Numerical Solutions of Ordinary and Partial Differential Equations

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Lecture - 5
Higher Order Methods/Equations

Good morning, in the last class we have learnt Range Kutta method for solving initial value problems, which is a single step method and we have learnt second order method. Now, let us see how higher order Range Kutta methods can be developed, and further we will see how to solve higher order equations as well.

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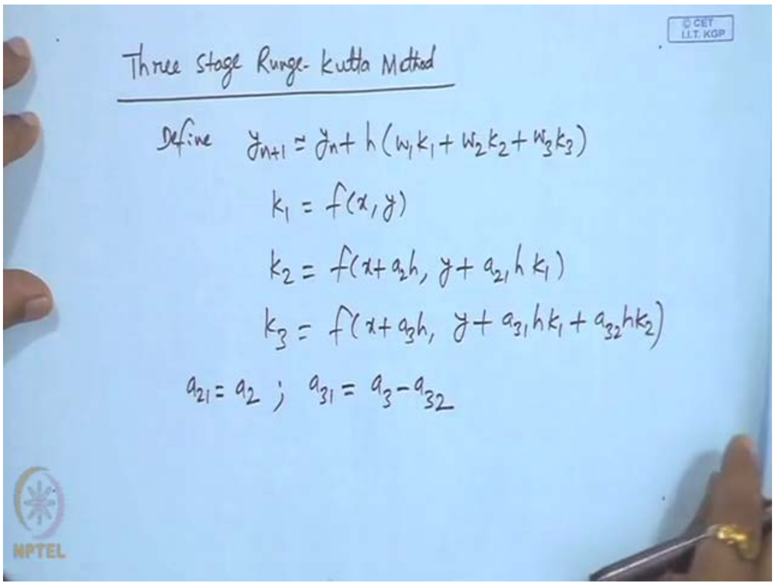

So, let us talk about, last time we talked about second order, so now let us go 1 step ahead, that is third order. So, that is 3 stage Range Kutta method, so that means we are using 3 slopes, so we define the method as follows W 1 K 1 W 2 K 2 S 3 K 3, where K 1 is f of x y K 2 a 2 h y plus a 2 1 h K 1 K 3 a 3 h y plus a 3 1 h K 1 plus a 3 2 h K 2. So, I would like to make remark here, sometimes in some books K 1, K 2, K 3 will be defined with h in front, in which case this h will be taken out. So, that means if you multiply h inside so h K 1 will be new K 1, h K 2 will be new K 2, h K 3 will be new K 3.

Now, how do we determine the coefficients W 1, W 2, W 3, a 2, a 3, a 2 1, a 3 1, a 3 2 so total how many 8 so one has to compare with the Taylor series method and then try to determine the coefficient. Since we are using three stage method, we expect that we should equate terms of 2 h cube. So, for the second order method I have shown you the detailed calculation, so I expect that if 1 would follow those detailed calculations, then we arrive at a system. Now, we have to make some arbitrary choice, so I make the following choice say a 2 1 is a 2, a 3 1 is a 3 minus a 3 2, so that means 1 2 3.

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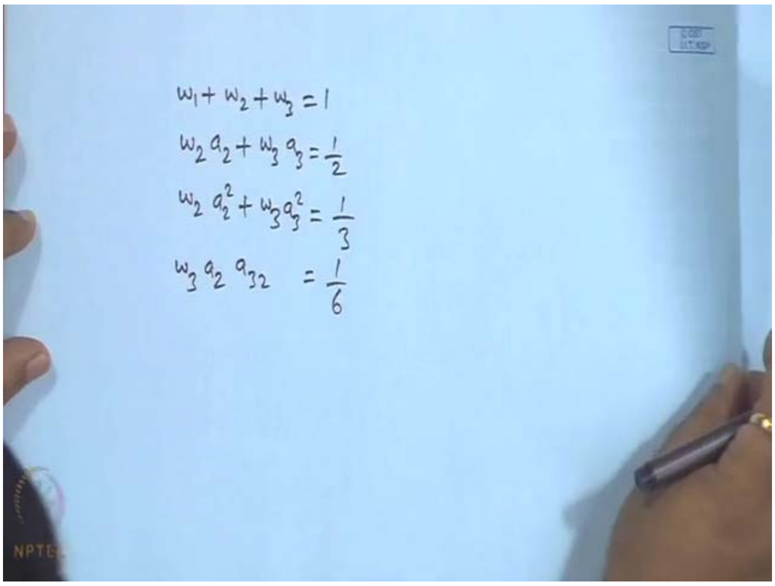

So, with this we obtain system of equations as follows, so the system of equations the weight coefficients they satisfy this. So, I repeat again by choosing the above arbitrary coefficients, we get system and that system has been reduced using this arbitrary, then this will be the reduced system so this should be an exercise for you to obtain this system. So, once we get the system what are the standard methods? One can have arbitrary choice like in second order method, we had different choices for example, alpha is something and beta is something half and half, so things like that.

