Probability and Statistics

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Lecture – 10
Independence of Events

So, this is good we have looking at the base theorem as a cause and effect relationship.

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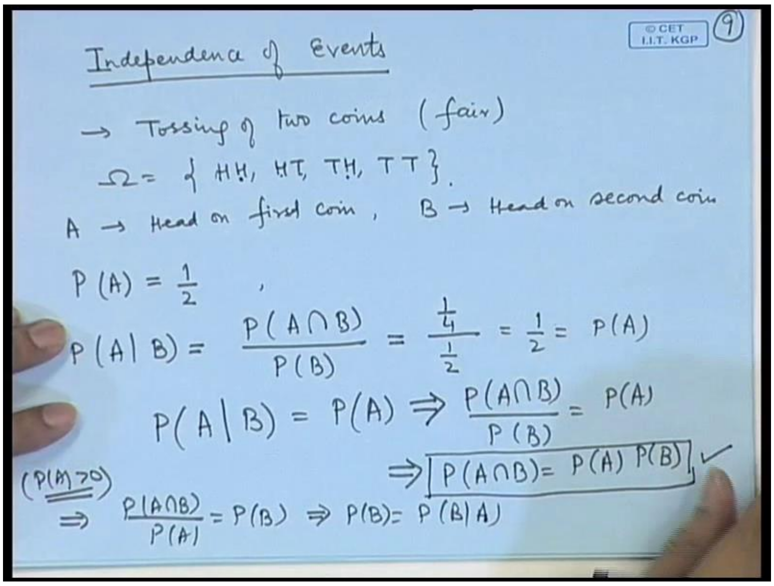

Now we have another concept called independence of events. Let us consider suppose tossing of two coins, suppose they are fair coins. So, my sample is base here is H H, H T, T H and T T. Suppose I consider my event A as head on first coin and B is the event head on second coin. If I look at the probability of say A head on the first coin is in H H and H T, so it is half. Suppose I consider probability of A given B then it is equal to probability A intersection B divided by probability of B.

Now probability of B is simply equal to half, because we are saying head on the second coin; head on the second coin is occurring in this and this so it is half. And probability of A intersection B means head on the first coin and head on the second coin which is happening a only in one case, so the probabilities 1 by 4. So, this is turning out to be half which is same as probability of A. Now this is different from by statements which I gave in the beginning of the conditional probability definition; that if I consider probability of

E and probability of E given B these two where different. It means that happening of B definitely affects the happening of E.

Whereas, in this one probability of A and probability of A given B is the same; that means, happening of B does not affect happening of A. This means the happening of A is independent of the happening of B. This is the concept of independence of the events. So, if we proceed from the definition of the conditional probability we must define event A to be independent of event B, if we have probability of A given B is equal to probability of A; which translates in to probability of A intersection B divided by probability of B is equal to probability of A; which is translating to probability of A intersection B is equal to probability of A into probability of B.

Now, if we also make assumption that probability of A is positive then this will translate to probability of A intersection B divided by probability of A is equal to probability of B, provided we have assume that probability of A is positive. So, this will again imply that probability of B is equal to probability of B given A. That means, B is independent of happening of A. So, provided I consider that probabilities of A and probabilities of B are positive the concept of independence of A with B or B with A is symmetric in nature. Therefore, to avoid unnecessary complications we can consider this as the definition of probability as the independence of the events A and B.

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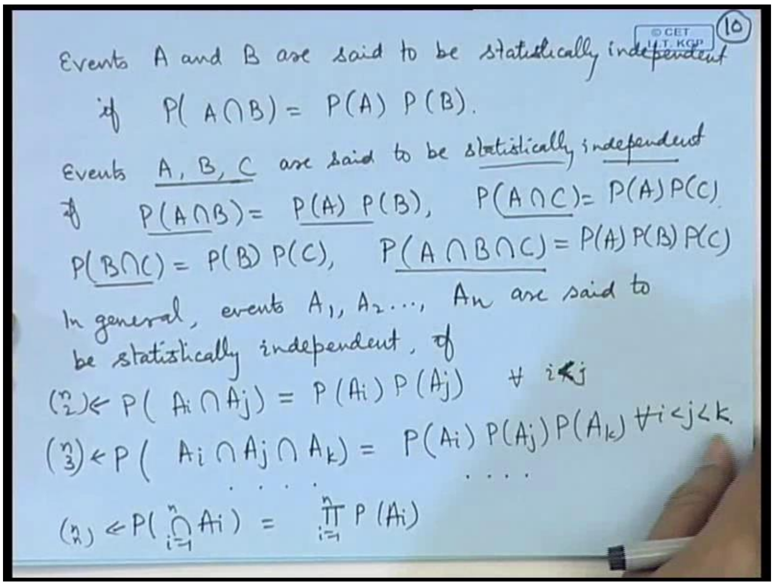

