Probability and Statistics
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Lecture – 14
Probability Distributions of a Random Variable – I

Next we will talk about the Probability Distributions of a Random Variable.

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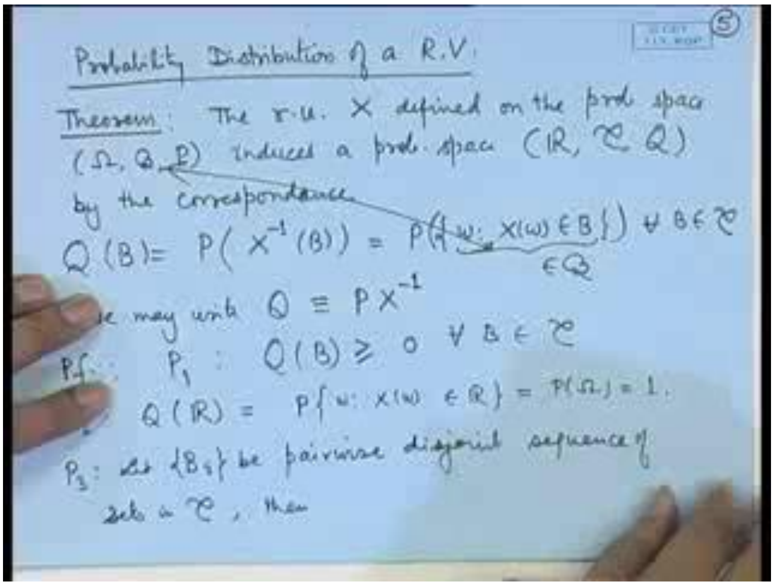

We have the following result; the random variable X defined on the probability space omega B P induces a probability space which we call R, C, Q by the correspondence; Q B is equal to probability of X inverse B, which is equal to probability of the; that the random variable takes values in the set B for all B belonging to C. Since it is a measurable function, X is a measurable function therefore this set is belonging to script B here, and therefore this probability is well defined. We may write Q as P X inverse, now immediately one should be interested to check whether it is a valid probability function or not; that means, it is satisfying the 3 axioms of probability.

We claim that this is true, let us look at the first axiom. So, Q B since it is equal to a probability in the original probability space, it is always greater than or equal to 0. If I consider Q of R then it is equal to probability of the random variable taking values in R, naturally it is probability of omega and that is equal to 1.

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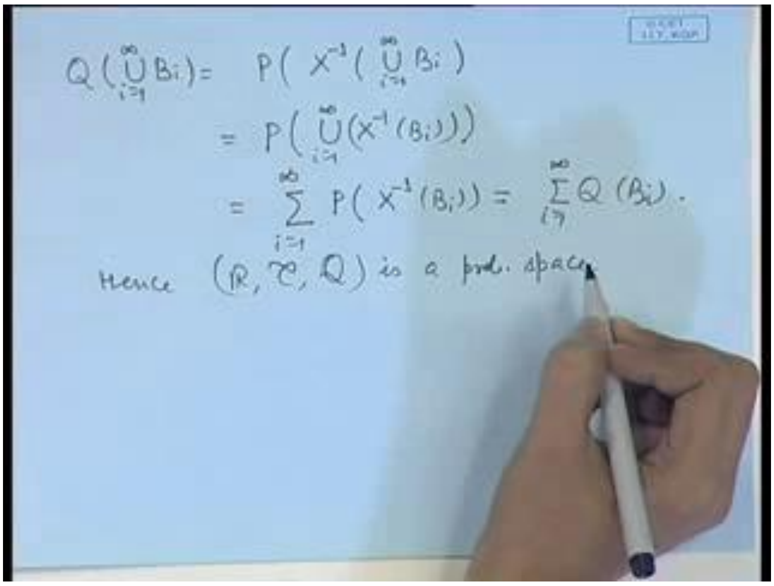

If we consider Bi's as pair wise disjoint sequence of sets in C then Q of union Bi, I is equal to 1 to infinity is equal to probability of X inverse union of Bi. By the definition of inverse, this is if Bi's are disjoint then X inverse Bi's are disjoint and therefore, by applying the countable additivity axiom and this is exactly equal to Q of Bi. So, all the 3 axioms are satisfied and therefore, R, C, Q is a probability space. So, this Q is called the probability distribution of the random variable. So, let us look at this example of head and tail.

