Probability and Statistics
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Lecture – 23
Poisson Process – II

Let us look at some applications of the Poisson Process.

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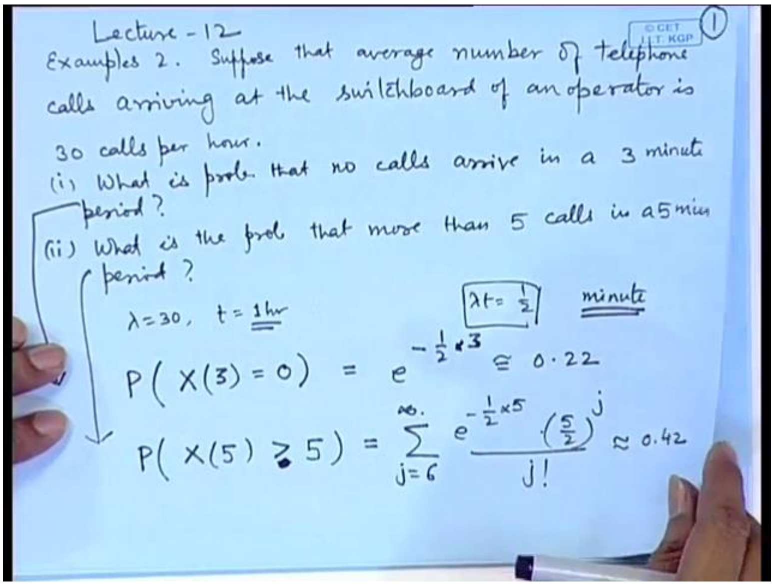

Suppose that average number of telephone calls having at the switchboard of an operator is 30 calls per hour. So, what is the probability that no calls arrive in a 3 minute period? What is the probability that more than 5 calls in a 5 minute period? So, here if we see lambda is equal to 30 and t is equal to 1 hour. So, if we consider the unit as minute then in 1 minute there will be lambda t is equal to half for if we are considering the unit of time as minute. So, if we say probability of no calls in a 3 minute period, this can considered as probability of X 3 is equal to 0. So, it is equal to e to the power minus 1 by 2 into 3. This 1 by 2 is the rate for 1 minute. So, in 3 minutes it will be 3 by 2. So, here I am taking lambda to be half and t to be 3. So, it is e to the power minus 3 by 2 it is approximately 0.22.

In the second one for example, what is the probability of more than 5 calls in a 5 minute period; that means, in a 5 minute period X 5 greater than or equal to 5. So, here the Poisson distribution will have half into 5. So, e to the power minus half into 5 then 5 by 2 to the power j by j factorial and summation from j is equal to a more than 5 calls, so we will put strictly greater than here in place of greater than or equal to. So, it is j is equal to 6 to infinity.

So, from the tables of 1 hour distribution or by calculation, we can check it is approximately 0.42. So, the probability that more than 5 calls will be received in a 5 minute period is 0.42, and probability of no call being received in a 3 minute period is 0.22. So, here you can see that when we want to apply the Poisson process then the parameter lambda is dependent upon the unit of time for which we are considering.

So, here initially it is given 30 calls per hour. So, if you consider the unit as hour then lambda is 30, but if you consider unit as minute, then lambda will become 1 by 2 because 30 by 60. So, this is the way of evolution in a Poisson process.

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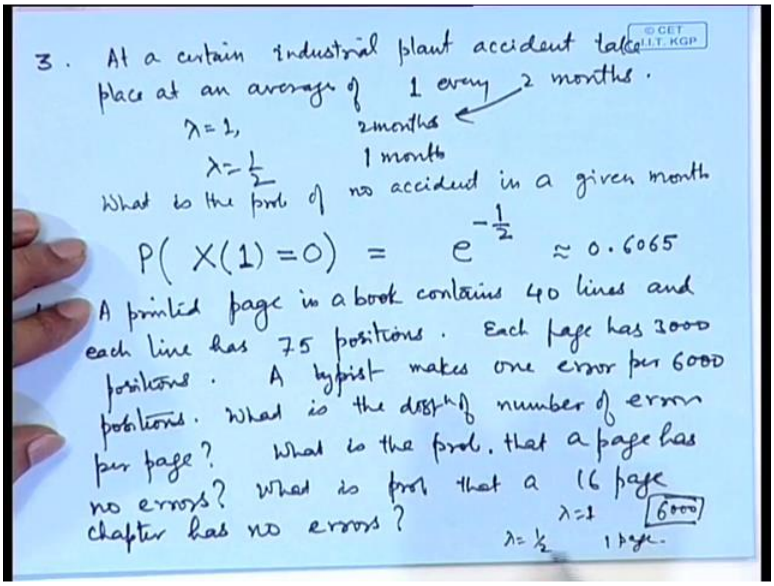

Let me take 1 more example here: at a certain industrial plant accidents take place at an average of 1 every 2 months.

