**Probability and Statistics**
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**Lecture – 01**
**Sets, Classes, Collections**

Welcome to this course on Probability and Statistics. This is an introductory course or you can say first course on the probability and statistics and it is quite useful for all branches of science and membrane.

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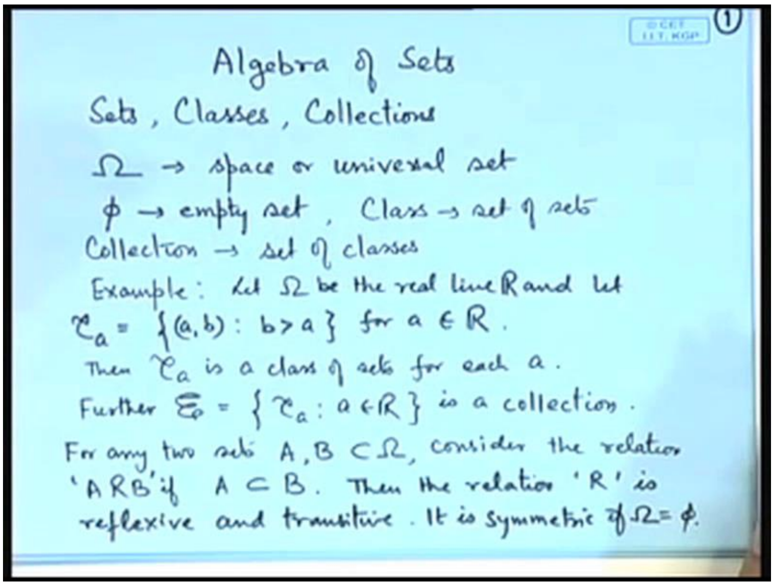


To begin with we introduce algebra of sets. This is required because the modern theory of probability is based on the set theory. So, here we introduce certain elementary concepts of sets, which will be directly used in our definition and concepts of probability. So, to begin with we start the discussion on what is set, classes and collections.

So, we start with the universal set, let us call it omega and then we also have the user notation for the empty set as phi, we call a set of sets to be the class and a set of classes to be a collection. As an example consider suppose omega is the real line R and we collect consider the collection of intervals is starting from a to b; where b is greater than a for every real number a then the C a is a class of sets for each a and then if we come collect all this classes C a into a set called descript E then this is a collection.

For any 2 sets A and B which are subsets of omega, let us consider the relation A related to B, if a is A subset of B then the relation R is reflexive and transitive, it is symmetric when omega is equal to phi.

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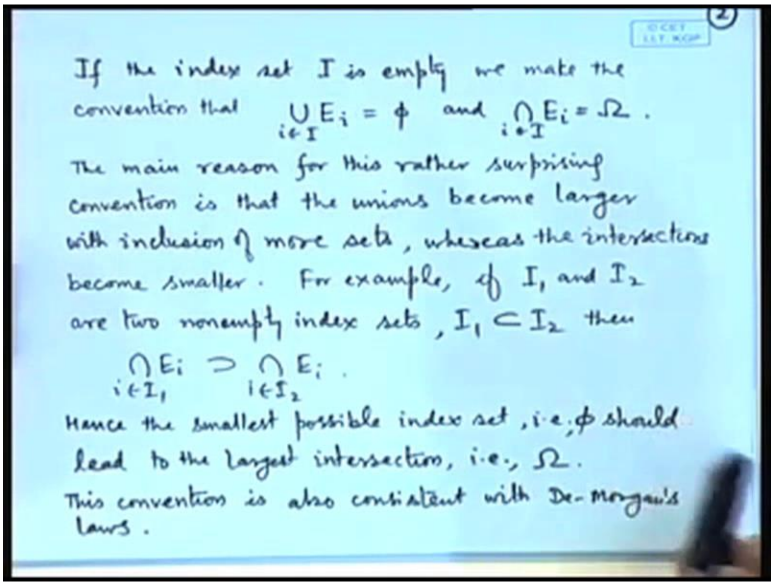


We introduce some further conventions and notations. If the index set I is empty, we make the convention that union of E i is empty and also intersection of E i is the full set omega. The second of this convention looks to be is rather surprising; however it is motivated by the fact that if we take more and more sets an intersection then it becomes a smaller.