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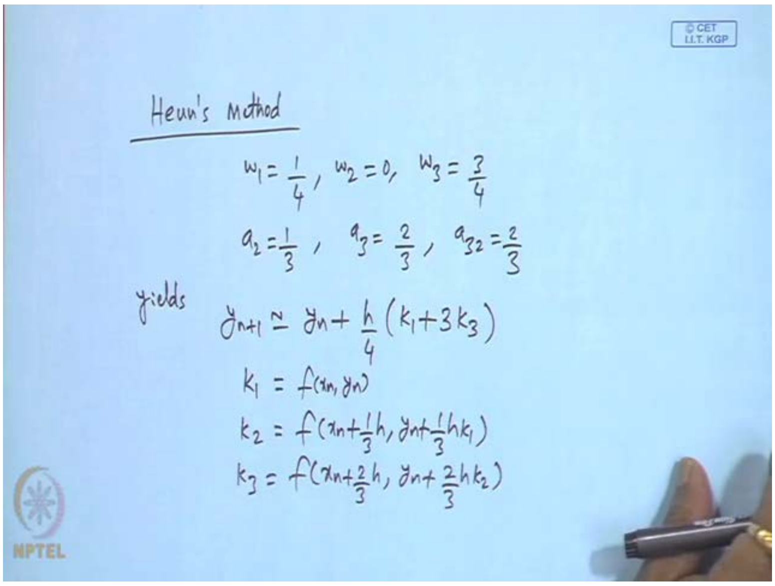

So, one can have arbitrary try choices, but unless you show that the method is really sensible, we do not make any arbitrary try choice. So, there are couple of standards methods, so what are they? So, the first one is Heun’s method, we make the following choice and a 2 is one-third a 3 is two-third. So, you may see this is a 2, we made a choice then a 3 a 3 2, so that will determine a 2 1 a 3 1 and in turn W 1 W 2 W 3, so this is the set.

So, this is the following method, since W 2 is 0 K 2 goes off, but then we need K 2 to compute K 3. So, this is K 2 and K 3, this is Heun’s method. So, we can see define K 2 we are using K 1 to define K 3 we should have used K 1 K 2, but since W 2 is 0, so K 2 does not explicitly appear.

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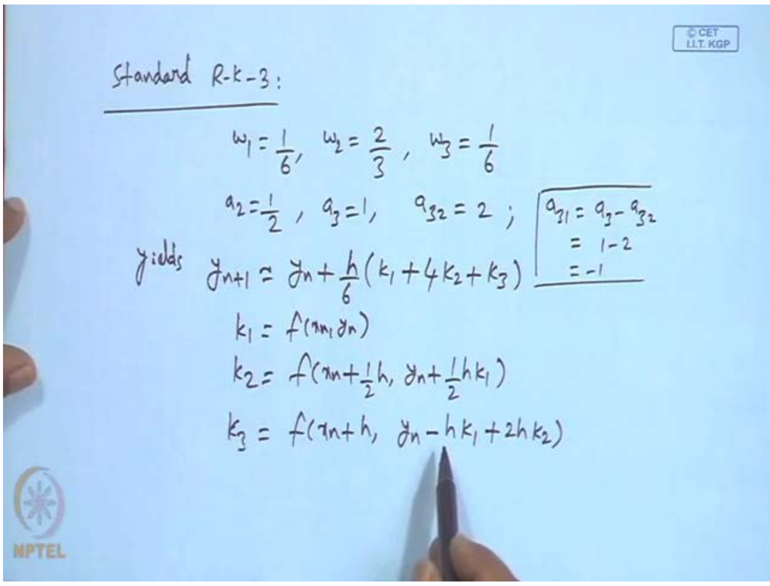

So, this would have been K 1 K 2, but we have other choice as well, so that is more standard R- K 3, so in short notation I am writing. So, for what is the choice, so this is called standard and this yields the following method correspondingly K 1. So, a 2 h a 2 is half therefore, half h K 3 observe the way we have defined that let to this kind of. So, this is the first time we have coming across minus sign in this, this is because we made an arbitrary choice.

So, what was our arbitrary choice, our arbitrary try choice was a 3 1 equals a 3 minus a 3 2, looking at these values a 3 1 is a 3 minus a 3 2, so this is 1 minus 2, which is minus 1 therefore, this we get a minus sign. So, how did we arrive at these methods, we have defined an approximation then we can Telesis expansion of the exact and then we consider the h expansion of the approximate formula, and try to match the terms and since we are using 3 slopes, so we have gone up to that. So, how to obtain the coefficients, I left it as an exercise, so I am sure making the second order method, one can obtain the system.

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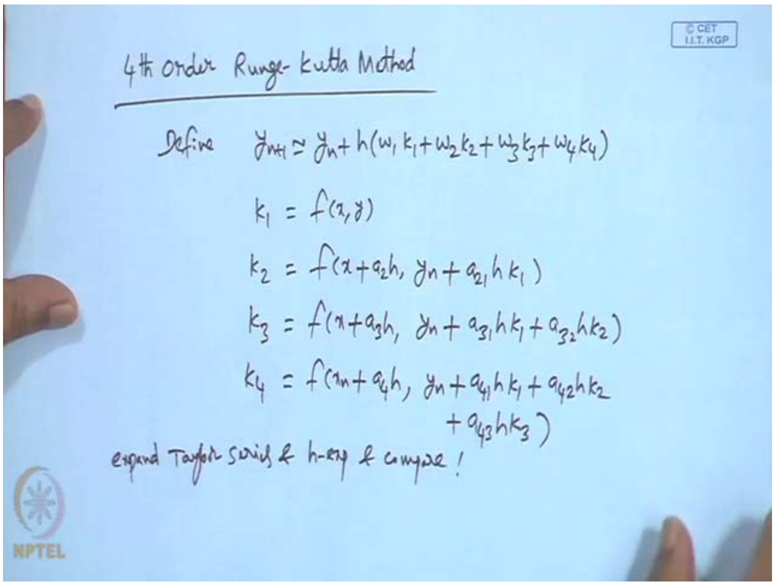

Now, let us proceed for the fourth order method. So, this is a fourth order R K method, so we define K 1, K 2, K 3, for K 3 K 1 and K 2 are used K 4, a 4 1, a 4 2, a 4 3. So, this is our definition, so this is aligned with the general method with l slopes. Now, again our duty is to compute the arbitrary coefficients up to some accuracy. So how do we do? Again we do this expand Taylor series and h expansion and compare. So, again I am living this exercise, so when we do this we obtain the following, it is a lengthy system then getting nasty as b plus c.

