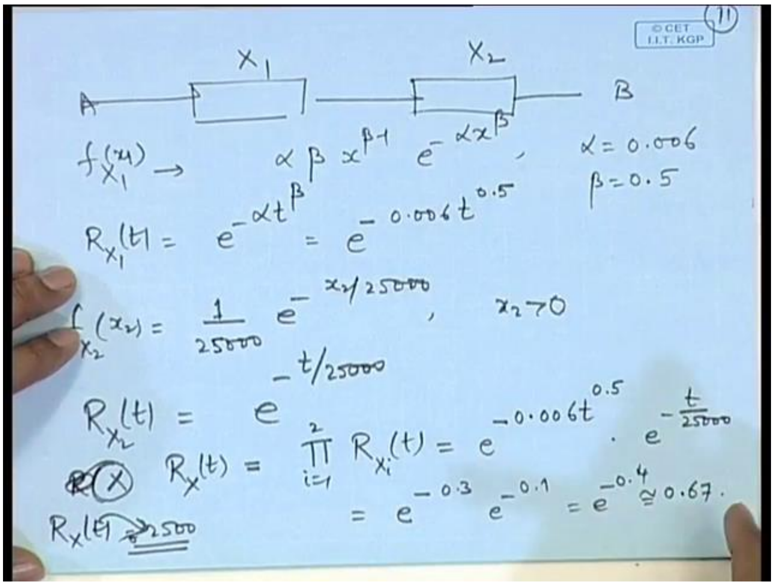

So, here the 2 independent components are connected in the series, let us call this as X 1 and this as X 2 the first one X 1 as a Weibull distribution. So, the density of is equal to alpha beta x to the power beta minus 1, e to the power minus alpha x to the power beta, were alpha and beta are given here. So, if we look at the reliability function of X 1 that is equal to e to the power minus alpha t to the power beta, that is e to the power minus 0.006 t to the power 0.5. If we look at the second component that is exponential distribution with mean 25000. So, the density function will be because mean of the exponential distribution with density lambda e to the power minus lambda x is 1 by lambda. So, if you are saying mean is 1 by 25000. So, the density function will become 1 by 25000 e to the power minus this. So, the reliability of this is equal to e to the power minus t by 25000.

Now, we are interested in the reliability of the system at 2500 hours. So, the system life suppose it is X. So, the system reliability at time t that is equal to 5 series system it is the product of the individual reliabilities. So, this is equal to e to the power minus 0.006 t to the power 0.5 into e to the power minus t by 25000. So, if we are calculating the system reliability at 2500 hours at t is equal to 2500, then this will be equal to after substitution this value is evaluated to be e to the power minus 0.3 and this one will become e to the power minus 0.1 that is equal to e to the minus 0.4, which is approximately 0.67. So, reliability of the compound system which is connected in a which consist of 2 components connected in a series, can be evaluated by multiplying out the reliabilities of the individual components at the given time.

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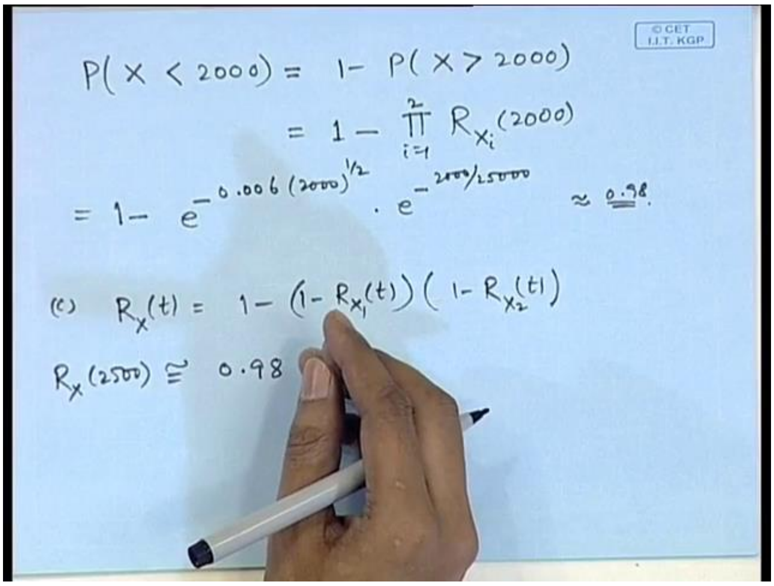

Now, the next part of it is, what is the probability that the system fails before time 2000 hours; that means, what is the probability that x is less than 2000. Now this can be written as o1 minus probability that x is working at 2000 hours, once again it is the product of the reliabilities at 2000 hours. So, this we can substitute the values 1 minus e to the power minus 0.006, 2000 to the power half e to the power minus 2000 divided by 25000. So, after some simplification this value turns out to be; this value is approximately 0.98. In the third part if the 2 components are connected in parallel, what is the reliability? Now if the components are connected in parallel, then we have seen that the reliability of the system is given by 1 minus 1 minus the reliability of x 1 into 1 minus reliability of x 2.