So, we formally define events A and B are said to be a statistically independent if probability of A intersection B is equal to probability of A into probability of B. So, you consider this definition is symmetric in the independence of A with B or B with A and more over we do not have to write down the condition that probability of A is positive or probability of B is positive, because this definition does not involve the division here.

Naturally a question arises that if I have more than two events then what is the meaning of independence. Let us consider say events A, B, and C. Then events A, B, and C are said to be a statistically independent if probability of A intersection B is equal to probability of A into probability of B probability of A intersection C is equal to probability of A into probability of C probability of B intersection C is equal to probability of B into probability of C.

Now, one may feel that these three conditions are enough for defining the independence of event say A, B, and C, but this is only payer wise independence of A, B, and C. We need one more condition namely; probability of A intersection B intersection C is equal to probability of A into probability of B into probability of C. Now this is required because it may happen that A is independent of B or A is independent of C, but then simultaneous occurrence of B and C may have something to do with A; or simultaneous occurrence of A and B may have something to do with C. Therefore, in order to define the statistical independence of three events we have four conditions namely; we take two at a time and then all the three at a time.

In general then, if we have events say A 1, A 2, An then they are said to be a statistically independent if probability of Ai intersection Aj is equal to probability of Ai into probability of Aj for all i not equal to j. You may write in one way only because there is no need to write for i less than j and then for j less than i, so you may put i less than j. Then taking three at a time probability of Ai intersection Aj intersection Ak is equal to probability of Ai into probability of Aj into probability of Ak for all i less than j less than k.

We may also consider four at a time and so on. Finally, probability of Ai intersection i is equal to 1 to n is equal to product of the probabilities Ai. The number of conditions you can see this is n c 2.

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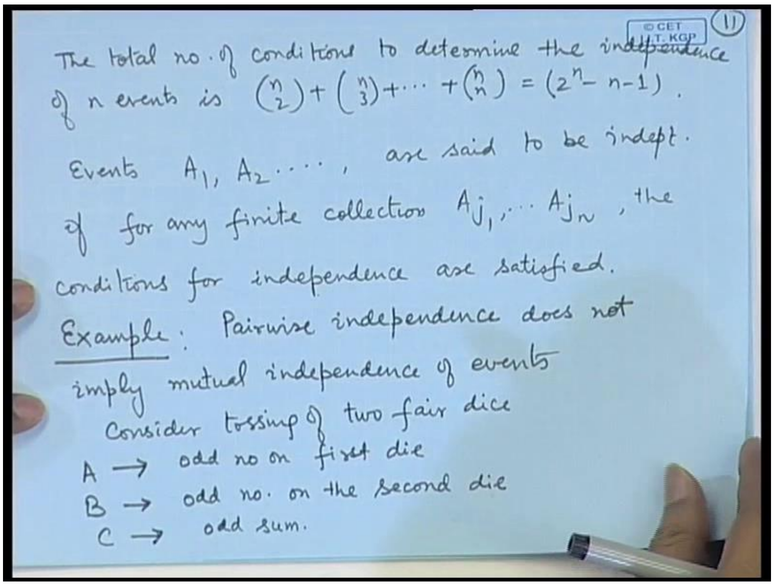

So, how many total number of conditions are there? Total number of conditions will become the total number of conditions to determine the independence of n events is; so let us look at how many conditions are there? Here we have n events and we are taking two at a time. So, total number of conditions is n c 2. Here we have three from n three at a time so the total number of conditions is n c 3. And finally, it is n c n or 1.