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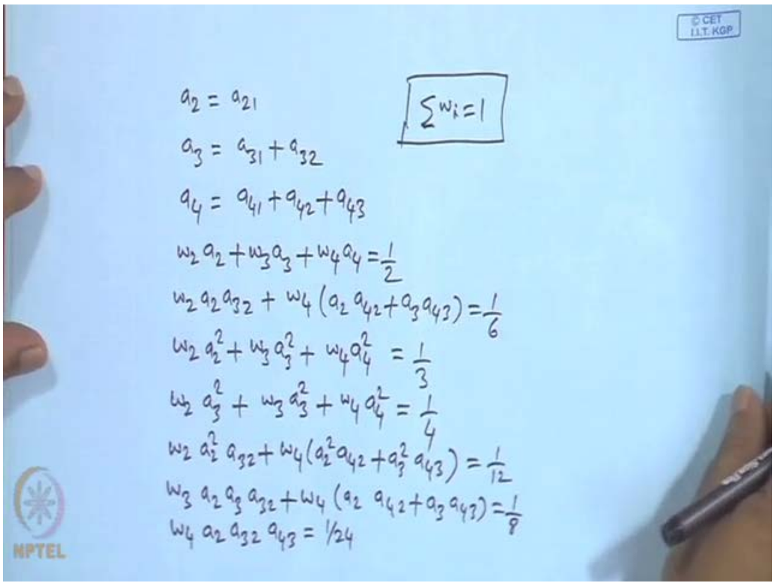

So, really very lengthy, but it is worth understanding because fourth order R K method is very popular and most of the calculations are done using fourth order R K method. And we have an additional condition, which we get this standard condition sigma W i is 1, this is also one of the equations out of that and further we have more equations. So, we I will try to write it here, so it is a very big, so this is we get and again one has to look for the arbitrary coefficients and then try to get specific method.

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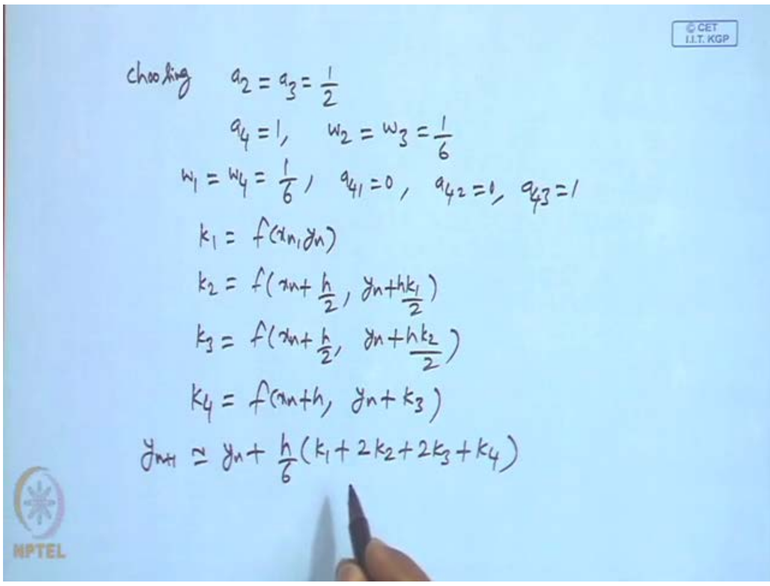

So, what are the standard choosing a 2 equals to a 3 equals to half we get, then W 1 W 4 once a 4 1 a 4 2, so this will define the method as follows. So, this is the coefficients and then the method is our fourth order R K method. Of course, the algebra is very tedious and we have done second order, so if we try to follow, we get the corresponding third order and fourth order methods.

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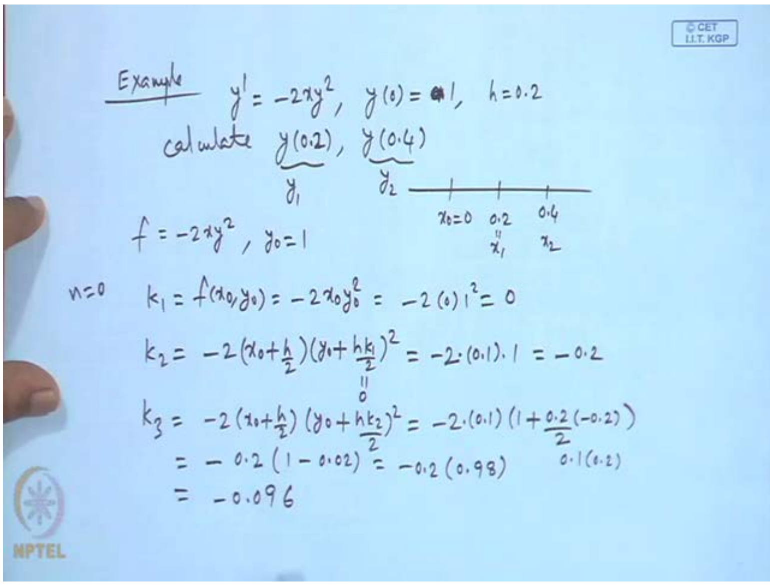