For example if I have two index sets I 1 and I 2 such that I 1 is subset of I 2 then intersection of E i where I belongs to I 1 contains intersection of E i where I belongs to I 2. Primarily because there more sets means the intersection becomes a smaller. Hence the smallest possible index set that is phi should lead to the largest intersection that is omega. This convention is also consistent with De Morgan’s laws.

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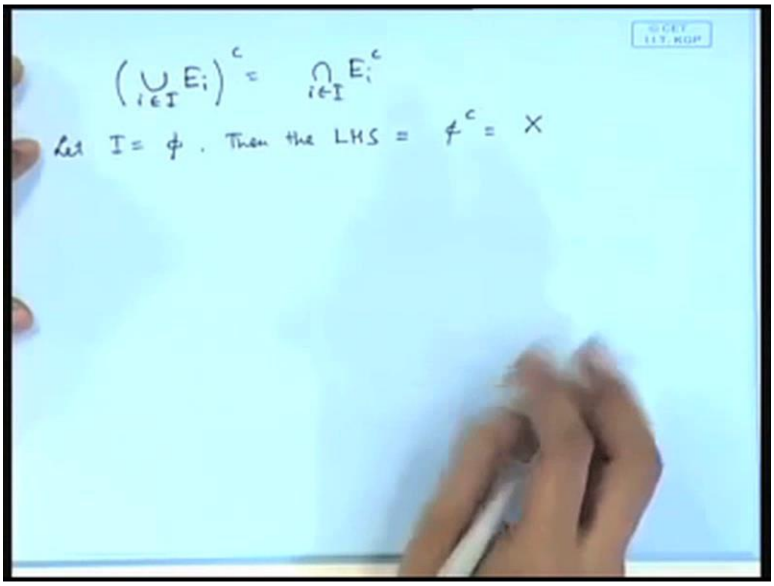


For example if I write union of E i i belongs to I, compliment is equal to intersection E i compliment I belonging to i.

Let us take I to be phi, then the left hand side is equal to phi compliment is equal to X this indicates that we should take intersection of E i when I is in a empty index set then this should be equal to the full space omega.

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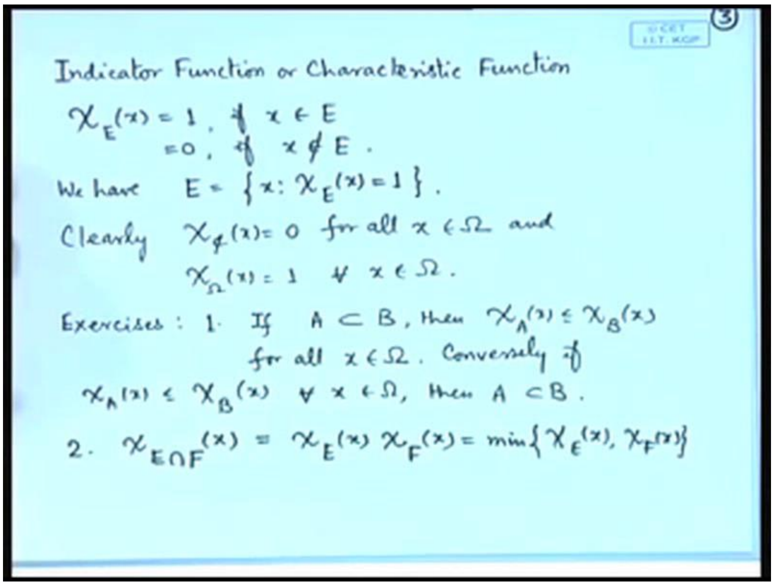


Let us introduce the concept of indicator function or characteristic function. The characteristic function of a set E is defined to be 1, if x belongs to E and it is 0 if x does not belong to E. Alternatively we can write the set E as the set of all those points, for which x chi E x is equal to 1. Clearly indicate a function of the empty set is always 0 and the indicator function of the whole space is always 1.

