Numerical Solutions of Ordinary and Partial Differential Equations
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Lecture - 39
Finite Difference Approximations for 1st order Hyperbolic PDEs

Hello, welcome back. So, we have reviewed last lecture, we have discussed second order hyperbolic and then method of characteristics for first order hyperbolic. So, let us see there are some finite difference methods as well, for first order hyperbolic curve PDEs. Both scalar as well as system of equations, that can be represented in terms of a vector equation. So, let us discuss the following methods.

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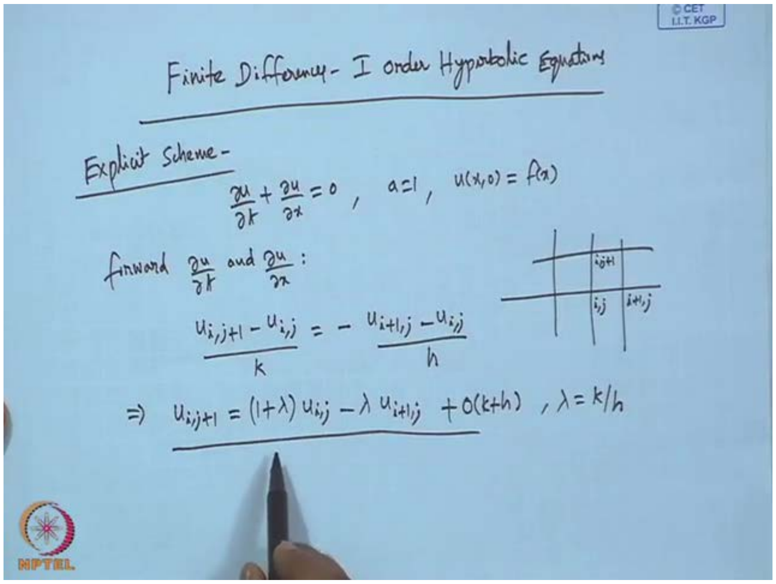

So, to start with explicit scheme, so we consider equals to 0, so sometimes we take a constant a, but in this case a is 1, so maybe we can write this. Then now if we go for forward and for both, what we get minus h or plus equals to 0. Now, from this u ij minus lambda u i plus 1 j, we get this and this is of order k, this is order h.

So, clearly where lambda is k by h, so this is explicit however we can see to compute and j plus 1 require jth level only, two points i and i plus 1. So, this is an important observation, so this appears like this to compute i j plus 1 it ask i j and i plus 1 j. So, it does not ask this point, so this is an important observation. So, now let us try to see the stability aspects of this.

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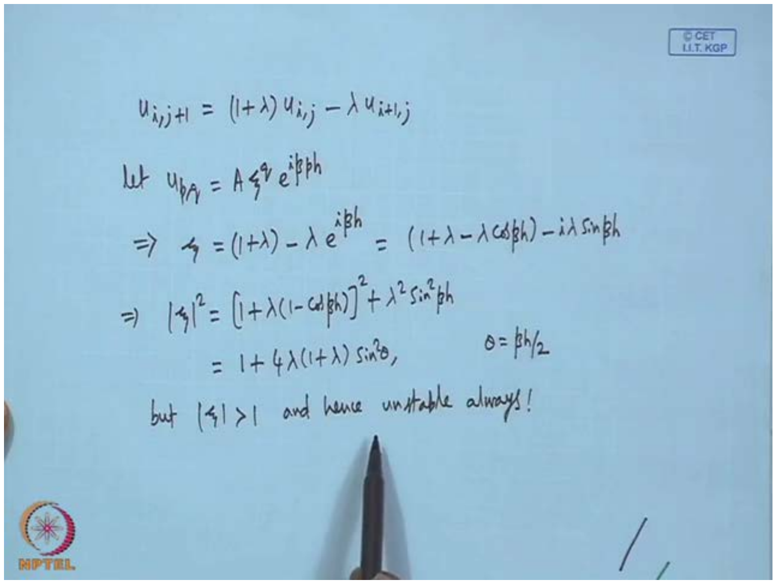

So, we have u i j plus 1, so then let u p q equals to, so if you substitute in the above equation, we get zeta power p plus 1. So, I am writing the simplified one, so zeta power p get cancelled throughout so we get zeta 1 plus lambda minus lambda, we have i plus 1. So, we get one excess here so this can be simplified as, now mod square is...