So, the rate is 1 accident in 2 months. So, if I consider lambda is 1, but time of unit of time is 2 months. So, if we consider unit of time as 1 month then lambda will become half. So, what is the probability of say no accident in a given month? So now, this means probability that now month is a unit, so probability that X 1 is equal to 0. So, in a 1 month lambda will be half. So, it is e to the power minus lambda t that is e to the power minus half that is 0.6065. So, there is a 60 percent chance that there will be no accident in a given month.

If we look at the conditions of a theorem, condition of the problem, here it is given an average of 1 every 2 month. So, you will feel that probability will be 50 percent of no accident of probability of 50 percent of 1 accident in a month, but it is not so. Actually the probability of no accident is more than that it is 0.60.

Let me take one more application of Poisson distribution here, a printed page in a book contains say 40 lines and each line has 75 positions, it is like 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, blanks are also counted here. So, each page has 3000 positions, a typist makes one error per 6000 positions, what is the distribution of number of errors per page, what is the probability that a page has no errors, what is the probability that a 16 page chapter has no errors?

Now, here you see lambda is equal to 1 for 6000 positions, if the unit of area are a space is 6000 is the position then it is 6000 positions then lambda is equal to 1. If we consider the unit a one page then in a page there are 3000 positions, then lambda will become equal to half if the unit is one page. So, in order to answer the questions here the unit is one page therefore, lambda will be half, what is the distribution of number of errors per page.

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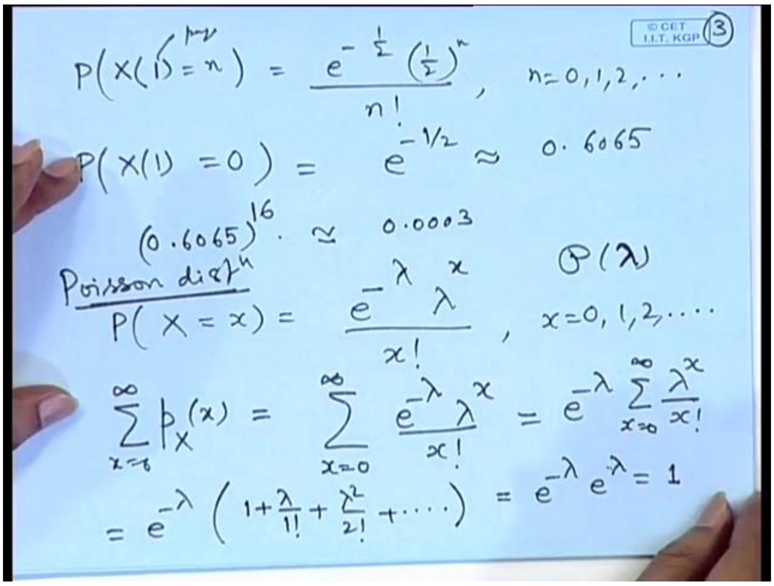

So, this will be probability of X 1. So, 1 means 1 page, X 1 is equal to n, and here lambda is becoming half. So, it is e to the power minus half, half to the power n by n factorial, n is equal to 0 1 2 and so on. What is a probability that a page as no errors? So that means, we want probability of X 1 is equal to 0. So, according to this formula it will be e to the power minus 1 by 2, which is approximately 0.6065, what is a probability that a 16 page chapter as no errors?

Now, here each page containing an error or not con be considered as a Bernoullian trial, now for 1 page not having any error is 0.6065. So, 16 page chapter having no errors can be considered as 0.6065 to the power 16, that is p to the power n which is approximately of course, 0.0003; which is quietly small which is obvious that since in one page the probability of an error is an not an error is 0.6. So, in 16 page chapters there will be no error the probability is naturally going to be very very is small.