There are certain exercises here for example, if set A is subset of B, then the indicator function of A is always less than or equal to the indicator function of B. Conversely if the indicator function of the set A is always less than or equal to the indicator function of B then A is the subset of B. A simple proof of this is that if we consider chi A x then it is equal to 1 for all x belonging to A and since A is a subset of B this implies that chi B x is also 1 for these points. Now for the points where chi A x is equal to 0 include certain points where chi B x may be 1 and for either points chi B x will also be 0. Therefore, chi A x is always less than or equal to chi B x.

We have certain elementary properties of the characteristics function for example, characteristic function of an intersection is equal to the product of the characteristic functions of the 2 sets, it is also equal to the minimum of the indicator function values for those 2 sets. The proof can be simply obtained by definition for example, chi of E intersection F x is equal to 1, if x belongs to both E and F and 0 otherwise. That means, it is when do we equal to 1 only if chi E x and chi F x both are 1 and in every other case it is going to be 0 so that means it is equal to the product.

In a similar way the minimum of chi E x and chi F x is going to be 1 only if both chi E and chi F x are 1; that means, chi E intersection F x is equal to 1.

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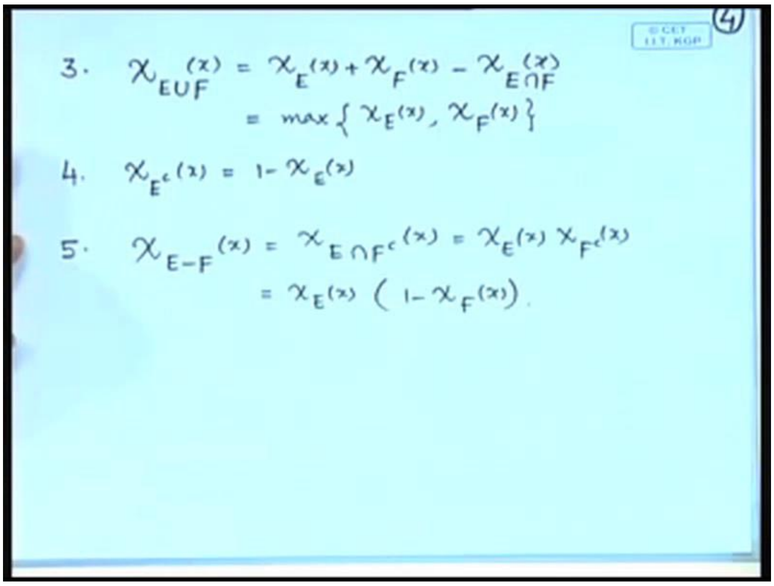


Similarly, the indicator function relations for the unions complementation and differences are also there for example, chi of E union F x is equal to chi E x plus chi F x minus chi F E intersection F x. It is also alternatively equal to the maximum of chi E x and chi F of x once again to look at the proof of this.

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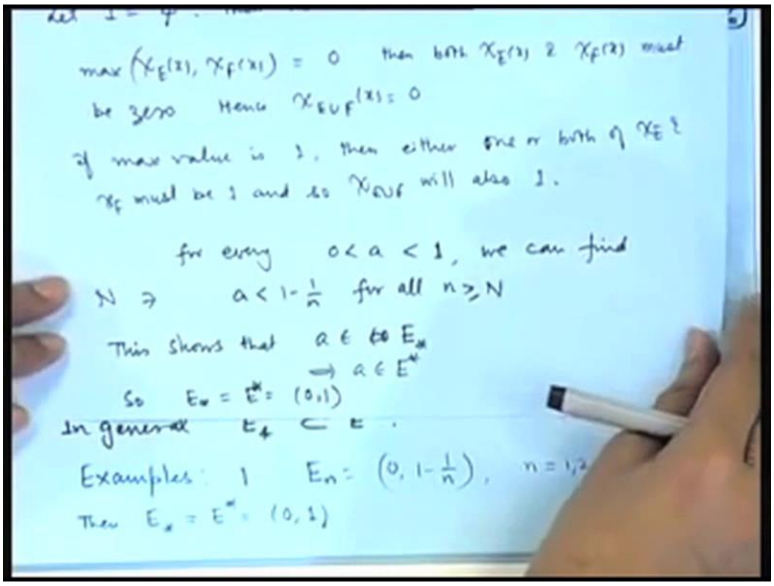