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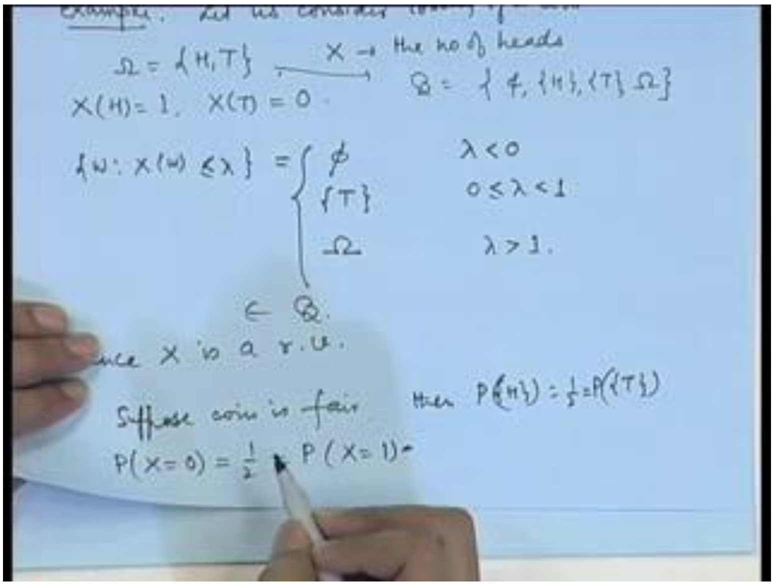

So here if we say that the coin is sphere, then you will have probability of H is equal to half and probability of tail is equal to half and therefore, if I say probability of X is equal to 0; that is equal to half and probability of X equal to 1 is half. So, this is giving you the probability distribution of the random variable X.

Since Q is a set function and here I am looking at the real numbers. So, there is another function which we consider continuously and it is called distribution function. So, we define probability of X less than or equal to X; as say F X of x, this is called cumulative distribution function of the random variable X, this is basically this is the probability of all the sample points such that X omega is less than or equal to x. So, this is an abbreviation we will be writing down like this. This function is quite useful in talking about all types of probability statements of the random variable x.

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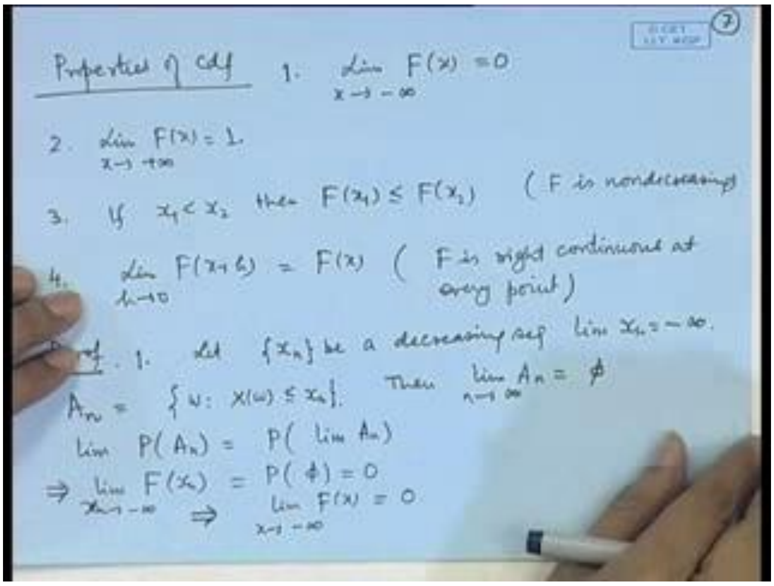

So, first of all this cumulative distributions function satisfies certain important properties, properties of CDF; the first property is that if I consider limit of F x as x tends to minus infinity that is 0. If you look at limit of F x as x tends to plus infinity is equal to 1, if x 1 is less than x 2 then F of x 1 is less than or equal to F of x 2. If I consider limit of F x plus h; as h tends to 0, then it is equal to F of x; that means, continuous from the right at every point; this is F is non-decreasing, right continuous at every point. To look at the proofs of this statement, we can consider the basic statements of the probability.

