Numerical Solutions of Ordinary and Partial Differential Equations
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Lecture - 37
Method of Characteristics for Hyperbolic PDE s- I

Hello, so as I have mentioned in the last lecture, we have discussed finite difference approximation for a hyperbolic PDE's. Now, let us ah discuss method of characteristics, so this is a very special method this is a kind of numerical treatment based on analitical approach, because for hyperbolic PDE's, generally we talk about characteristics and then capture the solution via characteristics. Now, similar technique we are using it however we develop a numerical scheme, so before we proceed we must understand what is a characteristic curves.

So of course, many of you when you have attempted your first course on PDE, they analytical one so you would have solved using separational variables and there you would have talked about this is a characteristic equation, and these are the characteristic curves etcetera. Let us review what is a characteristic curve first, and then slowly we understand.

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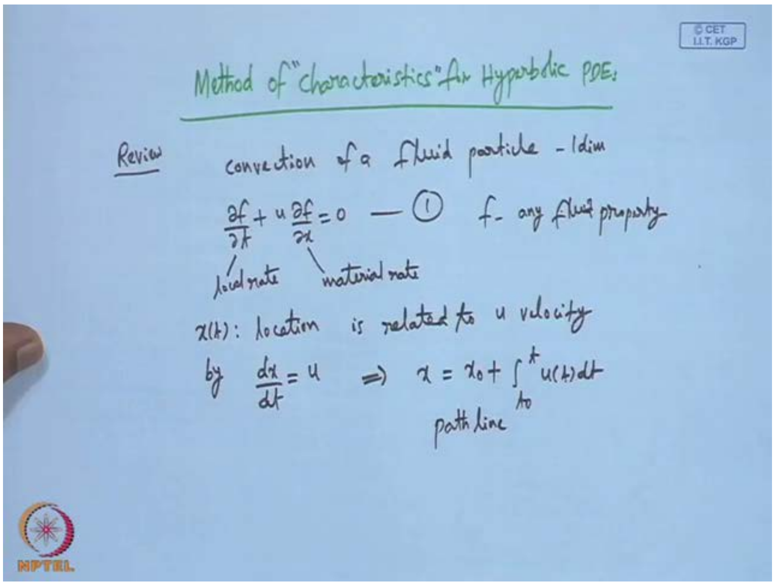

So, method of characteristics for hyperbolic PDE's, so when I say this, first we have to understand what is a characteristic? So, typically let us start with just a review, on first order then we discuss second order. So, review is it would be better to consider a physical example, so convection of a fluid particles. So, say we want to discuss in one dimensions, so the corresponding equation is equals to 0. So, where f is any fluid property, so this is local rate and this is material rate.

So, that means any fluid property this is a local change with respect to the time and this is how it percolates with respect to space. So, you can see so this is the velocity which is driving the material changes. Now, in this case x t which will be location is related to u velocity by d x by d t equals to u. Now, one can integrate this and obtain so u one can obtain which is path line.

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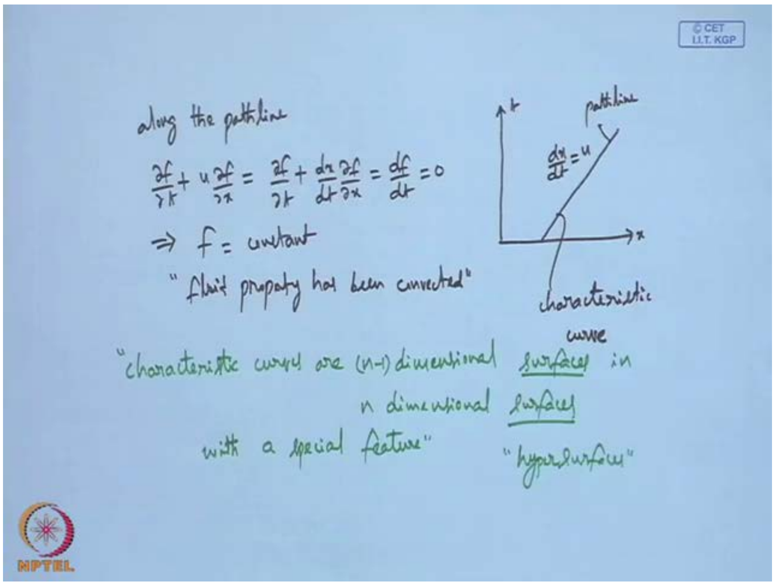