We can see that if we have; if we write maximum of chi E x and chi F x to be say 0 then both of chi E x and chi F x must be 0. Hence, chi of E union F x will be equal to 0; if maximum value is 1 then either 1 or both of chi E and chi F must be 1 and so chi of e union f will also be 1. Rather properties of chi function for example, chi of E compliment is equal to 1 minus chi E, chi of E minus F is equal to chi of E intersection F compliment x, which is chi of E into chi of F compliment from the property 2 and chi of F compliment from the above properties equal to 1 minus chi F x.

So, these are some of the useful relationships further in related function. We have all heard about the unit of sequence of numbers. For example, you easily understand that unit of a n is equal to a, if for every epsilon greater than 0 there exist capital N such that modulus of a n minus a becomes less than epsilon whenever n is greater than or equal to capital N.

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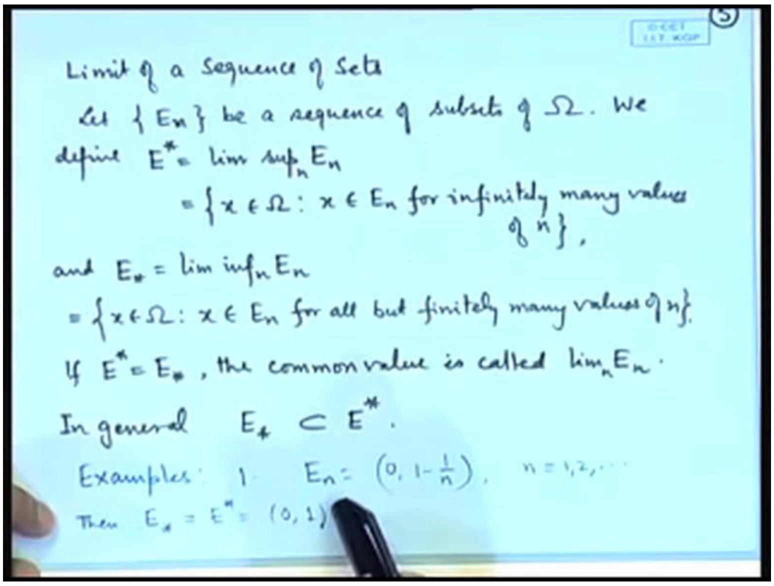


This shows that the sequence a n is closer to the number a the values of the sequence n become closer to the number a, as n becomes large. However, the similar concept is not available for the sequence of sets; here we define it in a slightly different fashion.

So, in other than then is the concept we introduce the concept of lim sup of E n and lim inf E n or called limit superior of E n or limit inferior of the sequence E n. So, consider a sequence of subsets of omega called E n, we define E upper star is equal to lim sup of E n, which is equal to the set of all those points which belong to infinitely many values of n, what does it mean? It means that from the whole space we take up those points which belong to E n for infinitely many values of n, the collection of such elements will be called limit superior of E n.

In a similar way we define limit inferior of E n called E lower star; it is the collection of those elements of omega, which belong to all except may be a finite number of values of n; that means, they belong to all most all the values of E n, except may be a for a few of 10. If E upper star and E lower star are the same sets then the common values called the limit of E n and we say that the limit of the sequence of sets exists; by the definition it is clear that in general we will have E lower star as a subset of E upper star, primarily because if x belongs to E lower star then already we are connecting that it is belonging to infinitely many values of n, because it is belonging to all accept a finite number that will definitely it is belonging to infinitely many values of n.