Basic properties of the probability, so let us see to prove the property one. Let x n be a decreasing sequence such that limit of x n is equal to minus infinity. If I consider the sets A n as the set of all those points; such that X omega is less than or equal to x n, then limit of A n is equal to phi as n tends to infinity because if x n is a decreasing sequence, then A n will be a decreasing sequence and if I take limit of A n as n tends to infinity; this will be the empty set. So, now limit of probability of A n is equal to probability of limit of A n. So, this is limit of probability of A n is nothing, but F of x n and on the right hand side I will have probability of phi that is equal to 0.

So, if I am considering x n tending to minus infinity then this is equal to 0. So, if along all the decreasing sequence has 2 minus infinity; limit of F x and as x n tends to be minus infinity is 0 then this implies that limit of F x as x tends to minus infinity must be 0. In a similar way, we can consider a sequence which is increasing to plus infinity.

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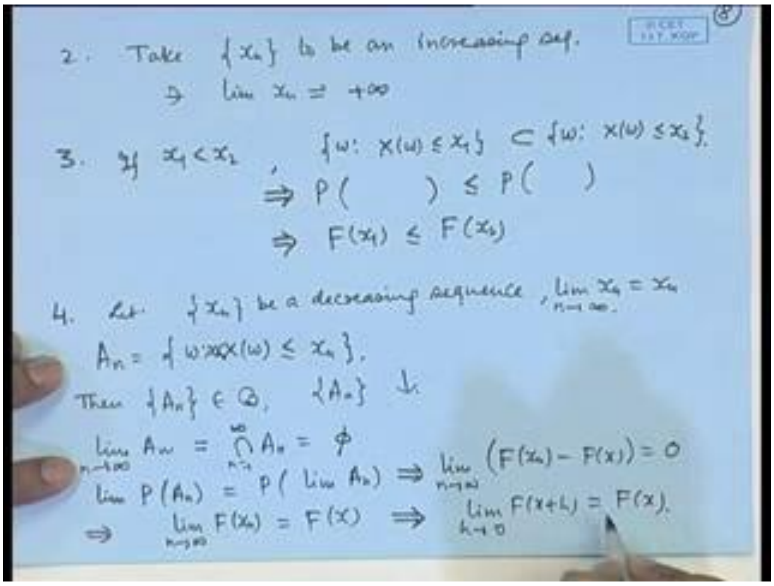

And then if we take the limit take x n to be an increasing sequence such that limit of x n is equal to plus infinity, if we take this and apply the argument given in the previous one then here we will have limit of A n is equal to fully space and therefore, limit of omega limit of A n will be omega and here you will get equal to 1. To prove the third property, if I have x 1 less than x 2 then the set omega such that X omega less than or equal to x 1 will be a subset of a set of all those omega such that X omega is less than or equal to x 2. So, by monotonicity of the probability function probability of this set will be less than or equal to probability of this set, which is equivalent to saying that F of x 1 is less than or equal to F of x 2.

So the cumulative distribution function is a non-decreasing function; to prove the continuity from the right, let us consider a sequence let x n be a decreasing sequence with limit of x n is equal to say x; as n tends to infinity. Now consider the set A n as the set of all those point such that X omega is less than or equal to x n. Then these are my label sets, A n is a decreasing sequence; they belong to be A n's are decreasing and if I consider limit of A n; as n tends to infinity then it is equal to; now you look at the definition here, I am considering this; however, here I should put x less than this, if I put this one then since this is a strictly x is strictly less than x omega. So, if I take the limit here this is going to be an empty set; that is intersection of A n as n is equal to 1 to infinity is an empty set.

Because if I consider any point, x is not included in any of them therefore, it cannot be included in the intersection. See if I take limit probability of A n is equal to probability of limit of A n, then this implies that limit F of x n minus F of x as n tends to infinity is equal to 0, which means limit of F of x n tending to infinity is F x. Now here x n was any sequence which was decreasing to x; that means, a sequence from the right side of X now if it is true for any arbitrary decreasing sequence to X then this implies that limit of F x plus h as h tends to 0 must be equal to F of x.

This proves that the function F is continuous from right, this cumulative distribution function is a quite an important function in order to define the distribution of a random variable. In fact, these four properties which we have given as properties of a CDF are also the characterizing properties of a CDF. That means, if there is a function which satisfies these four properties; then it will be CDF of certain random variable.

