Probability and Statistics
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Lecture – 11
Problems in Probability-I

Today we will discuss applications of the various rules of probability that we derived for example: addition rule, multiplication rule, the conditional probability, base theorem, the concept of independence of events etcetera.

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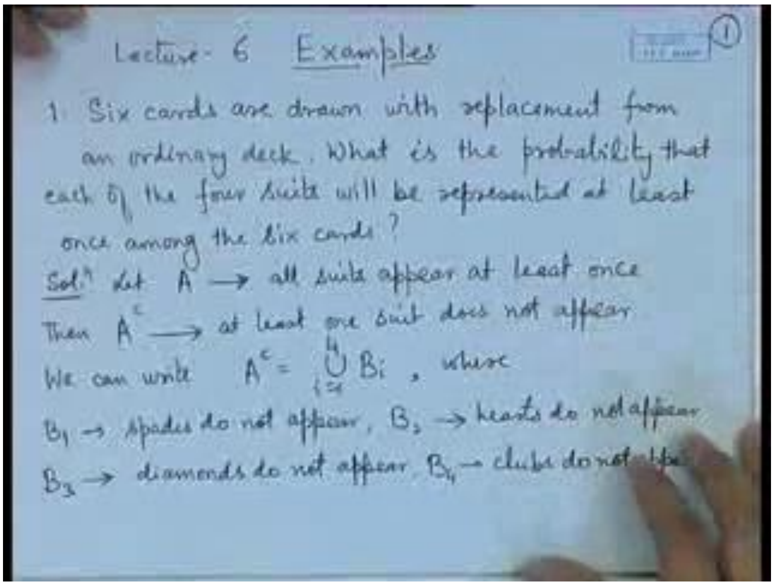

So, let me start with one application of the general addition rule; let us consider the problem. So, 6 cards are drawn with replacement from an ordinary deck. What is the probability that each of the 4 suits will be represented at least once among the 6 cards? So, here we are assuming that the deck of cards is well shuffled, there are 52 cards and 4 suits represent a spade, heart, diamond and club.

So, if you are drying; so we dry a card and we put it back. So, after note I down the colour of the suit of the card we put it back in the deck and again withdraw. So, this way it is called sampling with replacement. So, the event that we are interested is that out of 6 cards the 4 suits are represented at least once. Now if we try to find out the probability in the state forward fashion, the possibilities are too many for example, there could be 4 is spades and there could be one card of each and then the remaining two cards could be of any combination, they could be is spade, heart both could be spade, one could be is spade, one could be diamond and so on. So, the number of possibility is too many.

Here we will show that if we make use of the idea of complementation as well as the union of the events, then the problem is somewhat simpler. So, let us consider the event A as that all the suits appear at least once. Then A compliment denotes the event that at least one suit does not appear. Now once again if we try to de composite it directly by saying that exactly one suit does not appear, exactly two suits do not appear, exactly three suits do not appear, then once again it is going to be a complicated event. So, we represent this as a different union, union of B i, i is equal to 1 to 4 where B 1 denotes the event that a spades do not appear, B 2 denotes the event that hearts do not appear, B 3 denotes the event that diamonds do not appear and B 4 denotes the event that the clubs do not appear.