Therefore, in general E over star is always a subset of E upper star, when it is equal we say that the limit exist. Let us consider some simple examples of application of the concept of the limit of sequence of sets consider for example, E n say define to be an open interval 0 to 1 minus 1 by n, for n is equal to 1 to n so on. Here if you see if we calculate the limit in pdf, then if you consider any point in the interval 0 to 1 then since 1 by n goes to 0 as n tends to infinity, for every number a between 0 to 1, we can find for every 0 a less than 1 we can find capital N such that a is less than 1 minus 1 by n for all n greater than or equal to N.

This shows that a will belong to the interval; it a will belong to E lower star, this also shows that a will belong to E upper star. So, E lower star is equal to E upper star is equal to the interval 0. So, this is a case where the limit exists.

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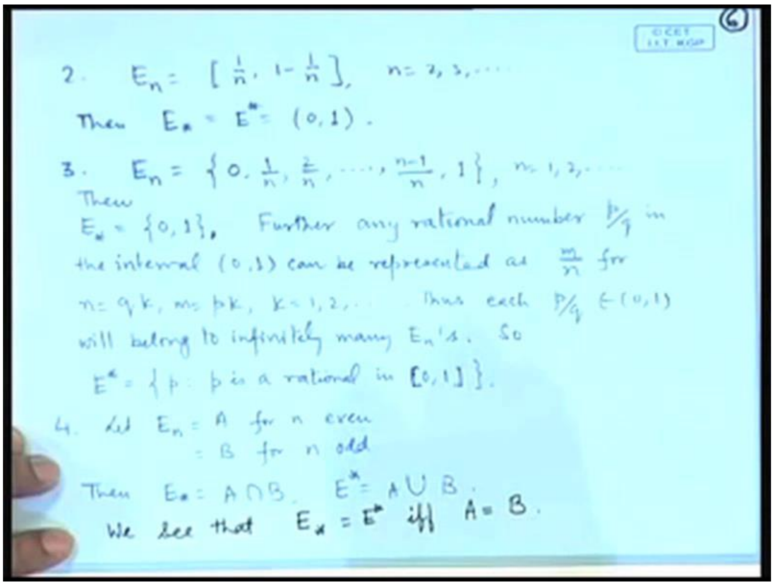


Let us take another example, the sequence of closed intervals from 1 by n to 1 minus 1 by n for n is equal to 2 3 and so on. Once again if we utilise the argument does given in the example 1, if we consider any element a between 0 to 1 then after a certain stage it will belong to the set E n for all n greater than or equal to some category n therefore, once again the E lower star that is limit inferior of this sequence will be the interval 0 to 1 and naturally E upper star will also be 0 to 1 because it is a super set of E star in general and it cannot be go beyond the interval 0 to 1.

Notice that here the point 0 itself and the point 1 itself do not belong to E lower star or E upper star, because for no value of n these points belong to the set; now I will give 1 example where the set E n is defined has a finite number of points and here in lower star n, E upper star are different. Consider the set E n given by 0, 1 by n, 2 by n, n minus 1 by n to 1, for n is equal to 1 2 and so on, then if this particular sequence can be explained like that.

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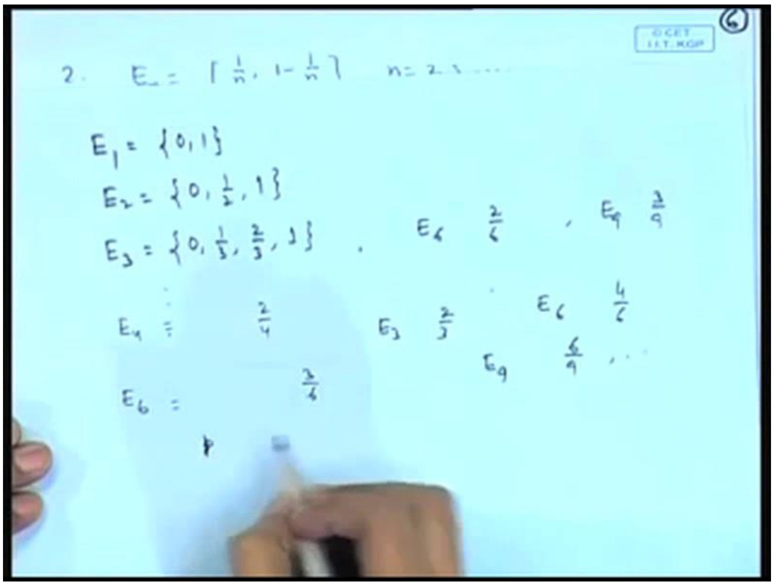