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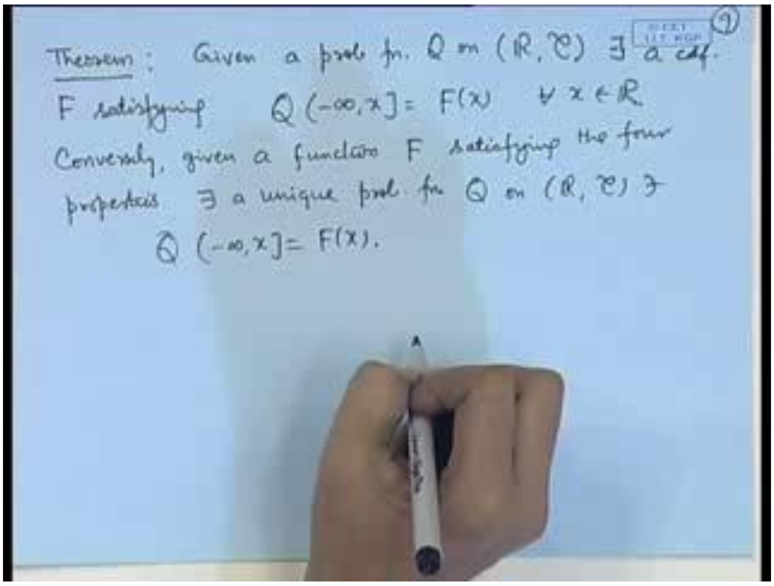

So, I state this in the form of the following theorem; given a probability function Q on say R, C; there exists a CDF, F which satisfies Q of minus infinity to x is equal to F of x for x belonging to R.

Conversely given a function F satisfying the four properties, there exist a unique probability function Q on R C; such that Q of minus infinity to X is equal to F of x. So, this shows that cumulative distribution function; it has a one to one correspondence with the probability function defined for a random variable and it can be actually treated as equivalent to the probability distribution of the given random variable.

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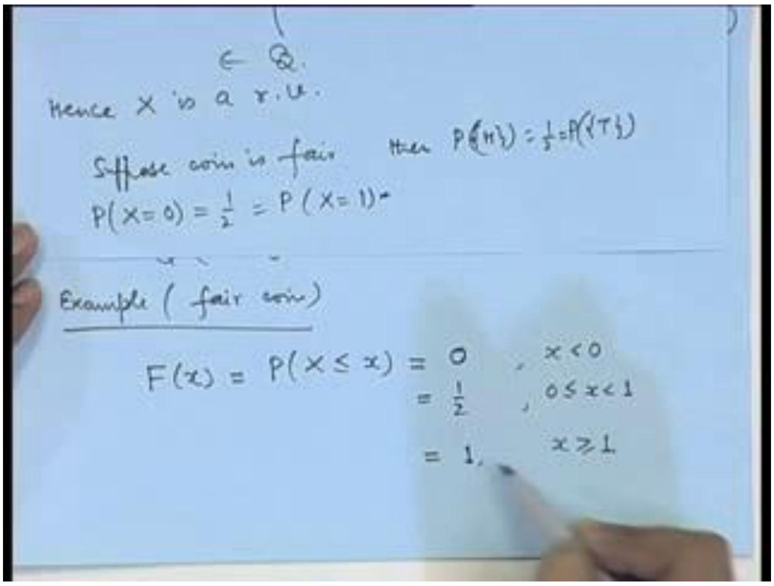

So, let us consider a certain examples here go back to the example of a head and tail and if we consider the fair coin example here; if we consider F x then you are looking at the probability of X less than or equal to x. So, since the random variable takes the smallest value as 0, this probability is 0 for x less than 0. The probability becomes half then x becomes 0 and it remains so till x is strictly less than 1, it becomes 1 for x greater than or equal to 1.

So you can see here this function is continuous on the interval minus infinity to 0 open interval; open interval 0 to 1 and open interval 1 to infinity. If we look at the change points that is at X is equal to 0, the right hand limit at X is equal to 0 is half; the value at X is equal to 0 is half. So, it is continuous from right at 0; if we look at the right hand limit at 1; it is 1 the value at X is equal to 1 is 1. Therefore, it is continuous from right at 1; at all other point is the function is a strictly continuous. The function is clearly non-decreasing as X tends from minus infinity F x is going to 0 and as X tends to plus infinity it is going to 1. So, all the conditions of the cumulative distribution function are satisfied here.