So, this can be simplified as, where theta is beta h by 2, but we can observe see this is positive. So, this is greater than 1 and hence unstable always, so there is a suitable explanation for this behaviour. So, what is happening in this method, as I mentioned this is an important observation to compute the value at this point, it is asking only these points. So, that means if the initial data is changed here, so then this is getting affected however, if there are some changes here, then this is not realising.

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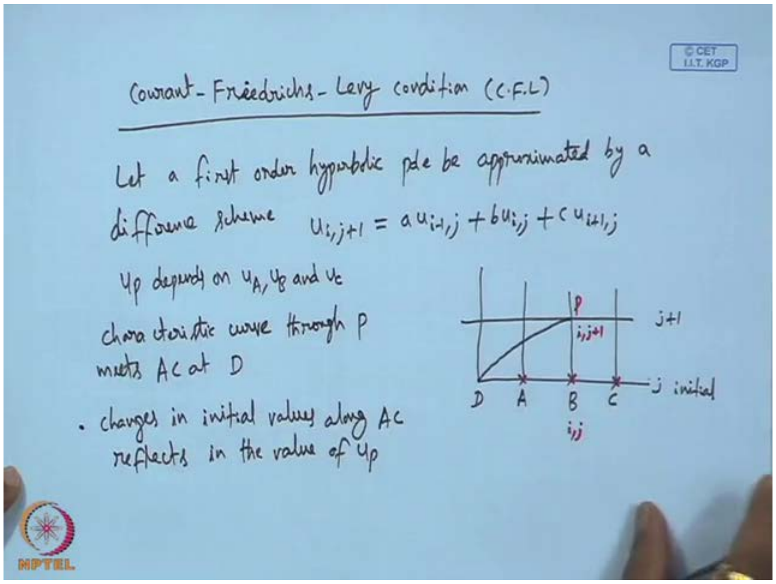

So, this can be explained suitably, this can be explained suitably via Friedrichs levy Condition so this is popularly known as CFL condition, Courant Friedrichs levy. So, now what is this states, so this discusses about such unstable behaviour. So, let a first order hyperbolic PDE be approximated by a difference scheme, then so this is A B C so then so this is i j so therefore this will be call it P. So, this is i j plus 1 so that means u P depends on u A, u B and u C, which are the mesh points. Now let us say, the characteristic curve through P meets A C at D, so this is initial line and let us say so the characteristic curve through P meets A C at D. So, then what happens, as I mentioned changes in initial values along A C reflects in the value of u P.

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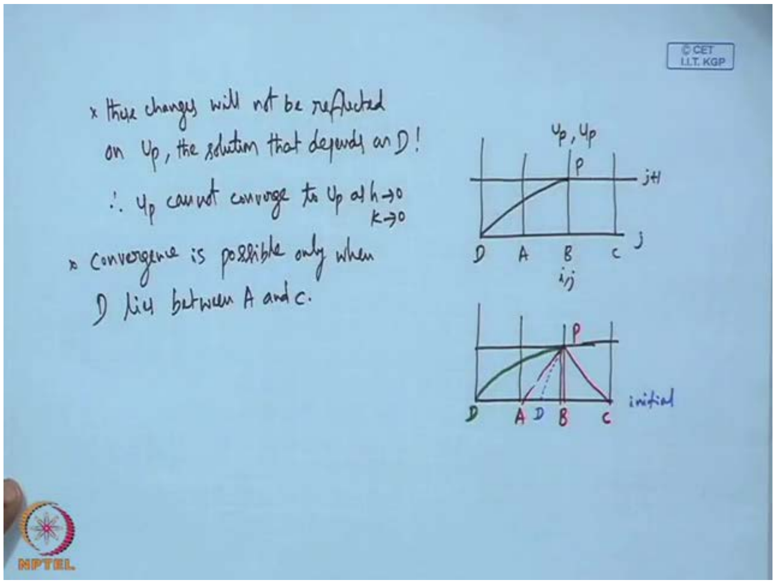

So, if you change, but unfortunately if you consider the point D so this is our P and D so this is i j. So, these changes will not be reflected on U P, which is the solution that depends on D. You see, when we change the initial data P is getting affected, but if you compute along the characteristic, the solution u P so we have u P and u P.

So, this is computed along the characteristic, but that does not know what is happening at A B C. So, therefore u P cannot converge to u P as H goes to 0, k goes to 0. So, now what should happen, if any changes in this they are percolated to P, so which means since we are computing the solution along the characteristic, the changes which are done A to C must be plugged in at a point D.