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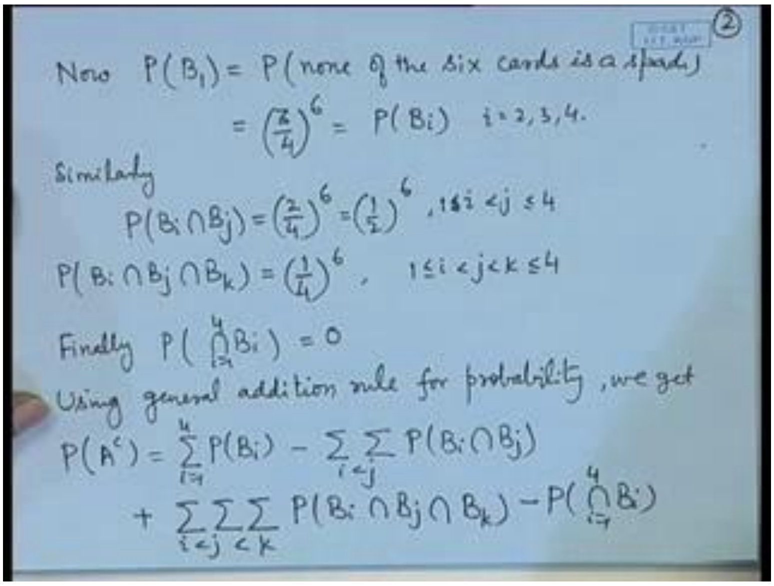

So, now by the addition rule probability of A compliment is equal to sigma probability of B i, minus sigma double summation probability of B i intersection B j, where i is less then j and the summation is going up to 4. And plus triple summation probability of B i intersection B j intersection B k, where i is less then j less then k and the sums are going up to 4, minus probability of intersection B i, i is equal to 1 to 4. So, we need to evaluate the probabilities of all the terms appearing in this expansion of probability of a compliment.

Let us consider say probability of B i. So, here I will need to consider probability of B 1, probability of B 2, probability of B 3 and probability of B 4. Let us consider probability of B 1. Now B 1 is the event then that none of the 6 cards is a spade. Now if none of the cards is a spade; that means, in one draw of a card there are 13 spades. So, if it is not a spade then the probability of that is 39 by 52 that is 3 by 4. Since the drying of the cards are independent and identical, because it is with replacement. So, every time there are 52 cards, the probabilities will be simply multiplied; it will be 3 by 4 into 3 by 4 6 times; that means, it is becoming 3 by 4 to the power 6. Now if you notice here that if we replace this word is spade by say club or by heart or by diamond then the argument remains the same. Therefore probability of B i for i is equal to 1, 2, 3, 4 is 3 by 4 to the power 6.

Now in a similar way if we consider the event say probability of B 1 intersection B 2, now B 1 means that by spade do not appear B 2 denotes the event that hearts do not appear. So, B 1 intersection B 2 means that in drying of the card is spade and heart do not appear now in a deck of 52 cards, 26 cards are for a spades and hearts. So, in a single draw if it is not a spade or a heart the probabilities half, therefore in 6 independent draws with identical set up, the probability becomes half to the power 6. Now this probability remains the same if we replace a spade by hearts, the diamonds by clubs etcetera.

So, for all the combinations of a spade heart, a spade club, a spade diamond, heart diamond, heart club and diamond heart, this probability of B i intersection B j is half to the power 6. Now in a similar way if we consider 3 of the suits do not appear then the probability will be simply 1 by 4 in a single draw and it will become 1 by 4 to the power 6 in 6 draws, Therefore, for all the combinations of i j k; for i less then j then less k lying between 1 and 4, probability of B i intersection, B j intersection, B k will be 1 by 4 to the power 6. The last term here is probability of intersection B i; however, what is the probability of intersection of B i? Intersection B i denotes the event that none of the suits appear. However, if you draw a card it has to be one of the suits therefore, the probability of intersection B i must be 0.

Now, we substitute these probabilities in the general addition rule. So, there are 4 terms of probability of B i each of them is equal to 3 by 4 to the power 6 then if we look at the second term which is having the probability of intersection of two events, then out of 4 there are two selections here, where i is less than j. So, it is C 2 combinations. So, there are 6 such cases which have probability half to the power 6; if we look at intersection of events 3 taken at a time, then out of 4 we can choose them in 4 C 3 that is 4 possible ways and the probabilities of these are 1 by 4 to the power 6.