Let us take one more example that is of a die throwing problem, where I defined the random variable to be the sum of the two dice.

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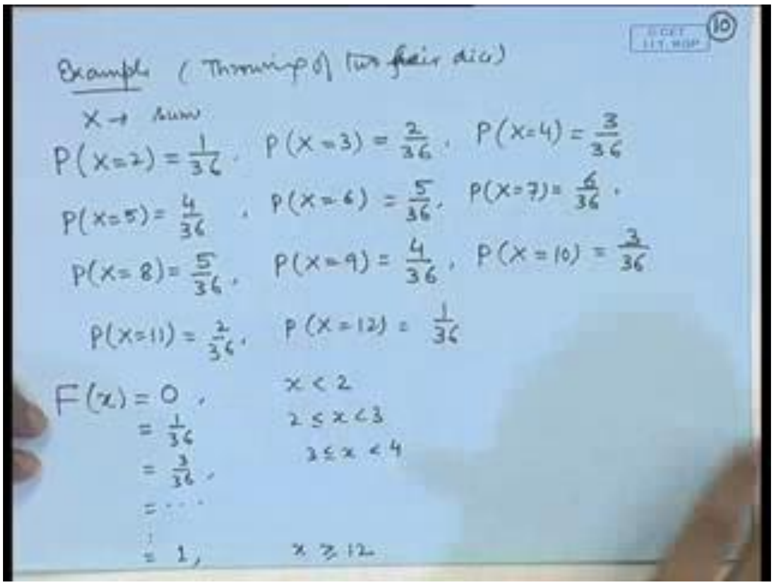

So in that example, if we make certain probability allotment; consider throwing of two fair dice and the random variable X was the sum. Let us look at the probability allotment here, so the probability allotment here will be probability of X is equal to 2, now this is corresponding to the event that the outcome is 1; 1.

So, let us look at the outcome 1; 1, so there is only one outcome of 36 outcomes. So, if it is a fair coin, the probability of this will be 1 by 36, if we look at probability of X is equal to 3, then there are two outcomes; 1; 2 and 2; 1 which are correspondent to the sum being equal to 3; therefore, the probability that x is equal to 3 will become 2 by 36.

If you look at X equal to 4; then we will have favourable outcomes has 2, 2, 1, 3 and 3 1 there are 3 possible outcomes. So, in this fashion we can write down probability X is equal to 5 as 4 by 36, the outcomes will be 1, 4, 4, 1, 2, 3, 3, 2 probability X is equal to 6 is equal to 5 by 36, probability X is equal to 7 will be equal to 6 by 36.

The favourable outcomes will be 1, 6, 6, 1, 2, 5, 5, 1 and 3, 4, 4, 3 then probability X is equal to 8; we will have favourable outcomes as 2, 6, 6, 2, 3, 5, 5, 3 and 4, 4 that is 5 by 36, probability X is equal to 9, we will have four favourable outcomes, so the probability will become 4 by 36, probability X is equal to 10 will be 3 by 36, probability X is equal to 11; will be 2 by 36 and probability X is equal to 12 will be equal to 1 by 36 corresponding to only the outcome 6, 6. Now if I want to write down the cumulative distribution function here then we look at the set up the values that the random variable can take, it is taking values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12 with the positive probabilities and all others are 0.

So, the starting point is x less than 2 then at equal to 2, you will have 1 by 36, it is equal to 3 by 36 for 3 less than or equal to x less than or equal to less than 4 and so on; it is equal to finally, 1 for x greater than or equal to 12. So, you can again see here that it is satisfying the properties of the CDF, as X tends to minus infinity it is 0 as X tends to plus infinity it is 1; it is a non-decreasing function and it is continuous from right.

Let us take the continuity from the right at the end points of the intervals where the value of the function is changing. Suppose I look at x is equal to 2 then the right hand limit at x is equal to 2 is 1 by 36, the value at X is equal to 2 is equal to 1 by 36. If I take the right hand limit as x tends to 3, the value is 3 by 36, the value at X is equal to 3 is 3 by 36. So, the properties of the cumulative distribution function are satisfied here.