Now, we should try to see what happens to the error in each case because we did not talk it, we will talk about this little later. So, before we go further, let us try to solve problem. Let us say calculate y f 0.1 that will y f 0.2 because h is 0.2 y f 0.4, see what is our so what is x 0, so this will be h 1, so essentially this is y 1, this is y 2. So, let us try to compute, so f is minus 2 x y square, so we should try to compute K 1. Obviously, our n is 0 we are trying to compute y 1 therefore n is 0, so the given from here y 0 is 1, so what is this K 1 is minus 2, x 0 is 0 and y 0 is 1 square.

So, this is 0 K 2 is minus 2 x 0 plus h by 2 y 0 plus h K 1 by 2 square because K is defined as x n plus h by 2 comma y n plus h K 1 by 2, so this is what we get. So, this is equals minus 2 x 0 is 0 h is 0.2 so we get 0.1 there and y 0 is 1 and K 1 is 0, so this is 0, so we get 1 there, so this will be minus 0.2, then K 3 minus 2 x 0 plus h by 2 y 0 plus h K 2 by 2 square so minus 2, this already we computed 0.1 and y 0 1 plus h by 2 K, so K 3. So, this will be minus 0.2 1 minus, so this will be 0.1, so this will be 1 minus 0.02, so this is 0.2 times 0.98.

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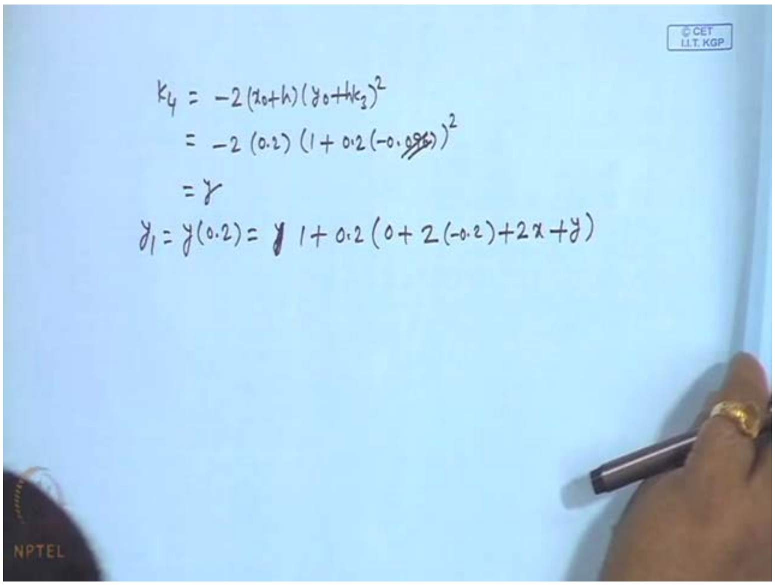

So, K 3 we have the value, so this is our K 3 then K 4, so K 4 is defined as minus 2 x 0 plus h y 0 plus h K 3 square, so K 4 x n plus h, h should be there. So, this is minus 2 x 0 is 0 h is 0.2 y 0 is a 1 h is 0.2 and K 3 is minus 0.096 square. Make a pardon; this is square missing here, so this value is incorrect, so we get a new value. So, we can use it accordingly, this is incorrect, so we get K 3 is some x then K 4 is some y, so then we get y 1 equals to y of 0.2 equals to y 0 that is 1 plus h is 0.2 times K 1, K 1 is 0 2 K 2 plus 2 K 3, K 3 was x plus K 4 some y, so we get the answer.

Make a pardon; we have to simplify and get x and y, this is just to explain the method so that is how we compute the solution. Now, having learnt third order and fourth order R K method, we can try some more examples, but this is higher order method. So, far we have solved an only first order initial value problem, which is y dash equals to f of x y y f x z equals to y 0. Now, let us go 1 step ahead. So, can we use these methods learnt to solve higher order initial value problems, so what are the methods we have learnt?

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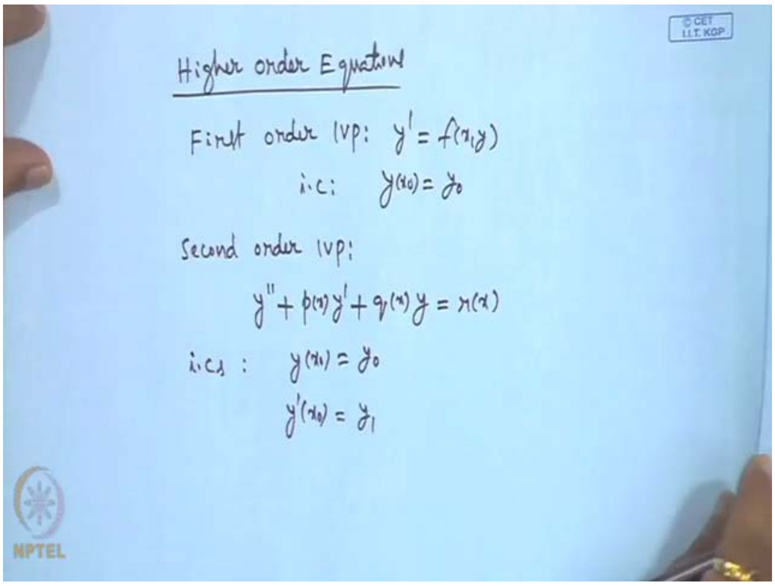