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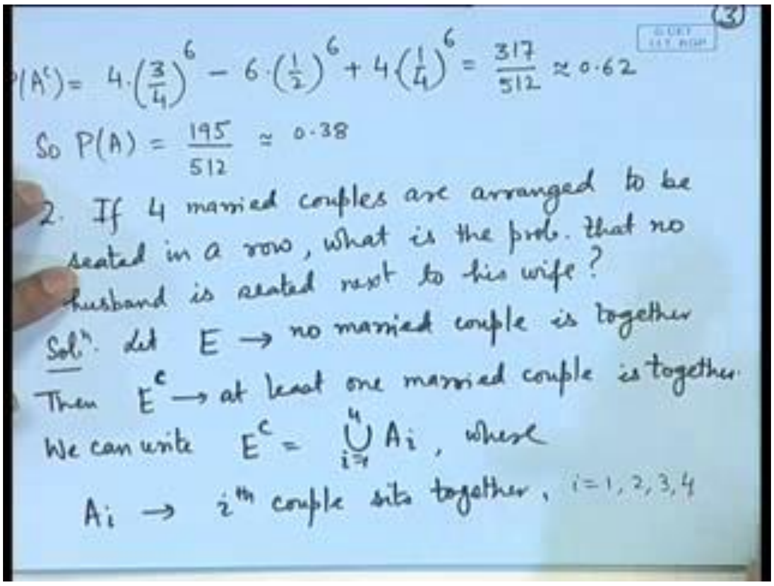

The last term is zero, therefore the probability of A compliment that is equal to this expression which after simplification terms out to be 317 divided by 512 and approximately it is 0.62 and therefore, probability of A becomes 1 minus this that is equal to 195 by 512 or approximately 0.38.

So, the answer to the question that each of the 4 suits will be represented at least once is 0.38, which is less than 40 percent basically. Let us look at one more application of this general addition rule, if 4 married couples are arranged to be seated in a row, suppose there is a long table where these people 8 persons, who are actually basically 4 married couples are to be seated, what is the probability that no husband is seated next to his wife?

So, if we analyze this event directly let us call the pairs 1, 2, 3 and 4 then the possibility that no husband is seated next to his wife, will lead to various combinations. For example, husband 1 seated next to wife 2, husband 1 seated next to wife 3, husband 3 seated next to wife 4 and so on. The total number of possibilities to many and it will be a enumeration problem. However, we can simplify this by considering complimentary event and then making use of the unions of events.

So, let us define the event E to be that no married couple is together; then E compliment denotes the event that at least one married couple is together. Therefore, E compliment can be written as union of A i, i is equal to 1 to 4, where A i denotes the event that it ith couple sits together for i is equal to 1, 2, 3, 4. Notice here that this is a clever way of representing the union because the other way of representing the union could have been union of B i, i is equal to 1 to 4 where B 1 even would have meant that one married couple sits together, B 2 would have men the 2 married couples sits together etcetera.

However, evaluation of the probability is of those events could be equally complicated whereas; here you will see that this representation leads to an easy calculation of the probabilities involved.

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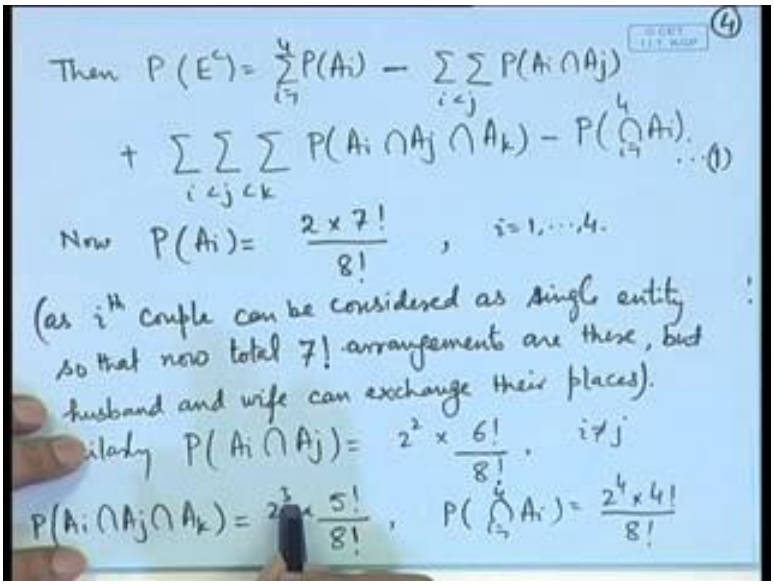