Let us look at the characteristics of the Poisson distribution. So, for convenience we will denote probability X equal to x as e to the power minus lambda, lambda to the power x by x factorial, x is equal to 0, 1, 2, and so on. So, here that lambda t we are replaced by lambda, because what happens that lambda is the rate of occurrence of the event when the unit of time area the space is taken as something which is denoted by t. So, that lambda we can merge into 1 and we can rise it as lambda again. So, this is a convenient way of expressing a Poisson distribution.

So, this form of the probability mass function is known as a Poisson distribution and we will use a notation Poisson lambda. So, this means that rate lambda here; if we look at sigma of P X x, x is equal to 0 to infinity that is equal to sigma e to the power minus lambda, lambda to the power x by x factorial, x is equal to 0 to infinity. So, this is equal to e to the power minus lambda summation lambda to the power x by x factorial, x is equal to 0 to infinity. If we look at the series sigma lambda to the power x by x factorial, this series is 1 plus lambda by 1 factorial, plus lambda square by 2 factorial and so on which is nothing, but the expansion of e to the power lambda.

So, this is equal to e to the power minus lambda e to the power lambda that is equal to 1 therefore, this is a valid probability mass function. Now this summation also suggest that how the moments of the Poisson distribution will be evaluated; that means, we will have to interpret or represent the infinity series as a expansion of e to the power lambda terms.

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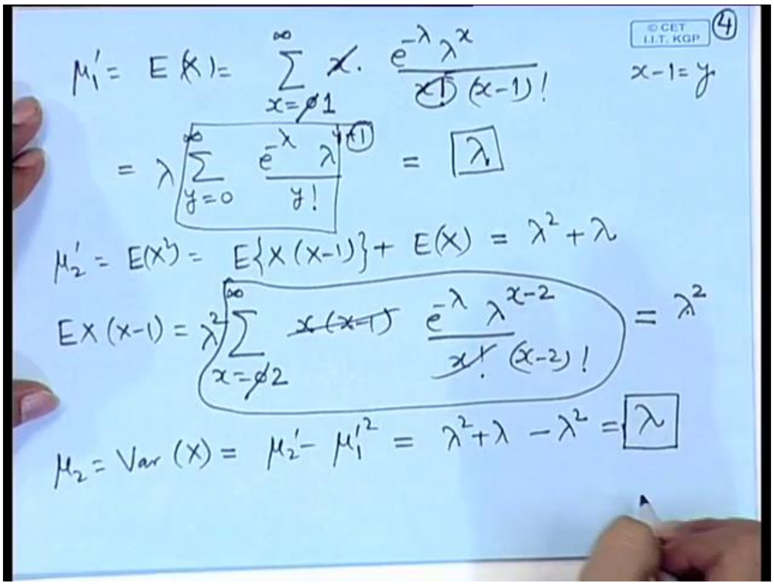

So, let us look at say mu 1 prime that is expectation of X. So, that is equal to x, e to the power minus lambda, lambda to the power x by x factorial, summation form x is equal to 0 to infinity.

As we have seen in the binomial distribution or hypergeometric distribution, when the factorial term is involved then we have to adjust that. So, notice here that corresponding to x equal to 0 this term is vanishing. So, in effect this is actually summation form x is equal to 1 to infinity. So, now, this term can be adjusted and we can write it as x minus 1 factorial. So, naturally we can substitute x minus 1 is equal to y, then this becomes summation form y is equal to 0 to infinity, e to the power minus lambda, lambda to the power y plus 1 divided by y factorial.

So, we can keep lambda outside and then this sum becomes 1 and therefore, the mean of a Poisson distribution is lambda; which is obvious because when we are saying lambda is the rate of occurrence, so in a particular unit of time t, the number of arrivals will be lambda t. So, when we are place lambda t by lambda, here the mean must be lambda. Now this suggests that in order to calculate say second moment, we will need to calculate the second factorial moment. So, if we apply the same argument expectation of X into x minus 1, it will be equal to x into x minus 1, e to the power minus lambda, lambda to the power x by x factorial x is equal to 0 to infinity.

Notice here that corresponding to x equal to 0 and x equal to 1 this term is vanishes. So, in effect this summation is from 2 to infinity and therefore, this x into x minus 1 term can be adjusted with this term and we get x minus 2 factorial therefore, this we can write as lambda to the power x minus 2, lambda is square and now this is nothing, but expansion of e to the power lambda. So, the second factorial moment becomes lambda square.