So, if we are calculating at 2500 hours, then after substitution of the values of R x 1 and R x 2, which we have evaluated here the R x 1 is this, and R x 2 is this, here if you put t is equal to 2500 and substituting this one, it turns out to be approximately 0.98. Compare the value which we calculated in the 2 parts; if the components are connected in series the system reliability is only 0.67 at time 2500 hours, where are if there connected in parallel the system reliability is pretty high that is 0.98.

This is the different because if we are insisting that each of the component should work then the probability becomes a smaller, when we are having the relaxation that if any of the system is working then the probability of system functioning will be much higher, that is why in the industries generally there are systems kept as redundant. So of course, that is part of the reliability studies, when we study k out of n system that the system will function if any k out of the n systems are working. So, we find out the probabilities of that kind of events.

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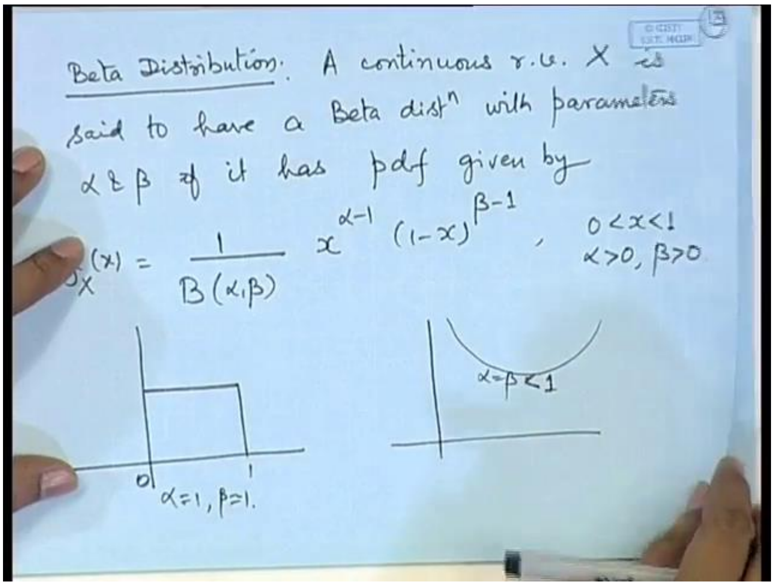

The next one is the discussion about a beta distribution; a continues random variable X is said to have a beta distribution with parameters a alpha and beta, if it has the probability density function given by 1 by beta alpha beta were beta denotes a standard beta function, x to the power alpha minus 1, 1 minus x to the power beta minus 1; for 0 less than x less than 1, alpha is positive beta is positive and 0 otherwise.

So obviously, this is called beta distribution because the it is consisting of a beta function here, the integral of this will give you beta alpha beta therefore, the ratio will become 1, the distribution is quite useful in describing various kind of phenomena where the range is bounded and if range is bounded we can limit it to the interval 0 to 1. In fact, we can look at the shape of the curves, if I take alpha is equal to 1 and beta is equal to 1, it is reducing to uniform distribution.

So, 0 to 1; if alpha is 1 beta is equal to 1. If we are having alpha beta both are less than 1, then it is becoming a function like this. Suppose I take alpha and beta equal, but less than 1; if we consider say beta is equal to 1 then this term will not be there and suppose I take alpha to be less than 1 then this will be a decreasing function.