Let us consider say the general addition rule. So, like in the previous problem probability of E compliment become some of the probabilities of individual events, minus double summation probability of intersection of 2 events taken at a time, plus triple summation probability of 3 events taken at a time, minus probability of intersection of all of them taken together. Now the next step is to evaluate probabilities of individual terms here. So, if you look at probability of A i then A i denotes that the ith couple sits together.

Now, in order to evaluate this we can make use of the classical definition of the probability, where we look at the favourable number of cases and the total number of cases. So, since there are 8 persons to be seated in a row, the total number of permutations in which they can sit is 8 th factorial. Now if I treat ith couple as one entity because if I am saying that they sit together then they it can be on the left or the right therefore, the total number of arrangements that we have to consider is for only 7 people because 6 persons and then 7 th and there that is the ith couple, it is considered as one in individual and we have to put that together somewhere along with those 6 people, so the total number of arrangements can be 7 factorial.

Now, here the place of husband and wife itself can be interchange that is in two possible ways. So, the total number of possibilities becomes 2 into 7 factorial which is favouring to the event that the ith couple sits together, and therefore, the probability of A i is simply 2 into 7 factorial divided by 8 factorial and of course, is argument is valid for any of the ith couple; that means, for i is equal to 1 to 4. Now if we extend this argument and consider the event A i intersection A j; that means, I am saying ith couple and the jth couple sit together and we are not concerned about the other couple. So, there are 4 persons left plus 2 couple which will be treated as 2 entities. So, there will be 6 persons and these 6 persons can be arranged in 6 factorial possible ways.

Since in this arrangement two of the persons are basically couple and therefore, each of them can inter change their places, which can be done into a square ways; because in one of it h pair the husband wife can inter change, their places in two ways and the j th pair the husband wife can interchange their places in two ways. So, 2 is square. So, the favourable number of cases becomes 2 square into 6 factorial divided by 8 factorial. So, and this will be true for all pairs of i and j. In a similar way if we consider ith, jth and kth pairs of couple sit together then the total number of arrangements could be only 5 factorial into 2 cube, let us look at this if we have fixed the 3 couples then one extra couple is left whom we are treating as separate. So, that is two persons and these 3 couple will be considered as 3 persons. So, the total number of persons will be 5.

Therefore, these people can be arranged in 5 factorial ways; now in each pair the husband wife can interchange their places and therefore, each of them will have two extra arrangements, so 2 into 2 into 2 3 times. So, the total number of cases of that ith, jth and kth couples sit together is 2 to the power 3 divided by into 5 factorial divided by 8 factorial. Finally, if I say that all the 4 couples sit together then basically it will be arrangements of only 4 persons, 4 factorial plus all the husband wife pairs can interchange their places among themselves. So, that is 2 to the power 4 divided by 8 factorial. Now if I look at here probability of A i is this term and this is appearing 4 times, probability of A i intersection A j is this term and this is appearing 4 C 2 times.

Probability of A i intersection A j intersection A k is this term and this is appearing 4 c 3 times and this term is single term.