So, this is possible only when, D lies beyond this that means within this range, if D is any point so then what would happen, the following is going to happen. If you consider so this is the P so this is our explicit method however, so this is via characteristic. So, now if D lies suppose you take D so then what will happen, any changes on the initial data D also gets changed and those changes are percolated to P. So, accordingly the solution capital U P also responds. So, therefore convergence is possible, only when D lies between A and C.

So, when we talked about, the earlier explicit method you can see, this point depends only on these two points. So, therefore if you compute along the characteristic so definitely it is not going to converge. Hence, we got unstability, hence this is unstable, so therefore what is the remedy?

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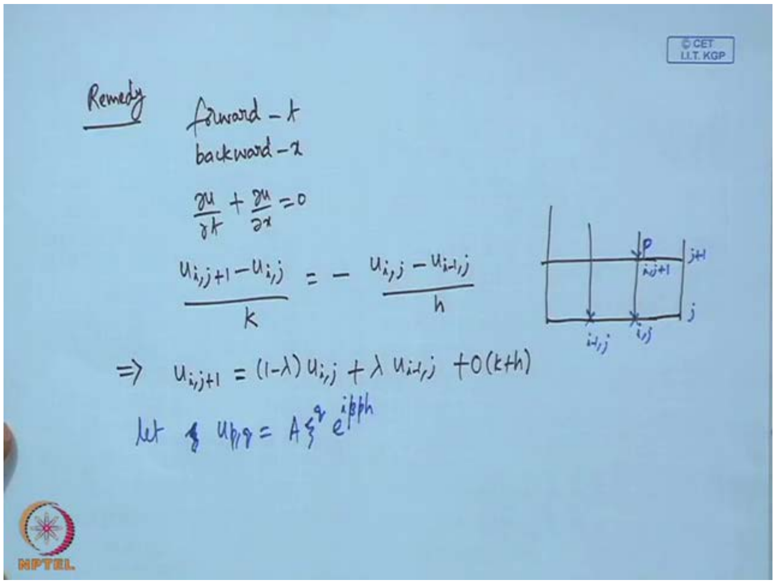

So, definitely we have to think of so some remedy. So, the remedy is forward in time, backward in space so consider this. So, you can see this is what we have so backward so now this implies so we get this. So, this scheme is using points as follows, this is using so i j plus 1 is i j and i minus 1 j. So, therefore what would happen any changes here so along characteristics within this we can get convergence. So, let us verify this, let u p q is.

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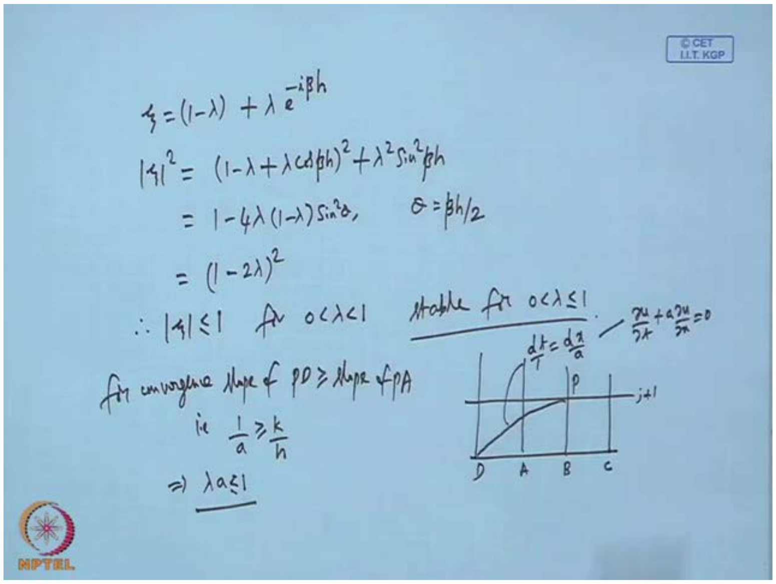

So, then when we substitute, the above equation we get so we get this. So, this is almost similar, but there is a sign change and considering the maximum value so therefore. So, less than 1 for, lambda less than 1, so this is stable for so this is the corresponding stability condition. So, this can be also explained via CFL condition because what is inferred using CFL condition this is P and this is D. So, what is inferred for convergence, slope of P D is greater than or equals to slope of P A. So, that is slope of P D is nothing but see, this is dt by 1 dx by a from where we are getting. So, that is 1 over a greater than or equals to k by h so that implies lambda a is less than or equals to 1. So, this is, there is a coincidence a is 1, in this analysis so therefore we have corresponding stability condition.