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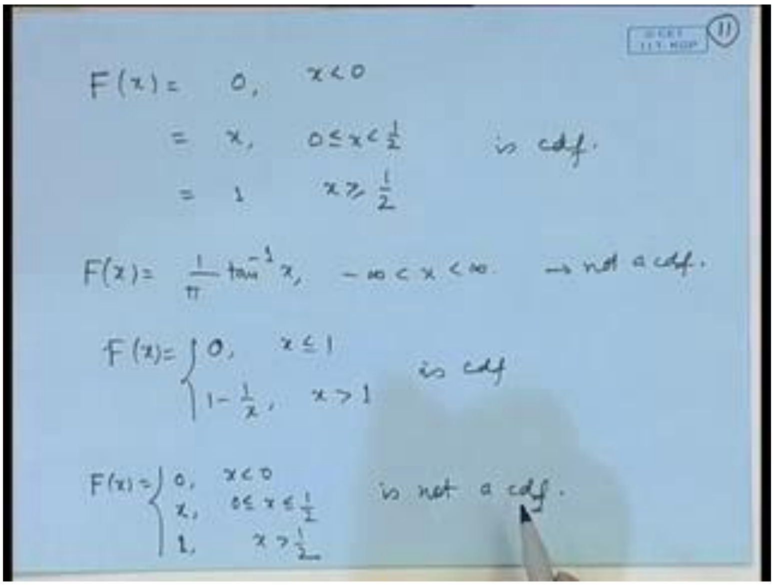

Let us take some problem say F x is equal to say 0 for x less than 0, it is equal to say x for 0 less than or equal to x less than half; it is equal to 1; if x is greater than or equal to half; is it a CDF. If we check the condition that x tends to minus infinity, it is 0 that is satisfied, if I take x tends to plus infinity the limit is 1; if I look at the non-decreasing nature of the function that is satisfied.

Let us check the continuity from the right, so for the interval minus infinity to 0, for the open interval 0 to half, for the open interval half to 1; it is continuous, if I check at x is equal to 0; the right hand limit at x is equal to 0 is 0, the value at x is equal to 0 is 0. So, it is continuous from right at 0, let us take the value x is equal to half. So, at the right hand limit at x is equal to half is 1 and the limit as x tends to and the value at x is equal to half is 1 and that the right hand limit at x is equal to half is also 1; therefore, the function is continuous from right.

So, it is defining the CDF of a random variable; if we consider say F x is equal to 1 by pi tan inverse x, if we take x tending to minus infinity; this goes to minus pi by 2. So, this is not tending to 0; so it is not a CDF; if I consider say F x is equal to say 1; 0 for X less than or equal to 1, it is equal to 1 minus 1 by X for X greater than 1. Once again check the conditions; condition 1, 2, 3 they are satisfied; the limit as x tends to 1, the right hand limit is 0; the value at x is equal to 1 is 0; therefore, it is a CDF, this is a CDF.

Suppose here I change to F x is equal to 0 for x less than 0, x for 0 less than or equal to x less than or equal to half it is equal to 1 for x greater than half then you see here the condition of continuity from the right at the point X is equal to half is violated because as x tends to half from the right the limit is1, but the value at x is equal to half is half, so it is not continuous from right, so this is not a CDF.

Now, we consider classification of the random variables from the examples that we have considered for example, tossing of a coin and the random variable was defined as the number of heads, so it was taking values 0 and 1. If we consider the tossing of two dice and we took the random variable x to be the sum of phases appearing upwards, then it is taking values 2, 3 up to 12.

However there may be some other random variables for examples I may be looking at the life of a bulb then the life of a bulb could be 1 hour, it could be 1 hour 5 minutes, it could be 56 seconds etcetera; that means, it is an interval. Therefore, the nature of the random variable is different from the once which we considered just now. Similarly suppose x denote the number of trials needed for the first successful experiment with a drug to treat certain disease.

Then the number of trails may be 1, 2 3 and so on; that means, the set of possible values of the random variable is countably infinite.