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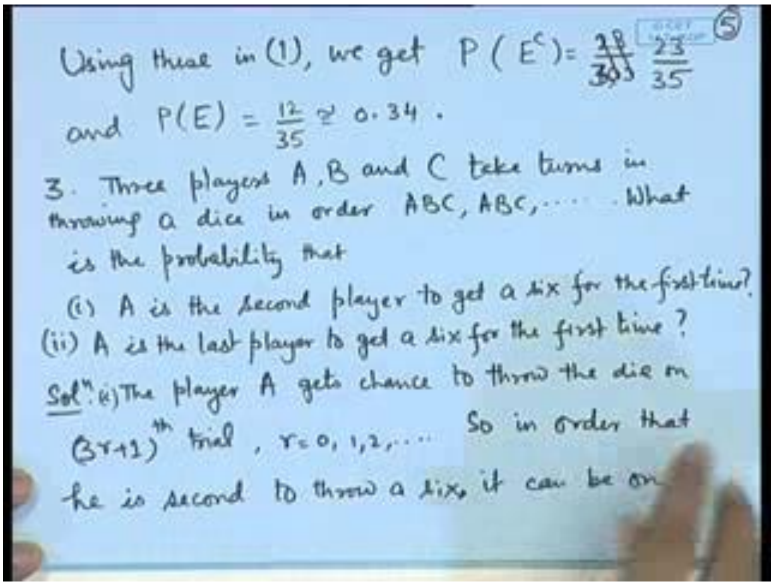

So, if you substitute all these values, probability of E compliment after some simplification turns out to be 23 by 35 or probability of E turns out to be 12 by 35 that is approximately 0.34. So, if we look at our original problem, what is the probability that no husband is seated next is wife is 0.34 which is quite high; that means, in a random arrangement of 4 couples, which are to be seated in a row on a long table, then the probability that none of the pairs are together is approximately 0.34 which is more than one-third. So, which is substantially high; let us look at one more problem where we consider the a splitting of the event into various possibilities and then using the concept independence etcetera.

So, consider a rolling of a die. So, 3 players A B and C they take terms in throwing a die in order ABC, ABC and so on; that means, firstly, the player A throws the die then player B throws the die then player C throws the die then player A throws the die then player B throws the die and so on. In this particular random experiment we are interested to find out the probability that A is the second player to get a 6 for the first time or A is the last player to get a 6 for the first time, what is interpretation of the first event? A is the second player to get a 6 for the first time; that means, either of B or C get a 6 before A in this sequence of trails. So, we analyze this event consider A to the throws of A. Now a gets a chance the first trail, the 4th trail, the 7th trail and so on.

That means he gets to throw at the die on 3 r plus 1 th trial, for r is equal to 0, 1, 2 etcetera. However, on the first trail itself he should not throw a 6, because he will not be then in second pair then he will become a first player. So, r is equal to 0 is ruled out for him to get a 6.

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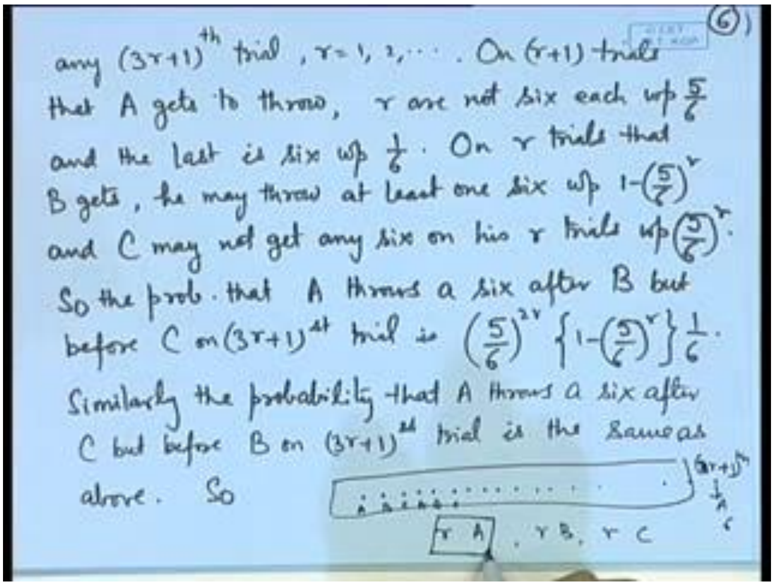

So, in order that he is second to through a 6, it can be on any of the 3 r plus 1th trial for r is equal to 1, 2 and so on not on r is equal 0. Now for r for the player A he is getting r plus 1 trials, because in 3 r plus 1 th trial is getting a 6; that means, before that he is able to get r trials each of A B and C are there get r trails. So, the player A he should not get 6 on first of the r trails, now in a single trial if we are assuming the die to be fair, A will not be able to throw a 6 with probability 5 by 6.