So, CFL condition helps because if you take any characteristic point, which is not on the initial interval so what is happening, the changes are not being percolated. So, whereas if you consider D within the range of the initial strip, then initial line segment, then the changes are being percolated hence, we get convergence.

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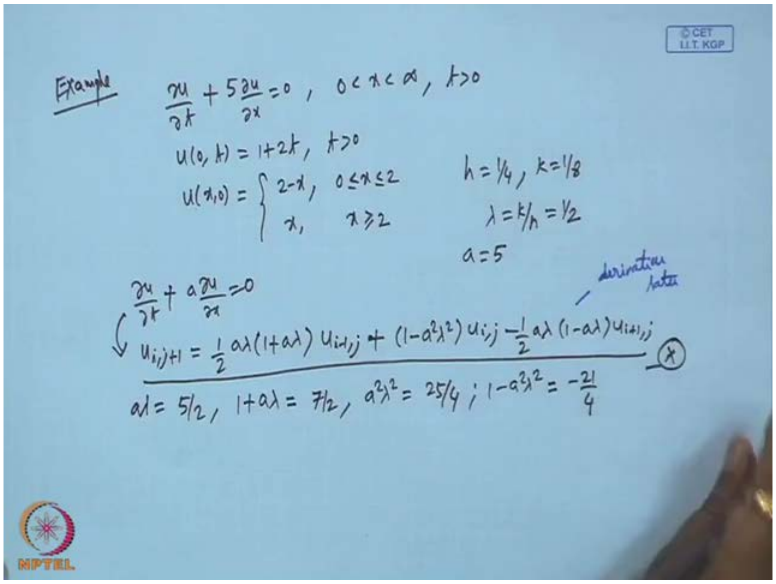

So, let us take an example and do the numerical calculation so this is the problem and h 1 by 3. So, comparing with the standard form, we have a equals to 5. Now, we require couple of quantities, so the quantities we require a plus lambda, a minus lambda etcetera. So, let us if we discretize the corresponding equation, we get so if we discretize this equation then, we get. So, let us assume for the time being such a method, we are doing. So, then we need various quantities a lambda, a lambda is 5 by 2 then 1 plus a lambda is 7 by 2, a square lambda square is 25 by 4, then 1 minus a square lambda square. So, this is minus 21 by 4 so now the derivation of this method, I will do it so I put a remark. So, for the time being you think we are solving this problem, which has been approximated by this scheme.

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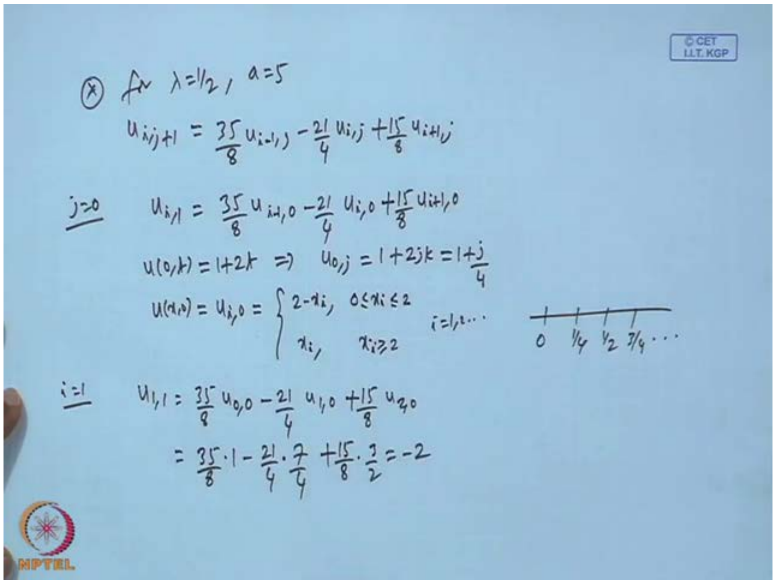

So, then star for lambda equals to half is given by u i j plus 1 and a equals to 5, for these values this is given by 35 by 8. So, it is given by this so now we have to compute at various grid points. So, we compute at various grid points so consider j equals to 0 so then at first time level, then we have 0, t equals to 1 plus 2t. So, this implies is 1 plus 2 j k which is 1 plus j by 4 because k is 1 over 8, then u of x, 0 is u i 0. So, now so on. So, consider i equals to 1 so then u 1,1 is 35 by 8 u 0,0. So, u 0, 0; so j is 0 so this is 1 and u 1 0 from here 2 minus x 1 so that would be 2 0 is 2 minus x 2.