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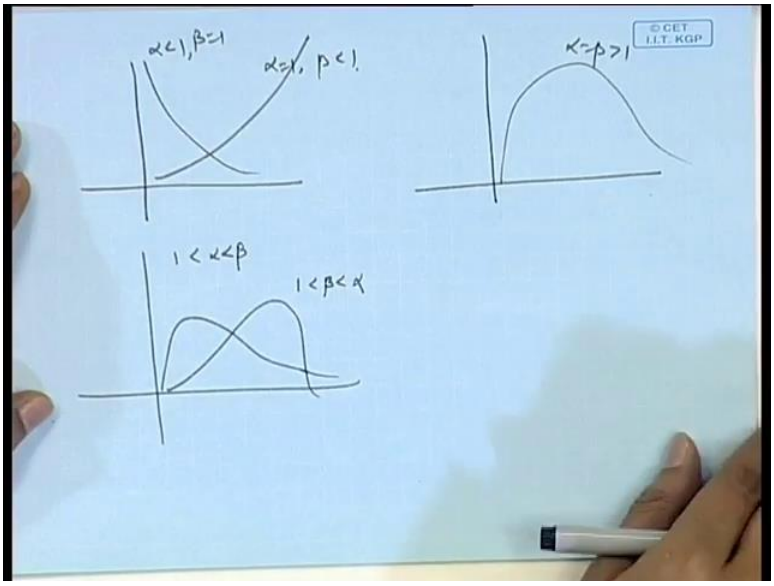

Suppose, I consider alpha to be less than 1 and say beta equal to 1. On the other hand if I take alpha is equal to 1 and beta to be less than 1 then it will become like this. If we take alpha is equal to beta, but greater than 1, then the shape will become something like this. If we are having say 1 less than alpha less than beta then the shape will be something like this, the shape will be something like this if we take 1 less than beta less than alpha.

So, here you can see that since here 2 parameter alpha and beta are there depending upon the values of alpha and beta various kind of shapes are coming; if you are having alpha beta equal to 1, which is reducing it to uniform distribution which is a symmetric distribution; if we have alpha is equal to beta, but less than 1 then it is having a convex shape; if we are having alpha and beta equal, but greater than 1 then it is having a concave shape, it may be positively skewed or negatively skewed depending upon various combinations of alpha and beta values, we can look at the moments of these distribution since its over a finite range it is clear that the movements of all orders will exists.

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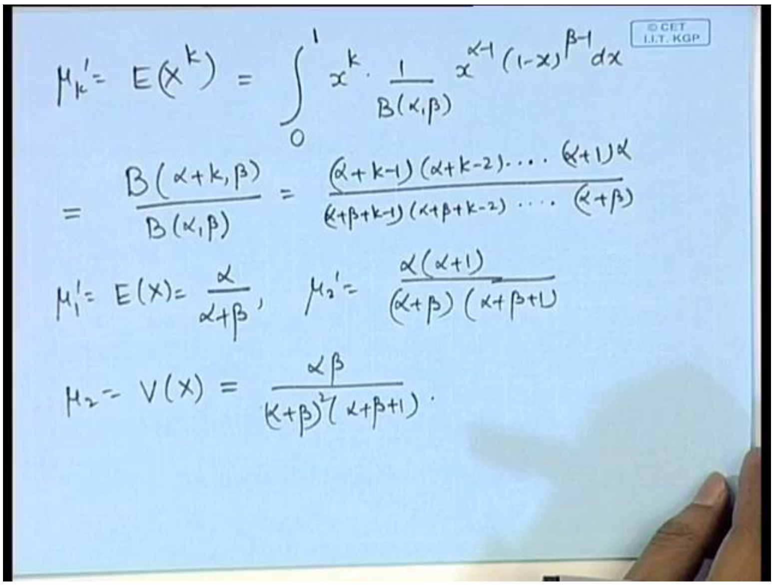

If we considers a expectation of X to the power k, this is equal to integral x to the power k 1 by beta, alpha beta x to the power alpha minus 1, 1 minus x to the power beta minus one d x from 0 to 1.

So, this is nothing, but the beta function, alpha plus k beta divided by beta alpha beta. So, this can be simplify it and we get the expirations as alpha plus k minus 1 alpha plus k minus 2 and so, on up to alpha plus 1, alpha divided by alpha plus beta plus k minus 1, alpha plus beta plus k minus 2 and so on alpha plus beta. So, if we calculate the mean of this distribution it is simply alpha by alpha plus beta, the second moment will become equal to alpha and 2 alpha plus 1 divided by alpha plus beta into alpha plus beta plus 1 and therefore, variance of beta distributions will be equal to alpha beta divided by alpha plus beta is square into alpha plus beta plus 1.

Of course the third and forth moment will decide about the symmetry, and the kurtosis of this distributions and we have already exhibited that it will depend upon the values of alpha and beta; here we give one derivation of beta distribution based on sampling from a uniform distribution.