Let us says Euler method, Taylor series method and R K methods, so can we use these methods to solve higher order equations. So, let us try to do that, so higher order equations, we have first order I v p y dash equals to f of x y and y of x 0 is y 0, then second order I v p, so I will try to write standard method y double plus p x y dash q x y is some r of x. And since this is second order to define second order initial value problems, so this is our initial condition, but here we need two conditions because this is a second order equation y of x 0 is y 0 y dash of x 0 is y 1.

So, how do we solve this? And we have learnt how to solve first order initial value problem, with that knowledge can we solve second order I v p. So, the answer is yes, certainly we can. So, how do we do this? So, we tried to do this by reducing the given second order 2, a couple systems of first order equations, so how do we do it? Let us consider an example.

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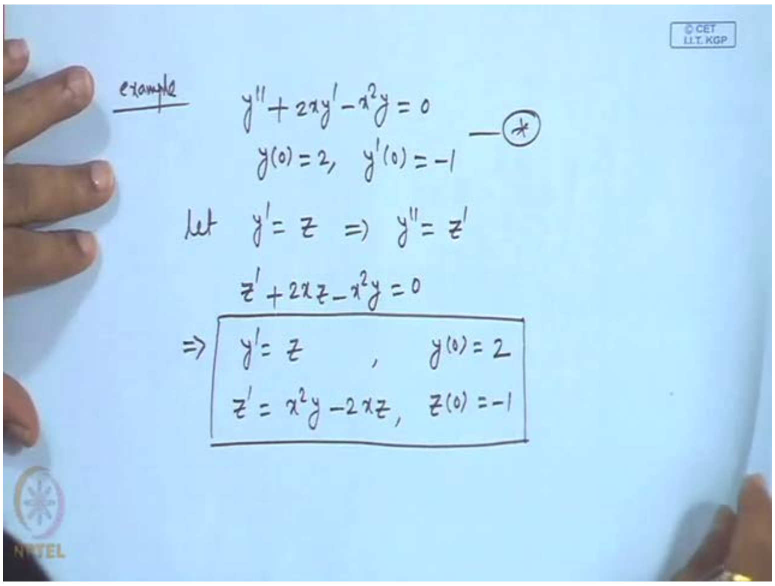

Suppose, here the equation is y double plus 2 x y dash equals to 0 and y of 0 is 2 y dash 0 is minus 1, so this is our star. Let us say now can we reduce this to couple system of equations? Yes of course let y dash equals to z then y double will be of course, x is our independent variable, y is dependent variable. Now, we have defined new dependent variable y dash equals to z and hence y double is z prime. Therefore, using this notation star reduces to y double becomes z and y double becomes z prime and y dash becomes z therefore, we obtain a couple systems as follows.

Now the dependent variables are y and z, so we need the corresponding conditions, we have y 0 is 2 and y dash of 0 is minus 1, but our y dash is z therefore, this will be supported by z of 0 is minus 1. So, our given second order initial value problem has been reduced to couple first order system and where is the coupling? You can see y dash is z and z dash is related, so this involves y so that means one cannot solve explicitly this unless we solve this and vice versa. So, given system independently one can solve or given second order by reducing to couple first order one can solve. So, let us try to learn how to solve system using Eulers method and if possible R K method.

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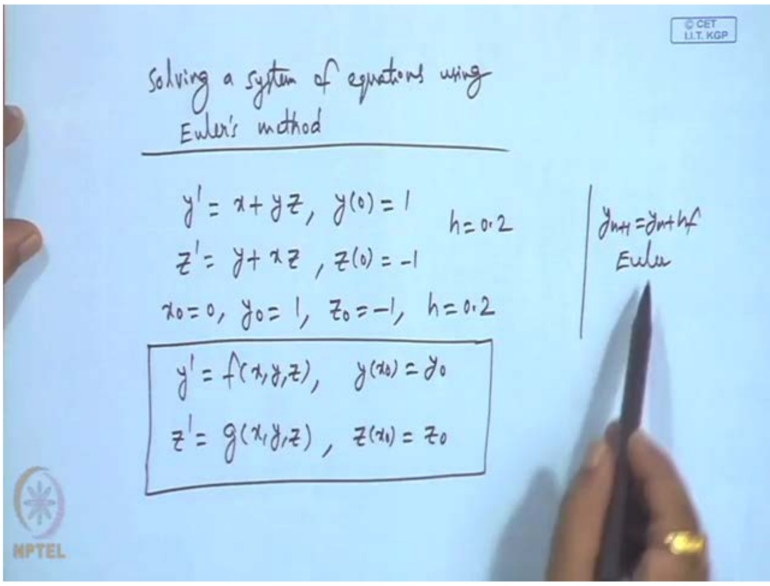

So, solving a system of equations using Euler's method, Consider the following system, so here x is independent variable, y and z are the dependant variables supported by the following initial conditions. Say h is 0.2, so x 0 is 0, y 0 is 1, z 0 is minus 1 h is 0.2, so these are all parameters. Now, we have to define Euler's method for this system, so if you recall we have y n plus 1 equals to y n plus h times f, this was our Euler. So, similar thing for this system, we have to write, but treating y dash equals to f of x y z, z dash equals to g of x y z and y of x 0 is y 0, z of x 0 is z 0. So, this is a general system, so we know f and g, for this we have to define Euler method, so how do we do?