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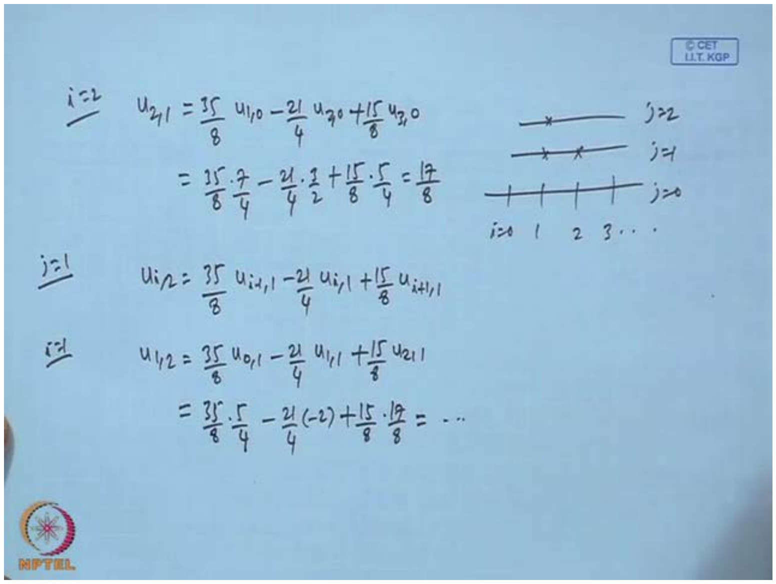

Now suppose, i equals to 2 so this 1 0 so one can compute so i 0, 1, 2, 3 so on so forth. So, now let us go for j 1 then we get from this formula, from this formula when j is 1 we get u i 2 1 plus. So, now i 1 so this will be u 0 1 then u 1 1 just now we have computed minus 2, u 2 1 also we have computed. So, we get some number, so that means we can keep on computing j 1, j 2 you can see, we have computed u 1 1, u 2 1 then we could go for u 1 2 so one can compute the values like this. However I made a remark, this equation can be approximated by this. So, let us see what kind of approximation gives this method.

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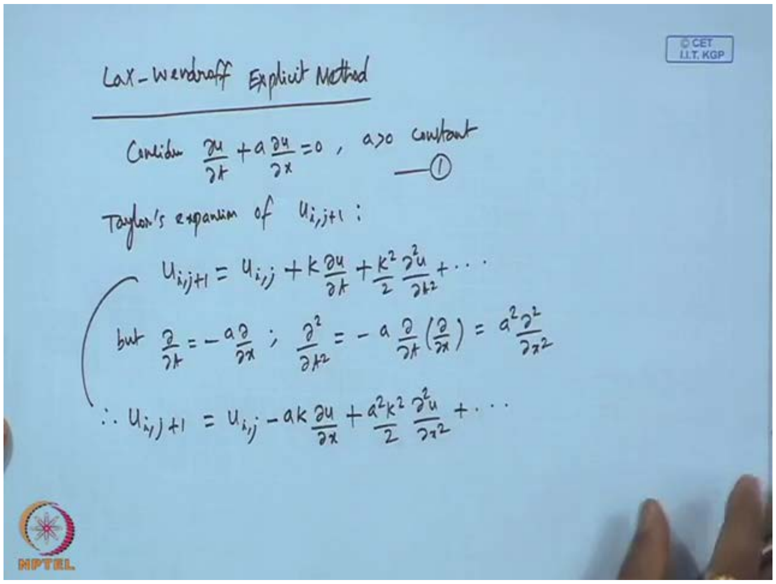

So, this is called Lax Wendroff Explicit method so what we are doing consider constant. So, now we would like to get explicit approximation, for this. So, Taylor expansion of u i j plus 1, so you consider this, but notice from 1 accordingly. So, therefore u i j plus 1, so we have this u i j minus we are substituting this, a k and for this.