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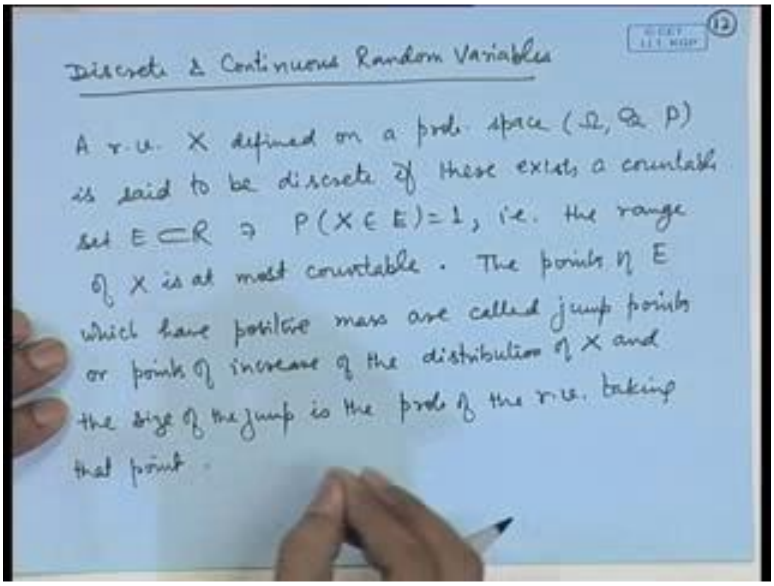

So, based on this we have the broad classification of the random variables discrete and continuous random variables. So, a random variable X defined on a probability space omega B P is said to be if there exists a countable set E such that probability of X belonging to E is 1; that is the range of X is at most countable.

The points of E which have positive mass are called jump points or points of increase of the distribution of X and the size of the jump is the probability of the random variable taking that point. So, the probability distribution of a discrete random variable is defined a mass function; that means, the probability of random variable taking a particular value say x i, where x i will belong to a countable set E, so it will satisfy the following properties.

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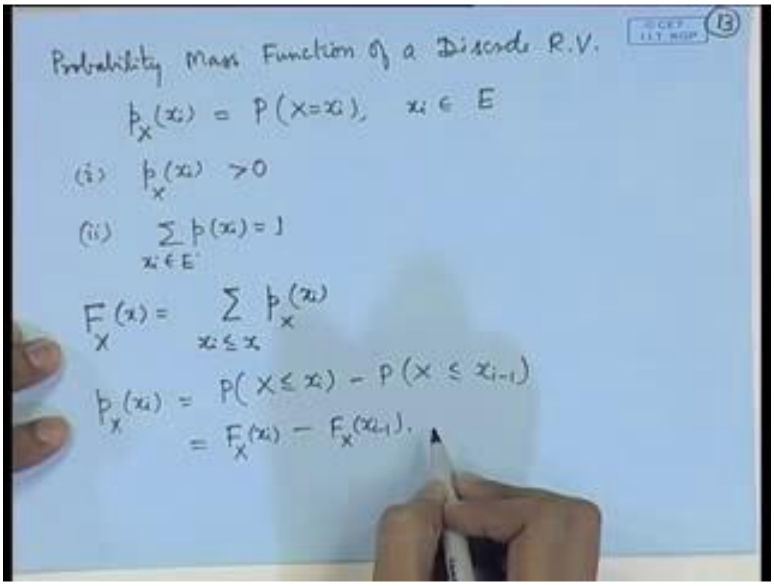

We call it a probability mass function of a discrete random variable, so it is defined by a mass function p x; x i, it will satisfy the properties. So, this is actually probability of X equal to x i, for xi belonging to the countable set E. So, we must have p x i is strictly positive and sigma probability of x i is equal to 1 for x i belonging to E.

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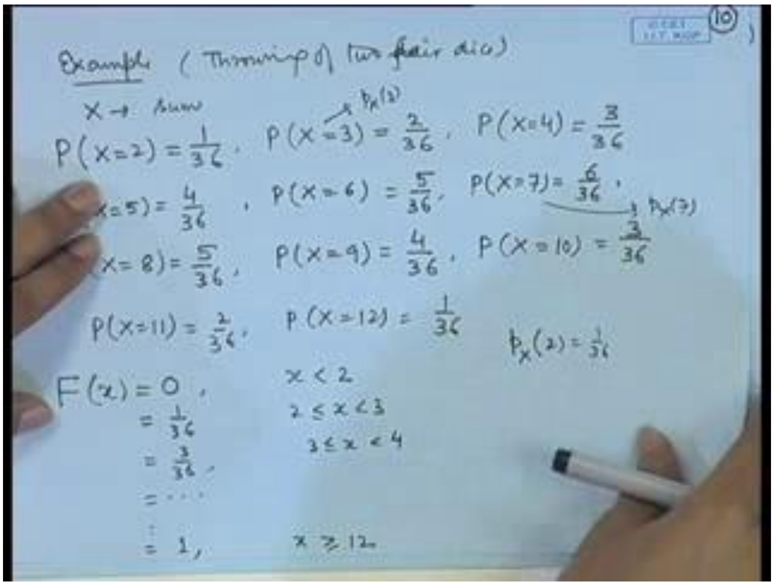