So, the total number of conditions is eventually equal to n c 2 plus n c 3 plus n c n, which we can write as 2 to the power n minus n minus 1. So, a total number of 2 to the power n minus n minus 1 condition are required to determine the statistical independence of n events. Of interest is also to look at infinite number of events and then talk about its independence. Then logically it means that if we consider any finite collection of those infinite numbers of events, then that finite collection must satisfy the conditions for the independence.

So, we define events A 1, A 2 and so on, that means the indexes are belonging to some set are said to be independent if for any finite collection let me call at say Aj 1, Aj 2, Aj n, the conditions for independence are. Here let me a specified at the conditions which are all mentioned here must we satisfied in order to have the actual independence of the events. For example, if I take the case k is equal to 3, then for 3 events there is statistical independence is defined in terms of pair wise independence conditions there are 3 conditions and one in which all the 3 are taken. Now one may say that if the pair wise independence conditions are satisfied then the last one automatically satisfy; actually it is not true. The satisfying of the first three does not imply the satisfaction of the forth one or vice versa, this may be satisfied but one of these may not be satisfied, let me give examples of this.

That is pair wise independence does not imply mutual independence of events. Consider say tossing of two fair dice. Let me consider the event A as, say odd number on first die, B as the event odd number on the second die, and c is the event odd sum. Let us consider the probabilities of A, B, and C.

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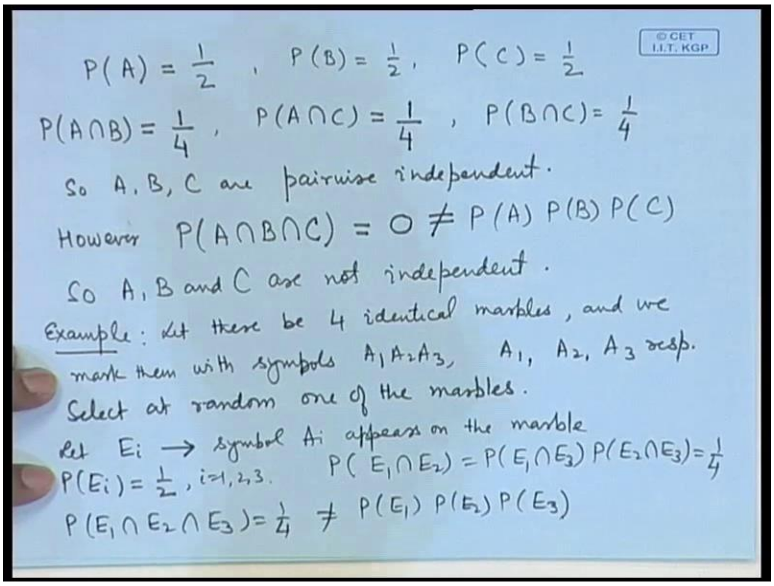

So if we look at probability of A, odd number on the first die it means you have 1, 2 and 5 on the first. The total number of possibilities here is 36 and odd number on the first means 1 1, 1 2, 1 3 up to 1 6. 3 1, 3 2, 3 6, 5 1, 5 2, 5 6; so total number of 17 possibilities are there. So, probability of A becomes half. Likewise probability of B is half and likewise probability of C that is odd sum is also half.

If we consider the probability of A intersection B then we are saying odd number on the first and odd number on the second, what are the possibilities? 1 1, 1 3, 1 5, 3 1, 3 3, 3 5, 5 1, 5 3, 5 5, there are 9 possibilities out of 36 possibilities. So, this becomes 1 by 4. If we look at probability of A intersection C, then we are saying that the first number is odd the sum is odd. It is possible 1 2, 1 4, 1 6, 3 2, 3 4, 3 6, 5 2, 5 4 and 5 6. So, this becomes

9 by 36 again 1 by 4. Similarly probability of B intersection C is 1 by 4. Naturally you can see that the conditions for the pair wise independence are satisfied here.