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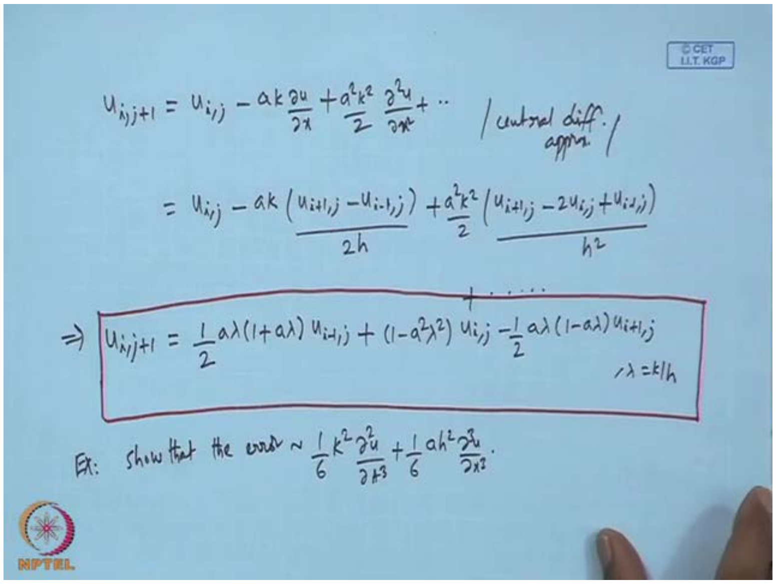

So, we have, we have this so now going for central difference approximations, we have for this term, then the next term. So, second order plus dot dot, so this implies, so we can adjust so this is our Lax Wendroff Explicit method. So, we have made use of a small trick what is that, we have considered the, straightaway we have considered the expansion at u i j plus 1. However, we made use of the equation and transferred the time derivatives to spacial derivatives and approximated by central. Then we got explicit expression, that involves all the three points. So, now exercise show that the error involved in the above approximation is, okay.

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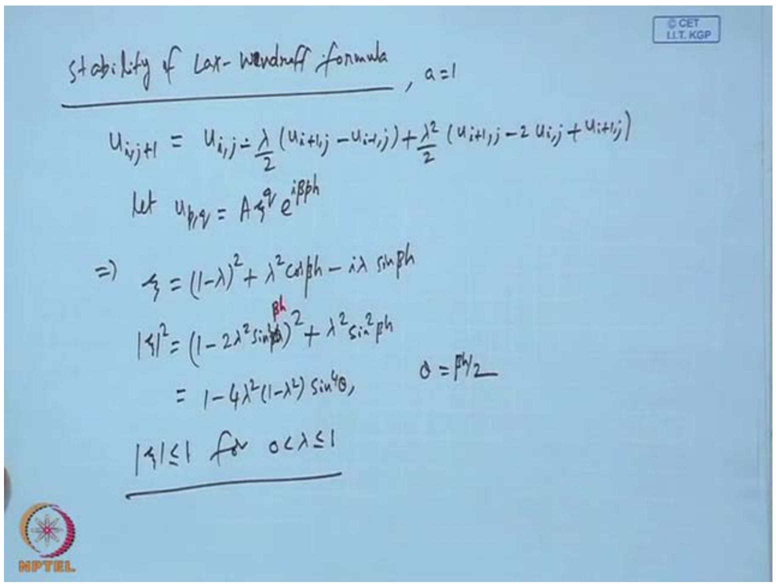

So, this can be shown so now let us talk about stability of this method. Formula or method whatever, u i j minus so we have considered a equals to 1. So, this is considered in raw form without adjusting so then let so this implies so this is sine square beta h so this is 1 minus 4. So, therefore one can less than or equals to 1 for, so this also tells that Lax Explicit method is a stable, for these values of lambda, which are coinciding with the earlier method, which we have described. We have considered forward time and then backward space so and then obtained a corresponding method, so that method and also this, both are stable for the same range of lambda. So, now as I mentioned at the beginning of the lecture, the corresponding first order hyperbolic equations can be extended to a system of hyperbolic equations.

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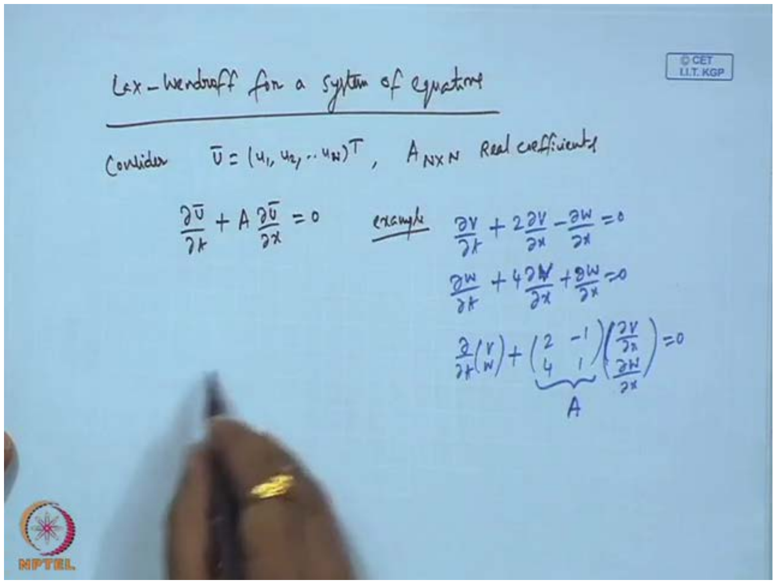