So if you look at the examples discussed earlier, the tossing of a coin, the throwing of a die etcetera then the probability distribution is quite clear. For example, let us look at this one; the die throwing problem. So, here it is exactly p x; we can write it in the terms 4,r

3, so we can write p x 2 is equal to 1 by 36; this is p x 3, this is p x 7 etcetera, it is clearly satisfying the properties that p x; xi is greater than 0 for x i belonging to the set here.

Here what is the set E, the set E is here 2, 3 up to 12, now from here you can establish a relationship between the cumulative distribution function and the probability mass function. The cumulative distribution function can be expressed as F x is the sum of all those values of the probability mass function for which x i is less than or equal to x, conversely p x; x i can be written as probability of X less than or equal to x i minus probability of X less than or equal to x i minus 1, so this will become F of x i minus F of x i minus 1.

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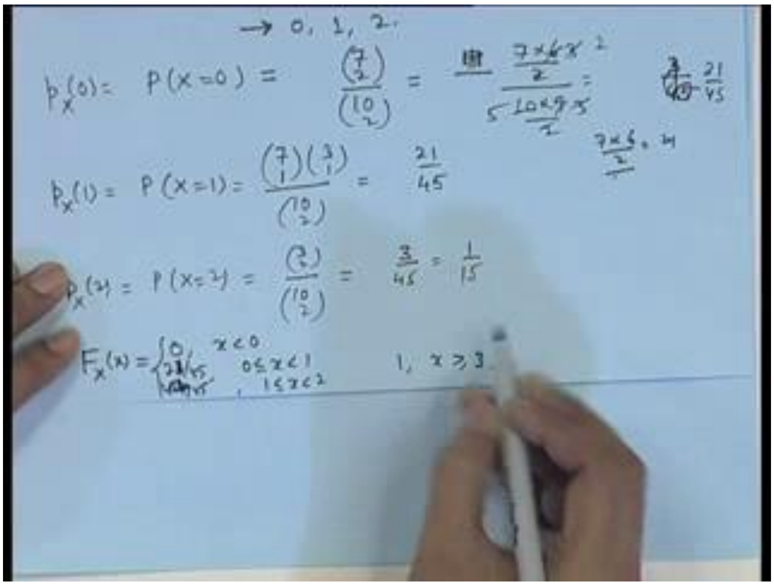

Let us take one example; a computer store contains 10 computers of which say 3 are defective. A customer buys 2 at random. Let X denote the number of defectives in the purchase then the possible values of X can be 0, 1, 2, so what is P X 0. Now this event corresponds to that out of 7 good P X chosen to divided by 10; C 2. So, this is equal to sorry 7 in to 6 by 2 divided by 10 into 9 by 2 which is equal to; if I consider probability X equal to 1, then it is equal to 7 C 1; 3 C one divided by 10 C 2. If we consider P X 2; that means, both P X purchased as defective then it is equal to 3 C 2 divided by 10 C 2. So all of these values can be calculated, 3 by 45 that is equal to 1 by 15, this is equal to 21 by 45 and that is equal to 3 by 45. Now here if I want to write down the probability cumulative distribution function, it is equal to 0 or x less than 0; it is equal to 3 by 45 or

0 less than or equal to x less than 1; it is equal to 22 by 45; 24 by 45 for 1 less than or equal to x less than 2 and it is equal to 1 for X greater than or equal to 3.

Now you look at the jump points of this distribution, so here at x equal to 0; there is a jump of size 3 by 45; which is equal to the probability of a random variable x taking value 0. I think we have made some mistake here it is not equal to this, so it is 7 into 6 by 2 that is equal to 21 and divided by 45, so 21 by 45. So, this is equal to 42 by 45 and 1. So, the probability of x equal to 0 and x equal to 1 is 21 by 45 and the probability of x equal to 2 is equal to 3 by 45 that is 1 by 15, which is equal to the size of the jump at the points 0, 1 and 2 respectively.

Thank you.