So, in the first r trails, he is not able to get a 6. So, the probability that he will not get a 6 is 5 by 6 to the power r and in the r plus 1 th trail he gets a 6. So, the probability of that is 1 by 6. Now out of this total 3 r plus 1 trial, player B and player C also get r trails to throw. Out of this either of B or C must get a 6, then only A will be a second player to throw a 6 for the first time. Let us consider the case that B gets a 6. So, if we consider the probability that B does not get a 6 it will be 5 by 6 to the power r, because in each trial he will not be able to get a 6 with probability 5 by 6 to the power 5 by 6. So, the total power will t that in r trails he does not get any 6 is 5 by 6 to the power r. So, if we consider 1minus 5 by 6 to the power r, this is denoting the probability that he gets at least 1 6.

So, now if we consider B is getting at least once is then c must not get a 6. So, on each of his r trials c will not get a 6 with probability 5 by 6 to the power r. Now the entire event can be split that you have A B C etcetera and this is the r 3 r plus 1 th trail. So, here a gets A 6 and before that there are r trails for A, r trails for B and r trails for C. In the r trails for A there is no 6 therefore, the probability of that is 5 by 6 to the power r; for c there is no 6, therefore the probability for that is also 5 by 6 to the power r. So, the probability becomes 5 by 6 to the power 2 r; for B he gets at least one six therefore, the probabilities 1 minus 5 by 6 to the power r, on the last trial 3 r plus 1 th trial A gets a 6 and the probability for that is 1 by 6.

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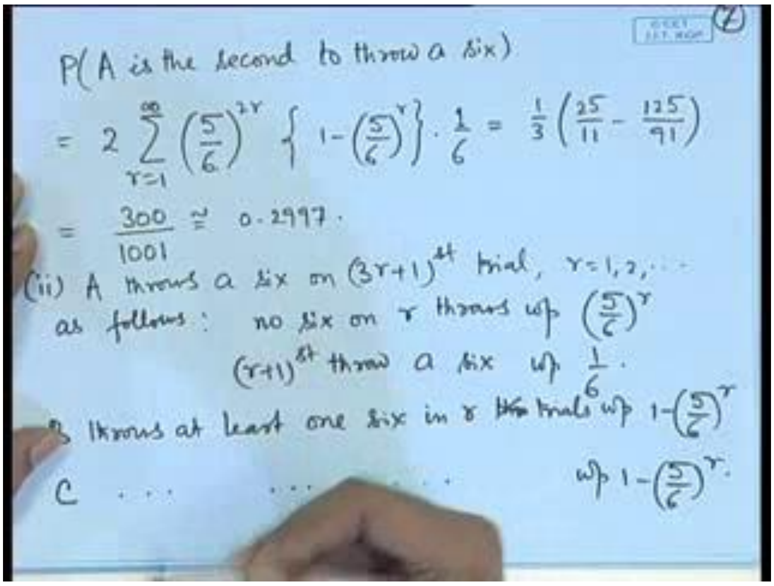

Now, here if we look at this probability, this is the probability that on one particular 3 r plus 1 th trial A gets a 6, before that B has got at least 1 6 and C has not got a 6 and A also does not get a 6 before that, this is denoting the total probability for this event. Now here you can notice here that we have made use of the concept of independence of the trails, because all the probabilities have been multiplied, this is total probability for the 3 r plus 1 th trail in this particular fashion that no sixes for A and no sixes for C and at least one six for B, and the last trial 3 r plus 1 th trial is a 6 for E. Now here r can take any values from 1 to and so on therefore, the probability that A is the second to throw a 6 after B, but before C. Now we can interchange the role of B and C here and we will get the same expression.