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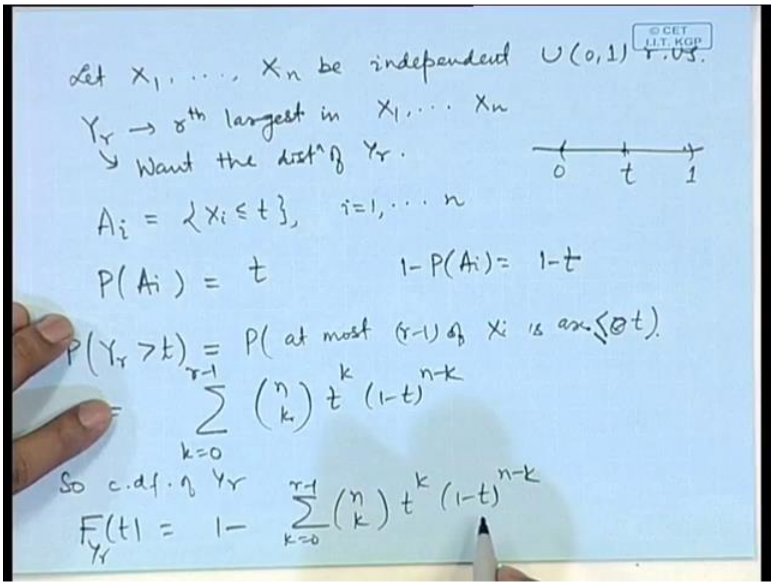

Let X 1 X 2 X n be independent uniform 0 1 random variables, let us consider y r the rth largest in X 1, X 2, X n what is the distribution of y r ?

Now, to consider these we may consider some point t on the interval 0 to 1, then any x i may be bigger than t that makes it maybe between t to 1 or it may be between 0 to t. So, if we consider A i to be the event that x i is less than or equal to t and if x i is are independent then A i is are independent events. So, observing of x 1, x 2, x n denotes a sequence of Bernoullian trials because each x i may satisfy a x i less than or equal to t or it may not satisfy; that means, A i may happen or may not happen and what is probability of A i? That is t where t is a number between 0 to 1, so 1 minus probability of A i is1 minus t. So, if we are looking at say probability of y r is greater than t, it is equivalent to the event that at most r minus 1 of X i is are greater than t sorry it is less than t.

So, this means this event is equivalent to sigma t to the power k 1 minus t to the power n minus k n c k, k is equal to 0 to r minus 1. So, let us analyze this what I am saying is that if the rth largest is bigger than t then at most r minus 1 of x i is will be less than or equal to t, because if they are bigger than if r of them are less than or equal to t then naturally it may happen that the rth largest will also become less than or equal to t. So, if we are saying rth largest is greater than t then at most r minus 1 of the x i is will be less than or equal to t.

So, now let us consider k of the x i' s are less than or equal to t, then for k of them there is a success and for the remaining n minus k it is a failure. The success probability is t and the failure probability 1 minus t. So, it is become like a binomial distribution out of n trails k success. So, the probability of the k success is n c k t to the power k 1 minus t to the power n minus k and here we are saying that k may be from 0 to r minus 1. So, the cumulative distribution function of y r is then obtained as 1 minus sigma n c k t, to the power k one minus t to the power n minus k, k is equal to 0 to r minus 1. So, the density function of y r can be obtained by differentiation of this term.

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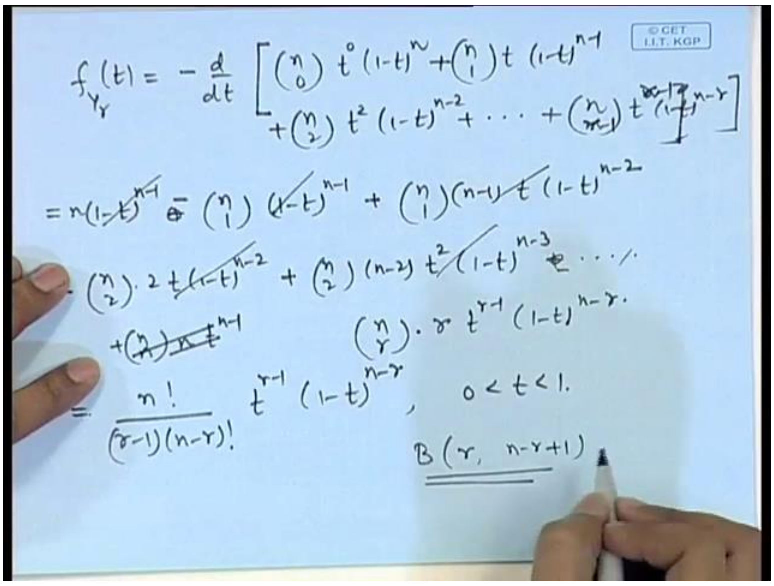

We will get f y r t is equal to minus d by d t; now here we have to write down the terms, so, it is n c 0, t to the power n plus n c 1, t 1 minus t to the power n minus 1 plus n c 2, t square 1 minus t to the power n minus 2 and so on plus n c n t to the power t n.