If we consider E 1 then it is the set 0 to 1; if we consider E 2 then it is the set 0, 1 by 2, 1; if we consider the set E 3 then it is equal to 0, 1 by 3, 2 by 3, 1.

Clearly the points 0 and 1 belong to all the sets therefore; E lower star that is a limit inferior must contain the point 0, 1. Now we say that no other element of the form say p by q can belong to all the sets E ns, in an after a certain stage for example, the point half it will belong to once again E 4 because 2 by 4 will come, it will belong to E 6 because 3 by 6 will come. So, it will belong to infinitely when we set, but it will not belong to all that finitely many. Similarly if I consider say 1 by 3 then it will be there in E 3, it will be there in E 6 as 2 by 6, and it will be there in E 9 as 3 by 9 etcetera.

So, it will once again belong to infinitely when we sets but it will not belong to all, but finitely many. Suppose we consider a point say 2 by 3, then it is belonging to E 3, then it will again belong to E 6 as 4 by 6, it will again belong to say 9 as 6 by 9 etcetera therefore, we conclude that all the rational numbers in the interval 0 to 1, all the rational numbers of the form p by q in the interval 0 to 1 will belong to infinitely many of the E ns therefore, E upper star will be the set of all those rationals in the interval 0 to 1 whereas, the limit inferior is equal to the interval and the set consistent of the points 0 and 1.

Another example is you consider the sequence as I split into two parts that for even ordered sets E n is 1 particular set say A and it is equal to B for n odd then E lower star is the set of points which are common to both A and B because if we consider a point which belongs to both of them then for all, but finitely many sets of E n, x will belong to them. However, if a point is not in one of A and B then for infinitely many values of n it will not belong therefore, the only points which can belong to E lower star are which are common to both A and B.

On the other hand any point which is belonging to either A or B will be there in infinitely many sets and therefore, the E upper star will be A union B; naturally you can see that E lower star is equal to E upper star if and only if A is equal to B.

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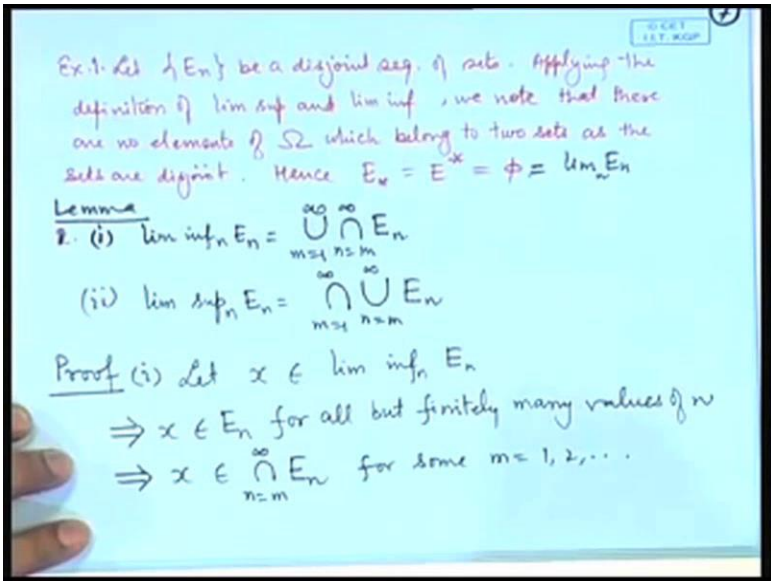


Lets us consider a it is joint sequence of sets, if we apply the definition of limit superior and limit inferior then we note that there are no elements of omega which belong to 2 sets, because the sets are disjoint therefore, no point in the set omega can satisfy the property of belonging to either infinitely many E ns r all, but finitely many points of E.