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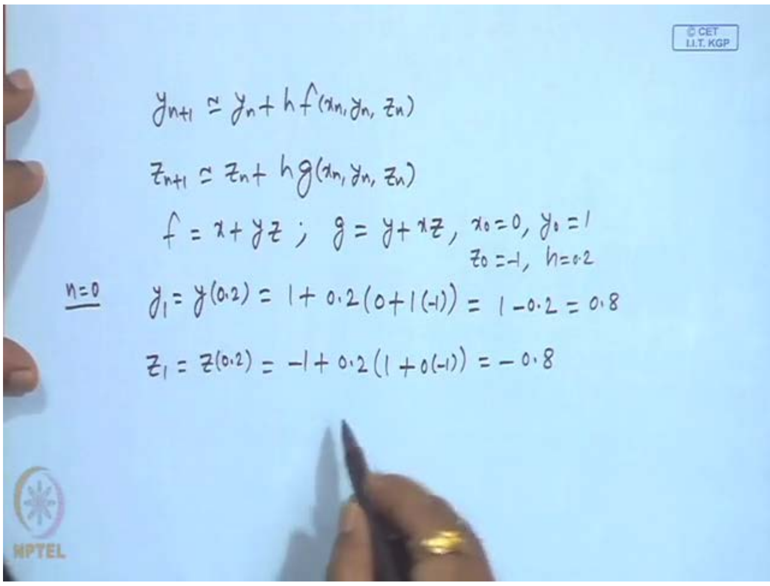

So, we define y n plus h and for Z g, now our h was 0.2, so for the given example f is this, g is this. So, let us write down our f is x plus y z, g is all that we need is this. Now, let to compute y 1, y 1 equals to y of 0.2 equals to y 0, so let us repeat all that x 0 is 0, y 0 is 1, z 0 is minus 1 h 0.2. So, y 0 is 1 h 0.2 and we have compute f of x 0 plus y 0 z 0 so that is x 0 is 0, y 0 is 1, z 0 is minus 1. So this is 1 minus 0.2, this is 0.8 then z 1, which is z of 0.2 is z 0 h, so we have to have g of x 0, y 0, z 0 so g is y 0 so 1 x 0 is 0 z 0 is minus 1. So, this whole thing is this is 0, so this is minus 0.8, so that means we got y 1 and z 1.

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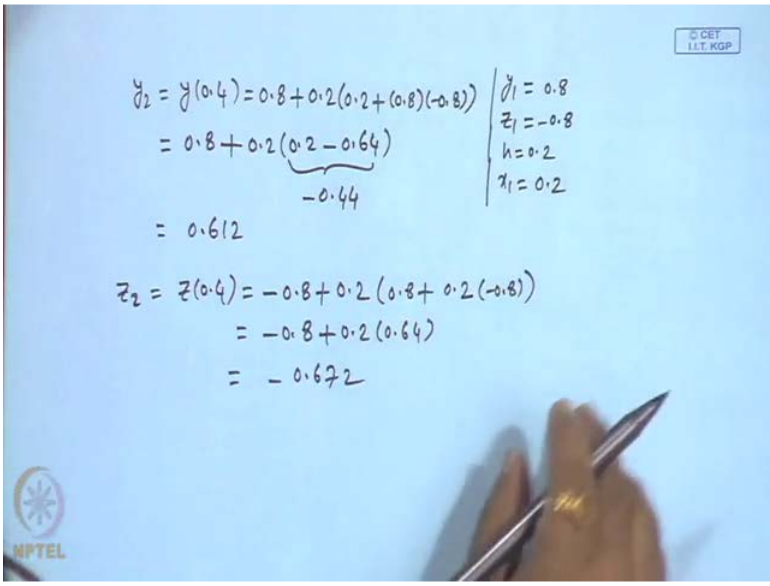

Now, let us proceed one step y 2, 0.4 is y 1, so y 1 just computed, so let us have the data here, so y 1 that was computed and x 1 we need y 1 plus h times f of x 1, y 1, z 1. So, y 1 plus h times, what was our f, this x 1 plus y 1, z 1, so x 1, y 1, z 1 this recalls. So, we can simplify we get, and z 2 z of 0.4, so z 1 that is minus 0.8 h then g of x 1, y 1, z 1 that is our g was this, so y 1 plus x 1 z 1. So, we get y 2 and z 2, so this is Euler method, so one can also try R K method. So, let us try to use may be points method or may be R K fourth order, we can try. So, another example because there is a slight change involved in solving system for R K method, so what is that trick, we will see.