Now, if we substitute this value in the expression for mu 2 prime, this is lambda square plus expectation x, that is lambda therefore, mu 2 that is variance of a Poisson distribution becomes that is mu 2 prime minus mu 1 prime square, that is equal to lambda square plus lambda minus lambda square, which is equal to lambda. So, we come to a surprising looking result here mean was lambda and now the variance is also lambda.

So, in a Poisson distribution the rate is mean as well as it denotes the variance of the distribution. Now the way the calculations have been done here since the factorials are involved, it will be easier to calculate factorial moments in the case of Poisson distribution.

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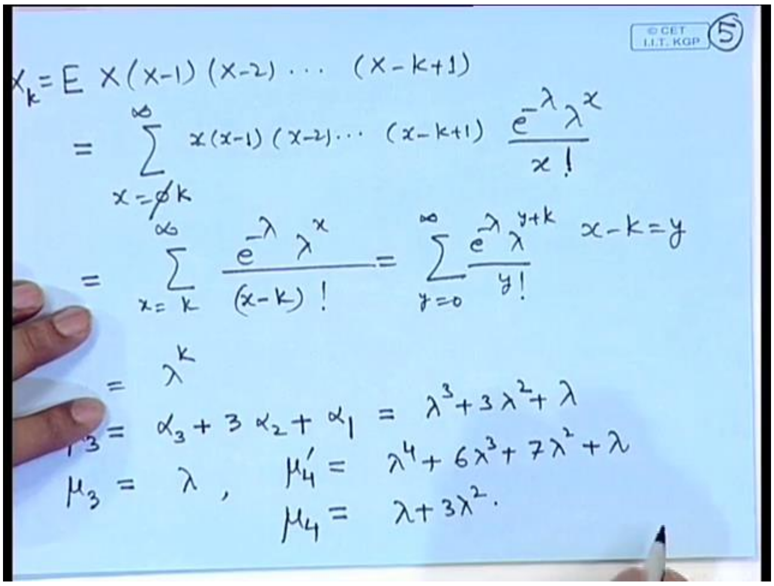

So, we may consider kth factorial moment that is expectation X into x minus 1, x minus 2 up to x minus k plus 1. So, this becomes summation x into x minus 1, into x minus 2 up to x minus k plus 1 e to the power minus lambda, lambda to the power x by x factorial x is equal to 0 to infinity.

So, noticing that for x is equal to 0 1 2 up to k minus 1 this term vanishes. So, this is basically an expansion from x is equal to k to infinity. So, naturally then we can cancel this term from expansion of x factorial and we will get e to the power minus lambda, lambda to the power x divided by x minus k factorial x is equal to k to infinity. So, if we substitute x minus k is equal to y, then this becomes summation from y is equal to 0 to infinity, e to the power minus lambda, lambda to the power y plus k divided by y factorial.

So, this is lambda to the power k into this term which will become actually 1. So, the kth factorial moment becomes lambda to the power k. So, we can use this and get the expressions for third forth non central moments and consequently the third and forth central moments of the Poisson distribution. So, for example, mu 3 prime; mu 3 prime will then become equal to the third let me denote it by say alpha, so alpha 3 plus 3 alpha 2 plus alpha 1.

Alpha 1 is expectation x. So, this is we can write alpha k. So, alpha 3 is equal to lambda cube plus 3 lambda square plus lambda, the third non central moment. Once again if we make use of the relationship between the central and non central moments, then mu 3 is equal to lambda that is lambda cube plus 3 lambda square plus lambda minus lambda cube minus 3 lambda square; that term is coming there so it is actually becoming lambda.

So, there is again surprising because here the third central moment the second central moment and the mean they are all the same; so in a Poisson distribution all the 3 are same. In a similar way we can look at mu 4 prime, mu 4 prime is lambda to the power 4 plus 6 lambda cube plus seven lambda square plus lambda and using that mu 4 that is a forth central moment is calculated to be lambda plus 3 lambda square.