So, let us see ah how the system looks like. So, we have so we consider so then we have the corresponding system, can be put it in this form. So, for example say we have so this can be put it in the system where, A dou u by dou x so this is our A then so this is matrix A. So, such kind of system can be also be discussed. So, now when we have such system, how do we approximate without going over the individual components.

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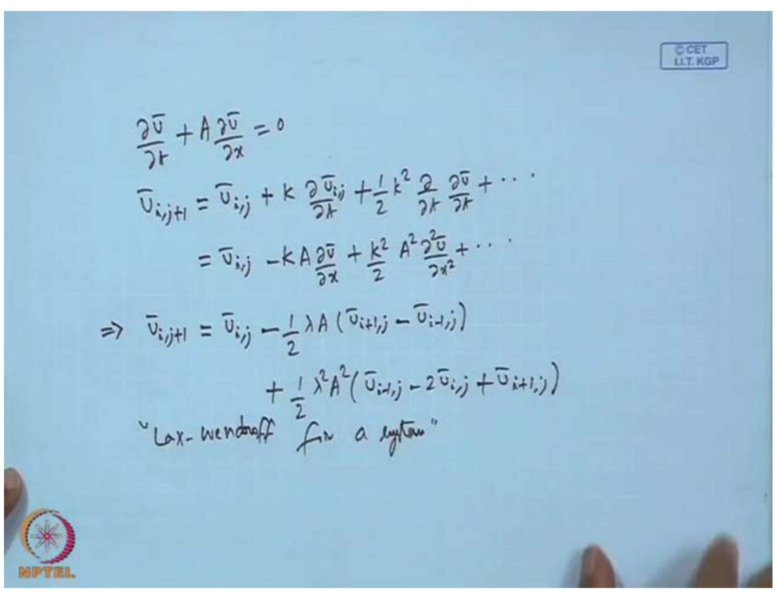

So, we follow similar method as we have done for a single equation. So, we consider so then ofcourse of i j, so this is so we make use of the equation. Then so this implies again we go for central so this is Wendroff for a system.

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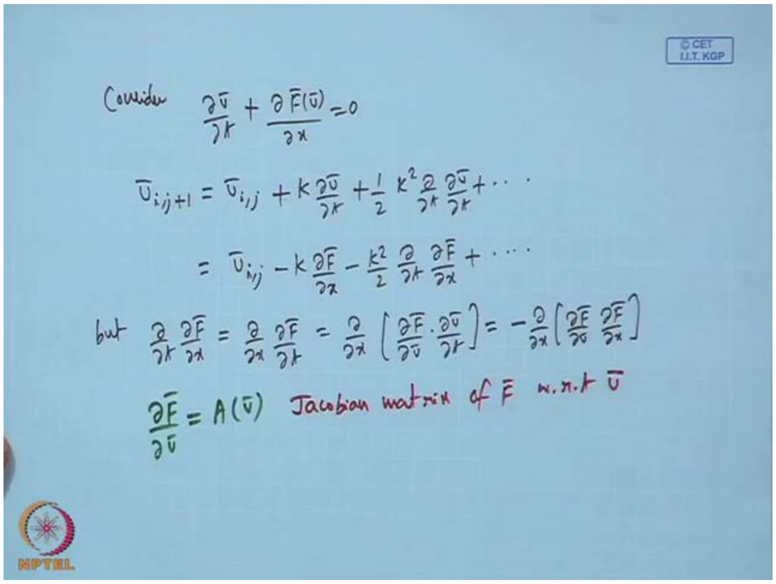

So, now let us consider the general case where, the equation is in more standard form. So, if this is in the standard form, then we get, again we make use of this. So, dou by dou t we substitute minus. Now consider this, this will be again we substitute so minus so now we have an interesting quantity, which one. So, this is nothing but our A of u which is the Jacobian of F with respect to u bar.