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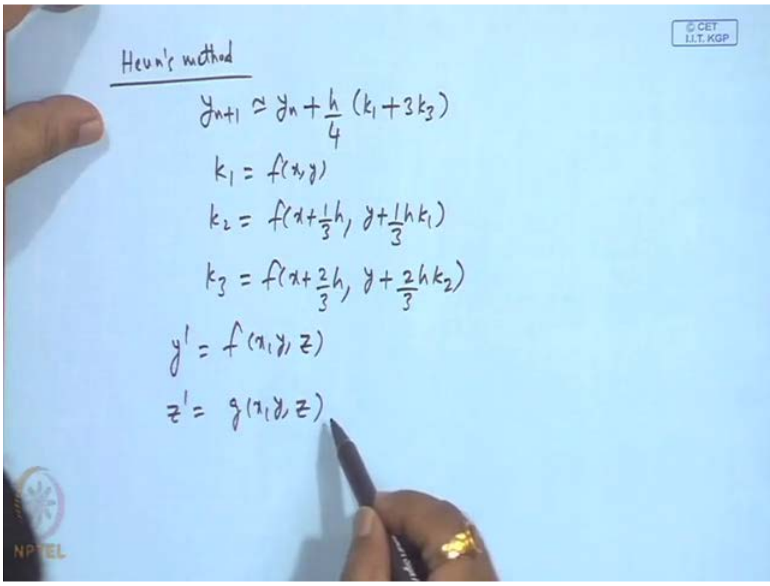

For example we talk about Heun's method, so what was the method y n plus 1 is y n plus h by 4 K 1 is f of x y K 2 is x plus y plus Heun's method, but we have a system y dashed is x y z, z dashed is g of x y z. So, that means we need to define similar set one set for this, one set for this.

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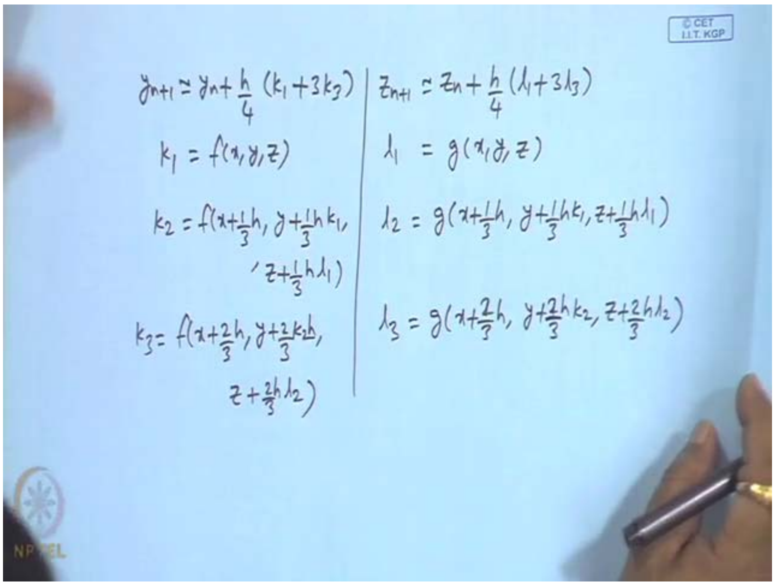

So, how do we do it, so we define y n plus 1 is y n plus h by 4 K 1 plus and K 1 is f of x y z. So, before we write for z, z n plus h by 4 some other notation and l 1 is g of x y z and K 2 f of x plus 1 by 3 h, y plus 1 by 3 h K 1 and z plus 1 by 3 hl 1. So, this is the difference while defining R K method coefficients, this is the difference because this is 2 dependant variables, so for y we are computing K 1, K 2, K 3 and for z we are computing l 1, l 2, l 3.

Therefore, in z the increment will be in the terms of l's and l 2 g of y plus 1 by 3 h K 1 z plus 1 by 3 h l 1 similarly, K 3 is f of x plus 2 by 3 h, y plus 2 by 3 K 2 and h h l 2 and l 3 2 by 3 h K 2 z plus 2 by 3 h l 2. So, you try to understand this because this is very important while solving system, since we have two dependent variables y and z. And since the system is defined as follows y dash is f of x y z, z dash is g of x y z, where x is independent variable and y and z are dependent variables. We are defining one set for y and another set for z and this is the set. So, you can see unless we compute one, we cannot compute the other. So, this time instead of taking a system, let us take y double plus with some conditions.

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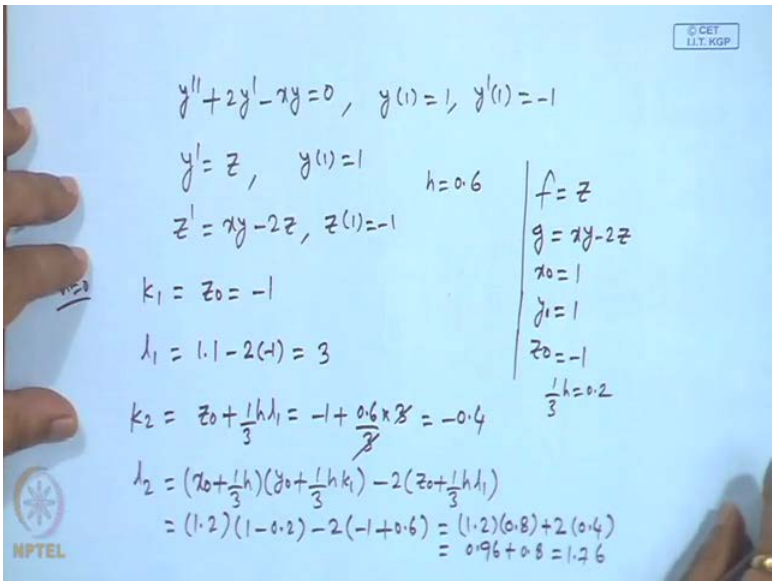