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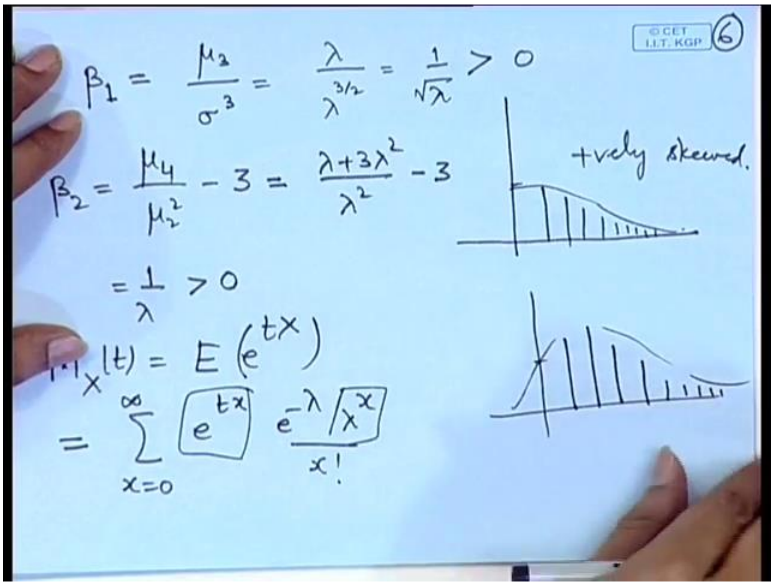

So, we can calculate the measures of estimation kurtosis; beta 1 is equal to mu 3 divided by sigma cube. So, here if we see sigma square is equal to lambda that is sigma is equal to lambda to the power half. So, if you substitute it here we get lambda divided by lambda to the power 3 by 2 that is equal to 1 by root lambda. Naturally since lambda is rate it is a positive parameter, so this is greater than 0; however, you can observe that as lambda increases this will converge to 0.

So, Poisson distribution is a positively skew distribution, which is obvious also because the terms of the Poisson distribution are given by e to the power minus lambda, lambda to the power x by x factorial. So, in the beginning if you see the first term is e to the power minus lambda, then lambda into e to the power minus lambda, then lambda square by 2 factorial into e to the power minus lambda; so as x increases the denominator it will be dominating x factorial term. So, the probabilities will rapidly decrease, it may increase a little bit in the beginning if lambda is greater than 1.

If lambda is less than 1 then from the first step itself the probability will start decreasing therefore, it is a. So, if lambda is less than 1, it will decrease quite rapidly so it is a positively skew distribution. If lambda is bigger than 1 then in the beginning may be it will increase, but there after it will start decreasing rapidly. So, the shape of the curve is positively skewed. Let us also look at the measure of kurtosis that is beta 2 mu 4 by mu 2 square minus 3. So, that is equal to lambda, plus 3 lambda square by lambda square minus 3, that is equal to 1 by lambda and once again it is positive.

So, the peak is little higher than the peak of a normal distribution; however, we can observe that as lambda increases 1 by lambda is approximately 0 and therefore, the peak will converge to a normal peak as lambda increases in a Poisson distribution. We can also look at the moment generating function of the Poisson distribution M x t that is expectation of e to the power t x, this is equal to sigma e to the power t x, e to the power minus lambda, lambda to the power x by x factorial, summation x is equal to 0 to infinity.

We have already shown how the evaluations of the moments are done that is by looking at the expansion of e to the power lambda term therefore, this term should be combined with this and we get e to the power minus lambda, lambda e to the power t to the power x by x factorial, x is equal to 0 to infinity.

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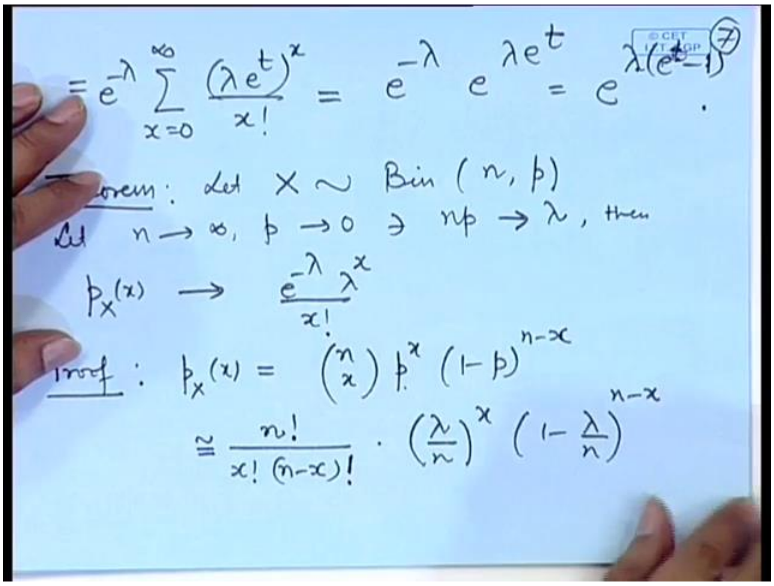

So, this is equal to e to the power minus lambda, e to the power lambda, e to the power t this terms is an expansion of e to the power lambda e to the power t, therefore the moment generating function of a Poisson distribution is e to the power lambda, e to the power t minus 1, which is existing for all values of t.