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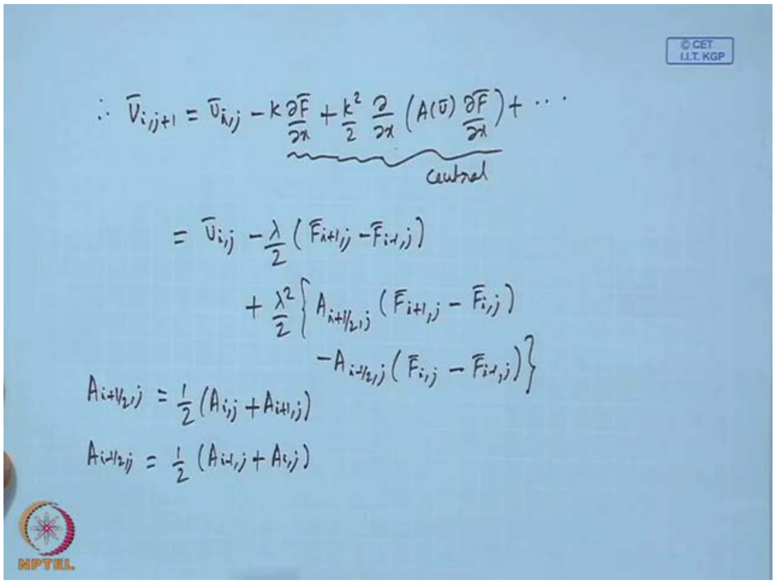

So, now the method becomes so therefore is. So, now we go for central difference approximations so this would be so this I am doing for the completion sake. Otherwise, if you consider a specific example and try to do this, then you would understand much better. You see for this central would involve, midpoints, so you may refer any book on numerical analysis so where midpoints. So, this is the case for a system however, this is very abstract unless, you consider a system and then try to follow the notations and maybe if you write a code. So, then one would feel more confident.

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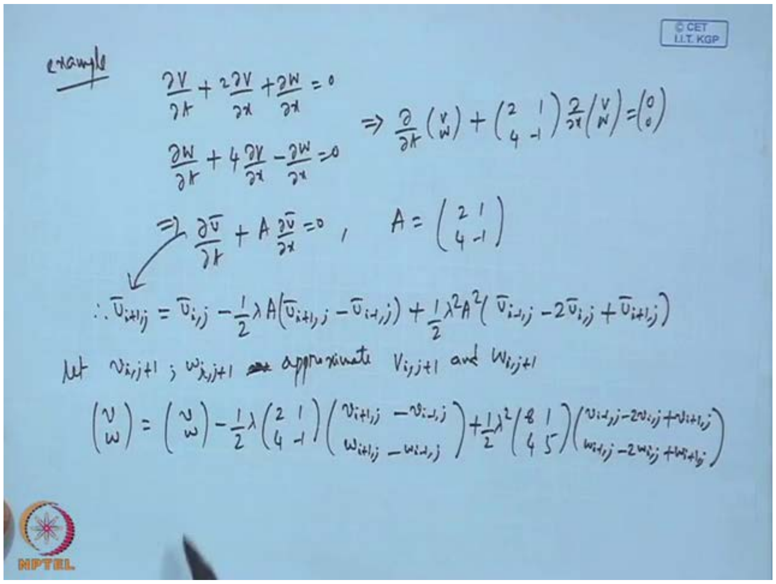

So, let us see a simple system and see how it looks like. So, this can be expressed as follows so this implies so where. So, accordingly we can use our, we can use this now, let v i j plus 1, w i j plus 1 approximate, let them be approximating this. So, then from the above structure we get the following so each component wise I just would like to give you an idea and similarly. So, if you consider such a system making use of the approximation, for this system in a general notation, we get each component and one can solve them. So, this is a simple example however, if you take a real numerical example where, you have coupled hyperbolic systems, it would be better to write a code and implement, so that one can understand better.

So, with this we have almost covered topics in PDE for example, we started with a parabolic type where, we discussed explicit and implicit methods for parabolic PDEs, in particular with respect to heat conduction equation. And then we moved to elliptic where, we discussed about Laplace and Poisson and then we moved to hyperbolic where, we considered wave equation in case of second order. And in case of first order hyperbolic, we discussed scalar equations as well as systems.

So, this course, main idea of this course is to get exposed to various finite difference approximations, to both ODEs and PDEs. So, in case of ODEs as you know, we started with single step methods and then multistep methods. And then we have also gone for finite difference methods, for the boundary value problems. So, you get some idea to start with then, once you are done with this course, you can definitely understand advanced techniques. So, for example alternate directions and then maybe higher dimensions and staggered grid and things like that not of course finite elements. So, we have almost come to end of this course, so we have one lecture left and then we will discuss maybe, some summary and maybe some important directions until then, bye.