Therefore the actual probability that a is the second player to throw a 6 will be 2 times this, because it incorporates the possibility that C is first, A is second and B is third etcetera also. Now we can simply this expression this is 1 by 3 into sum of 1 geometric series minus sum of another geometric series infinite geometric series with the common ratio either 5 by 6 square or 5 by 6 q. So, we can evaluate this and after simplification it turns out to be 300 by 1001 which is nearly 0.3; that means, in this particular sequence a will be the second player to throw a 6 for the first time is nearly 0.3.

See here we can also look at the event that a is the first player to throw a 6, then a must be able to throw it on the first trial on the fourth trail etcetera, if we throws on the first definitely it is 1 by 6.

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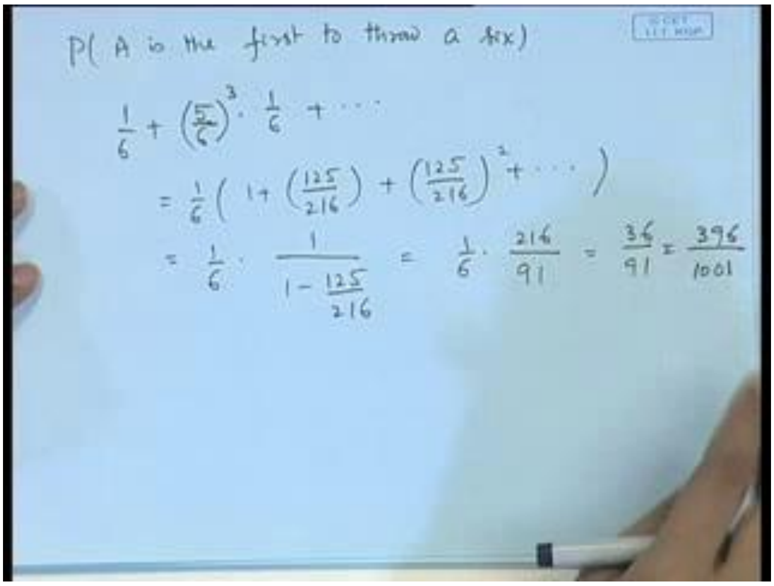

If we throwing on say fourth trail then before that none of the other players must be able to throw a 6; that means, A himself is not able to throw B is not able to throw C is not able to throw. So, this probability is simply a infinite geometric series that is equal to 1 by 6, 1 y 1 minus 125 by 216 that is equal to 1 by 6 and 216 minus this is 91 that is equal to 36 by 91. That is probability that A is the first to throw a six. If we try to compare it with this one then it is 11 11 then 99 99 plus 100 that is 396 by 1001.

So, you can see that the probability got reduced. Since a is the first paired to get a chance for throwing the probability that he will be the first to get a 6 is much higher, that is 396 by 1001 corresponding to as A is the second that is it is 300 by 1001 the probability is reduced. Let us also see A is the last to through a 6; if he is last to throw a 6 then once again he will be able to throw a 6 on 3 r plus 1 th trial, for r is equal to one to n so on. On r is equal to 0 he must not throw a 6. So, once again no 6 on r throws that will be 5 by 6 to the power r and r plus first throw is a 6 with probability 1 by 6. And if we he is last to throw a 6; that means, both B and C must be able to get at least one six in their r trails, which are held before the 3 r plus 1 th trial.

So, using the argument which we he gave in the first part of this problem, probability that B throws at least one six in r trails, that will be 1 minus 5 by 6 to the power r. And in a similar way probability that C throws at least one six in r trails, that will be with probability 1 minus 5 by 6 to the power r.

Thank you.