You can observe here that at t equal to 0 this is equal to 0. Therefore, it becomes e to the power 0 that is 1. If t is positive then e to the power t is greater than 1 therefore, this will be positive and therefore, this term is greater than 1. If I am considering t to be negative then e to the power t will be less than 1 therefore, this term will become negative and since lambda is positive this term will become less than 1 actually it will be between 0 and 1.

So, this is the way the moment generating function of the Poisson distribution behaves. Now we have already given you the way a Poisson distribution arises in natural process; so if we are looking at number of arrivals or number of occurrence, during a process which satisfy certain conditions, then the distribution of the number of occurrence during a specified time interval or during a specified area or during a specified portion of a space follows a Poisson distribution. However, it has also connection with the distribution which arises out of Bernoullian trials.

let us consider say X following a binomial distribution with parameter n and p, let n tend to infinity, p tend to 0 such that n p tends to some number say lambda, then the distribution of x converges to e to the power minus lambda, lambda to the power x by x factorial; that means, the binomial distributions probability mass function converges to the mass function of the Poisson distribution under the conditions that n tends to infinity p tends to 0 such that p tends to lambda.

A physical interpretation of this is that in a sequence of Bernoullian trials if n becomes very large, then it means that the probability of a single occurrence becomes very small. So, we were considering event such as probability of making a mistake in typing a certain a section of a chapter, probability of an occurrence of some accident at a particular traffic crossing. So, here you can consider this as Bernoullian trial in the sense that happening of an accident or not happening of an accident.

So, there are thousands of vehicle passing and one of them may be involved in the accident therefore, what will happen is that, the number of trails suppose 1000 vehicles are there and 1 accident takes place. So, may be 1 or 2 of the vehicles are involved in the accident. So, the probability of 1 vehicle meeting with an accident that is p is very small as compared to n. However, there will be a fixed proportion of the number of occurrences, which we call the rate. So, this seems logical that the binomial distribution should converge to Poisson distribution.

Let us look at a proof of this fact; we may consider the P x and it is equal to n c x, p to the power x, 1 minus p to the power n minus x. Now here the limit process involves n and p; however, since n p itself converges to lambda; that means, in the limit there is a relation between n and p. So, we can write it as n factorial divided by x factorial, n minus x factorial and this p we can write as lambda by n,1 minus lambda by n to the power. So, this is already I have taken the approximation of p as lambda by n, because n p converges to lambda.

So, p can be replaced by lambda by n and in the long run. Once again in order to take the limit we have to look at since n tending to infinity, so here factorials are involved, we have to simply these terms.

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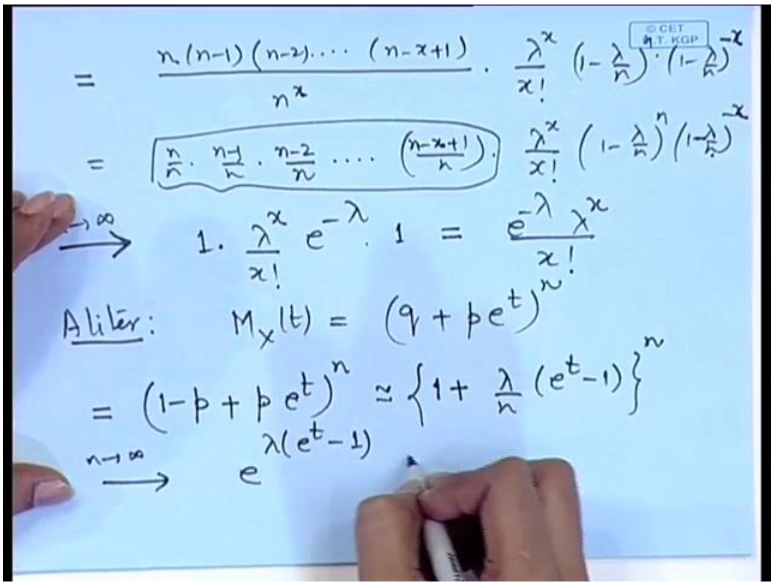

So, we can write it as n into n minus 1, n minus 2 up to n minus x plus 1. So, this term has been adjusted; in the denominator we have n to the power x. So, this term we put here. So, we are left with lambda to the power x by x factorial, then we have 1 minus lambda by n to the power n and 1 minus lambda by n to the power minus x.