So, A, B, C are pair wise independent. However, if you look at probability of A intersection B intersection C, then if we have odd number on the first die odd number on the second die naturally the some will be even, and therefore there is no case where A intersection B intersection C is satisfied. Therefore, this probability is 0. Naturally this is not equal to probability of A into probability of B into probability of C. So, A, B, and C are not independent, although they are pair wise independent.

Let me consider one example in which the condition for the simultaneous that is the third condition may be true, but one of the additional conditions may not be true. Consider say let us consider the set; let there be 4 identical marbles. And we mark them and we mark them with symbols. That means, on the first one we write A 1, A 2, A 3 and A 1, A 2, A 3. That is on the first marble we mark A 1, A 2, A 3 and the second one we put A 1 and the third one we put A 2 and fourth one we put A 3.

We draw one of the marble identically select at random one of the marbles. Let Ei denote the event that symbol Ai appears on the marble. Let is look at probability of Ei; it will be half for all i's. If I look at probability of say E 1 intersection E 2 probability of E 1 intersection E 3 probability of E 2 intersection E 3, each of this is going to be 1 by 4 because only one possibility is there. And therefore, if I look at probability of E 1 intersection E 2 intersection E 3 then that is also 1 by 4 and it is not equal to probability of E 1 into probability of E 2 into probability of E 3.

So, this is another example where the events are pair wise independent, but they are not independent.

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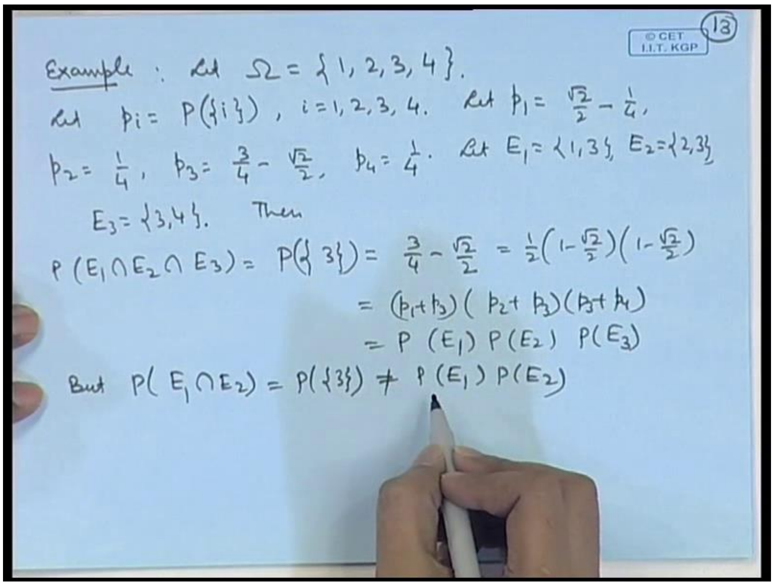

Let me take one more example: consider say omega is equal to 1, 2, 3, 4. Let p i be the probability assigned with the set consisting of i; i is equal to 1, 2, 3, 4. Let us define say p 1 is equal to root 2 by 2 minus 1 by 4, p 2 is equal to say 1 by 4, p 3 is equal to 3 by 4 minus root 2 by 2, p 4 is equal to 1 by 4. Let E 1 be equal to 1 3, E 2 be equal to 2 3, and E 3 is equal to 3 4. Then if I look at probability of E 1 intersection E 2 intersection E 3 then it is simply the probability of three that is p 3 that is 3 by 3 minus root 2 by 2; which I can write as half into 1 minus root 2 by 2 1 minus root 2 by 2 which can be written as p 1 plus p 3 into p 2 plus p 3 into p 3 plus p 4.

So, this is equal to probability of E 1 into probability of E 2 into probability of E 3. So, the last condition for the independence of E 1, E 2, E 3 is satisfied, but if I look at probability of E 1 intersection E 2 then that is also equal to 3. And therefore, it is not equal to probability of E 1 into probability of E 2. Therefore, the conditions for the independence are not satisfied here.

So, in effect it means that if you want to check the independence of events then we must check the probability of simultaneous occurrences by taking all the combinations by taking 2 at a time, 3 at a time and so on.

Thank you.