Therefore limit inferior as well as the limit superior both are the empty set and it is equal to limit of the sequence of the sets. In order to there are if the limit inferior and limit superior using mathematical arguments, one may look for the alternative representations for these sets for example, limit inferior can be written as union intersection E n; n is equal to m to infinity and m is equal to 1 to infinity. In a similar way limit superior can be written as intersection union of E n, n is equal to m to infinity, m is equal to 1 to infinity; to proof this statements let us take the first case consider a point x belonging to limit inferior of E n.

By definition of the limit inferior, it implies that x belongs to E n for all, but finitely many values of n. This implies that from a given m onwards x will belong to all the sets E n; that means, x will belong to the intersection of E n, n is equal to m to infinity for some m is equal to 1 2 etcetera; clearly this implies that x belongs to union intersection E n n is equal to m to infinity and m is equal to 1 to infinity.

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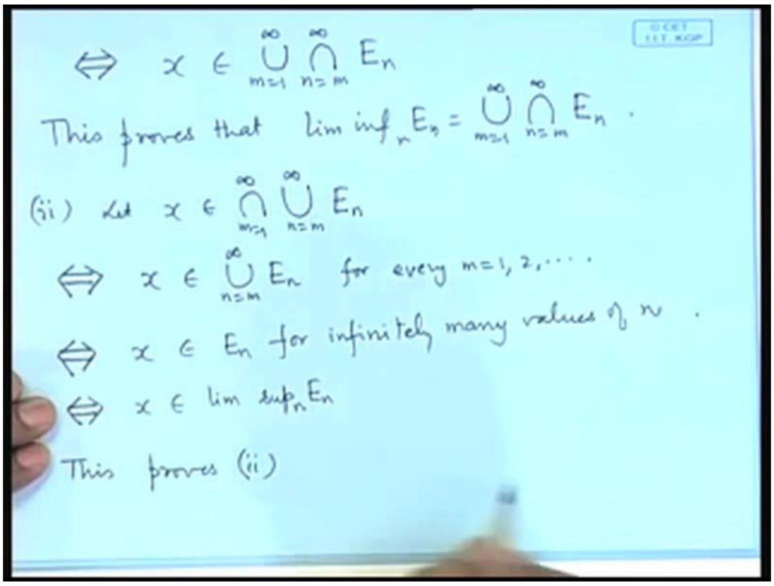


On the other hand if we take x belonging to union m is equal to 1 to infinity, intersection n is equal to m to infinity E n, then this statement implies that x belongs to intersection of E n for some m is equal to 1 to etcetera. Now this statement once again implies that x belongs to E n for all but finitely many values of n, which was actually the definition of limit inferior therefore, we have both the implication for all the statements and this proves that limit inferior of E n is equal to union intersection E n, n is equal to m to infinity m is equal to 1 to infinity. If we apply the same argument for the limit superior then we get that if x belongs to intersection union E n, n is equal to n to infinity, m is equal to 1 to infinity, so I am taking the right hand side first.

Now, this implies that x belongs to union E n, n is equal m to infinite for every m is equal to 1 to etcetera. Now if we say that x belongs to the union is starting from every m, how you say m large may be? This definitely implies that x belongs to infinitely many values of n, because if it was belonging to a finitely many value only then after a certain stage x will not belong to the union of ens; however, it is belonging for every value of n. This means that x belongs to E n for infinitely many values of n, which is actually the definition of limit superior. Now to look at the argument in the reverse way, if x belongs to the limit superior then this implies that x belongs to E n for infinitely many values of n.

Now, once again if x belongs to infinitely many values of n, then no matter for which value of m you start with x will belong to certain sets after that, and therefore this statement implies this. Now since this is true for every m this statement implies that x belongs to intersection union of E ns. This proves the statement two.