So, the derivative of this will give us. So, if we consider the derivative the term by term differentiation will be there, each of the term is a product of 2 term except the first and the last. So, in the first one we will get 1 minus t to the power n minus 1, there is a minus sign here, plus n c 1. Now derivative of this one will gives simply 1 minus t to the power n minus 1 and there is a minus sign so this will become minus and then you will have plus n c 1 into n minus 1, t 1 minus t to the power n minus 2 minus n c 2 into 2 t 1 minus t to the power n minus 2, plus n c 2 n minus 2 t square 1 minus t to the power n minus 3, plus n c n n t to the power n minus 1.

Again we can observe that the terms are telescopic in nature, that is the first term cancels with the second term, here if we look at n c 1 that is n into n minus 1 here it is n into n minus 1 by 2 into 2, so that is again n into n minus 1, so this again cancels out. So, likewise all the terms will cancel each other and we will be left with. So, here the last term was n c r minus 1, t to the power r minus 1, 1 minus t to the power n minus r.

So, this were we wrote (Refer Time: 21:46) here the last term will give us n c r, r t to the power r minus 1, 1 minus t to the power n minus r, that is equal to n factorial divided by r minus 1 factorial n minus r factorial t to the power r minus 1, 1 minus t to the power n minus r; this is because the cancellation of all the terms after the n c k into n minus k and that is same as n c n k plus 1 into k plus 1 this is nothing, but the beta distribution the parameters r and n minus r plus 1. So, this beta distribution arises in sampling from a uniform distribution with rth the distribution of the rth largest.

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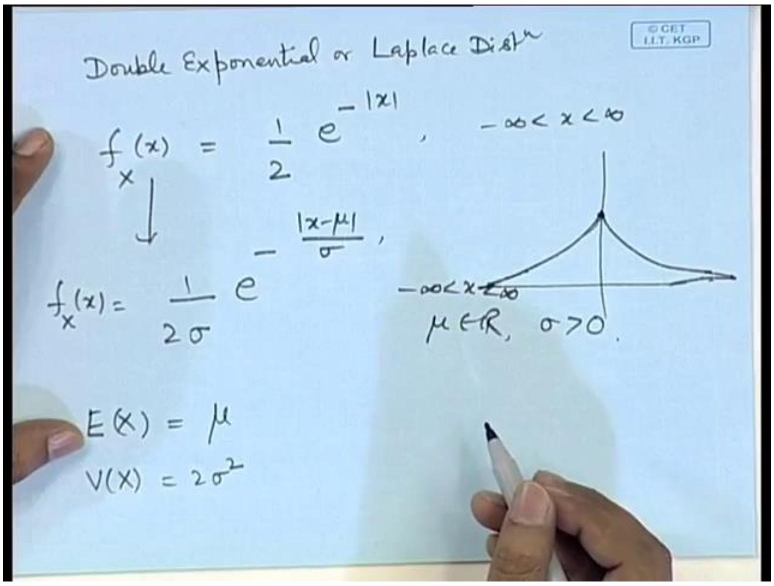

Next we introduce a double exponential or Laplace distribution; if we remember the exponential distribution, in the exponential distribution the density was one the right side of the axis, if we consider the density on both the sides of the x is a this is known as the double exponential distribution of course, we may introduce parameters here, we may consider 1 by 2 sigma e to the power minus x minus mu by sigma, where mu is a positive parameter mu is a real and sigma is a positive parameter. So, this is known as a double exponential distribution. The distribution is quite useful in various studies, where exponential is restricting to the positive side alone, but if we have some values on the left side also we may this distribution finds (Refer Time: 24:25). If we consider the mean this is equal to mu, which is obvious because it is a symmetric distribution, if we look at the variance of this is 2 sigma square; we may look at the measures of a skeweness which will be again 0, and the measures of kurtosis will be dependent on the value of the sigma here.

The moment genetic function of this distribution will exists, because if we are looking at a expectation of e to the power t x, then it is integral of this term into e to the power t x. So, it will exist for all values of t. In the next lecture we will introduce one of the most important distributions in statistics, which is known as the normal distribution and we will also show why it is important here. So, we will stop here.

Thank you.