If we look at this ratio these are x terms each of the terms are divided by n. So, here you look at the first term that is n by n, second term is n minus 1 by n which goes to 1 as n tends to infinity, the next term is n minus 2 by n as n tends to infinity this also to goes to 1 and so on and n minus x plus 1 by n as n tends to infinity this also goes to be 1, because x is a fixed number between 0 1 to n. Therefore, for a fixed x as n tends to infinity n minus x plus 1 by n will go to 1. Then we have lambda to the power x by x factorial, 1 minus lambda by n to the power n, 1 minus lambda by n to the power minus x.

So, when we take the limit as n tends to infinity, this entire block if this converges to 1, then lambda to the power x by x factorial, then 1 minus lambda by n to the power n converges to e to the power minus lambda and here x is fixed therefore, lambda by n goes to 0 and this term goes to 1. So, this is nothing but the probability mass function of a Poisson distribution with parameter lambda.

This proof can also be given using moment generating functions. If we look at the moment generating function of the binomial distribution: that is q plus p e to the power t to the power n, this we can consider as 1 minus p plus p, e to the power t to the power n. Now here you notice that the P and n both are involved here and we have to take the limit, therefore we can replace p as approximately lambda by n. So, this becomes 1 plus lambda by n e to the power t minus 1. So when we take the limit as n tends to infinity, this goes to e to the power lambda e to the power t minus 1.

So, by the uniqueness of the moment generating function we can say that the probability mass function of the binomial distribution converges to the probability mass function of the Poisson distribution. So, if we are having a certain problem to be solved for binomial distribution, where n is large p is small, such that n p converges to fix number lambda, then we may make use of the Poisson approximation.

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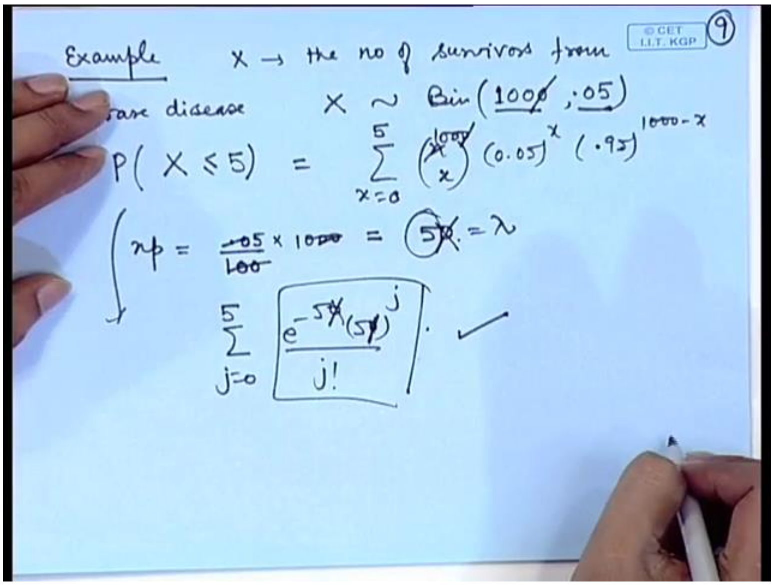

Let us consider one application here. So, let x denote the number of survivors from a rare disease; the probability let say 0.05. And out of 1000 patients what is the probability that say X is less than or equal to 5? Now if we directly want to calculate this 1, then it is equal to n c x 0.05 to the power x, 0.95 to the power 1000 minus x, here n is 1000, x is equal to 0 to 5. So, if you observe this term it can be calculated using certain extensive calculations.

But this involves heavy approximations, because 0.95 to the power say 1000, 0.95 to the power 999 etcetera. So, this will lead to lot of computational errors. However, here if we observe n is large and p is a small; so here n p that is 0.05 into 1000. So, this can be considered as; that is lambda. So, we may make use of e to the power minus 50, 50 to the power j by j factorial j is equal to 0 to 5. Suppose this n was 100 in place of 1000, then this will become slightly simpler, this will be 5 here and this will be 5 and this can be easily evaluated.

In fact, we will talk about this also that when lambda is large in a Poisson distribution, then what happens and when we discuss normal distribution.

Thank you.