So, let y dashed is z then this becomes z dashed equals x y minus 2 z from here, and y 1 is 1 z 1 is minus 1 and say h is 0.6. So, we have to compute say K 1, K 1 equals 2, so before we compute, we can write what is our f, f is simply z and g is x y minus 2 z. So, K 1 is now n equals 0, K 1 is z 0, which is we can write x 0 is 1, y 0 is 1, z 0 is minus 1. So, this is minus 1 then l 1 we do is parallel. Why we have to do this because to compute K 2, you need K 2 if it is function of z, you need l 1. So, unless you compute l 1, you cannot compute K 2, l 1 is g of x y z, g of x 0 y 0 z 0.

So, our g was this therefore, x 0 y 0, x 0 is 1, y 0 is 1 minus 2, z 0 is minus 1. So, this is 3 then K 2, K 2 is defined as f of x y z dependency like this, but for the given case, since f is only z, so we get z plus 1 by 3. So, z 0 plus 1 by 3 h l 1, so z 0 is minus 1 and h is 0.6 by 3 and l 1 is 3. So, this is minus 0.4 then l 2 is g of corresponding increments. What are the increments, x plus one-third h y plus one-third h K 1 and z plus one-third h l 1, so we have to have x 0 plus one-third h y 0 plus one-third h K 1 minus 2 times z 0 plus one-third h l 1.

So, this is l 2, we can simplify x 0 is 1 and one-third h, h is this so one-third is 0.2, so 1.2 then y 0 is 1 and one-third h is 0.2 and K 1 is minus 1. So, 1 minus 1 minus one-third h is 0.2, so let us write one-third h is 0.2 and K 1 is minus 1, so minus 0.2 minus 2 times z 0 is minus 1, one-third h is 0.2 l 1 is 3. So, this can be simplified 1.2 times 0.8 there minus 2 times so this becomes plus 0.4, so you get 0.96, so this will be 6 so this is l 2.

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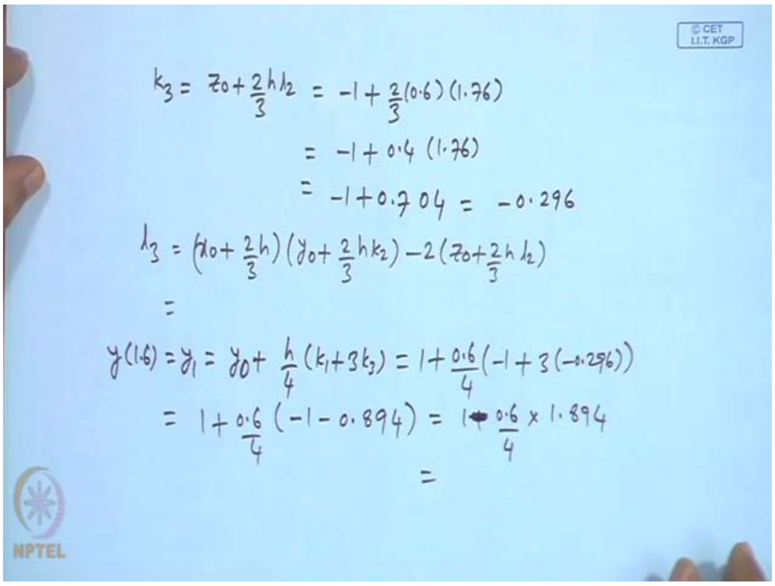

So, now having completed l 2, the next step is K 3. So, K 3 is z 0 plus two-third h l 2 so that is z 0 is minus 1 plus two-third h l 2 is 1.76. So, this minus 1 plus l 3, l 3 is little lengthy x 0 plus two-third h y 0 plus two-third h K 2 minus 2 times z 0 plus two-third h l 2, so we can get the value. So, for example having computed K 3, we get y of 1.6 equals to this is our y 1 and the formula defined is y 1 y 0 plus h by 4 K 1 plus 3 K 3. So, y 0 is 1 h 0.6 by 4 and K 1 minus 1 plus 3 K 3 plus 3, so this is minus sign. So, we simplified we get the answer.

So, this is the r K method for solving system of equation, the important of R K is how to define the coupled system like this, the coefficients. This is very important in R K method, so you can see here two dependant variables y and z. Therefore, one system for one dependant variable, other system for other dependant variable, but they are coupled and the coupling forces that you compute K 1 then l 1 will be computed, unless l 1 is computed K 2 cannot be computed, unless l 2 is computed K 3 cannot be computed.

So, in this sense you may try the R K fourth order for a system of equation, which is definitely a little bit of algebra is involved. Now, in the next lecture we try to learn for example, somebody gives this is an approximate method say R K method only, but how do you determine what will be the error corresponding to this approximation, so that is an important task. So, we should try to estimate what should be the order for a given approximate method. So, then we try to learn little bit of as I mentioned for computing y 1, some error is involved.

Now, we are using y 1 to compute y 2 so the error will be propagated. Now, if small disturbances are creating a small error at early stage creates large disturbances then definitely the system is not good for us because such an approximation is not good, that means this is the concept of stability. So, we try to discuss all this things in the next lecture.

Thank you.