So, here in x t space so we have, so these are the lines and this is the path line. So, now along the path line dou f by dou t u dou f by dou x equals to along the path line, so this is d f by d t and we have this equals 0. So, that means f can be obtained as constant so that means the fluid property has been converted. So, we make use of these in these case this path line is called characteristic curve. So, what do you mean by a characteristic curve? So, these are characteristics curves are say n dimension n minus 1 dimensional surfaces because in n dimensional surfaces with a special feature.

So, this is a typical definition when I say surfaces, these are hyper surfaces so those who do not you need not worry it is like more than three dimensions kind of x y z and t. So, that means if you have two dimensions the characteristic curves are one dimensional. If it is three dimensional the characteristics curves are two dimensions with a special property, what is the special property? You can see along which that that data gets converted the data gets percolated, so that means in one dimensional case in a such a simple physical example one can see any fluid property f has been convicted. So, along a path line and hence this path line is very special for us, and this is a called characteristic line.

Now, similar concept if we would like to extend it to second order so then we have what is a scenario there we have a higher order derivatives to call you have u x x u x y u y x etc. Now, when information percolates along these lines then for example, if there are some discontinuous let us say a places where initially you do not have derivatives, derivatives does not exist initially and same information gets percolated along those. So, this is a what we are trying to do in the method of characteristics is, find out such lines and try to compute the solution along those lines. So, this is the typical thumb rule kind of function. Now, let us see for second order how the characteristics come into picture.

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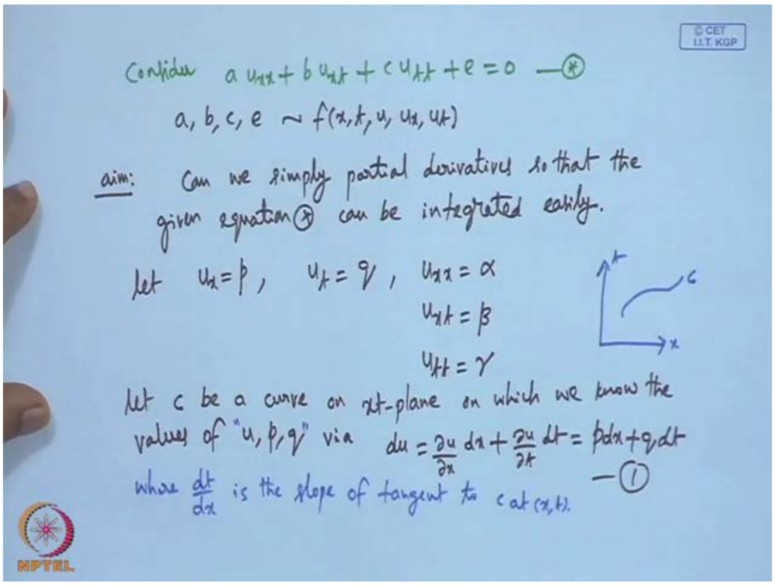

So, we consider second order, so where a, b, c, e all of them are functions of x, t, u, u x, u t. So, I did not write each one of them, but they are now what is aim? Aim is as I mentioned can be simplified partial derivatives, so that the given equation star can be integrated easily. So, this is relative what you mean by easily one may ask, in some sense by converting it to the curves that means by reducing the dimension in this case. So, let so the answer for this is yes of course, so that is the aim. So, how do we do it?

So, this is a typical u x equal to p, u t equals to q and u x x is alpha u x t is beta, u t t is gamma we introduce. Then let c be a curve on x t plane on which we know the values of u p q via d u equals to dou u by dou x, which is equals to p dx plus qdt so call this 1. So, let c be a curve. So, this is x this is t so this is any curve on which we know u p and q via this relation. So, this is a in analogy to the one dimensional case so on this curve let us say we know via this relation so here where d t by d x is the slope of tangent to c x x t, so what is our aim?

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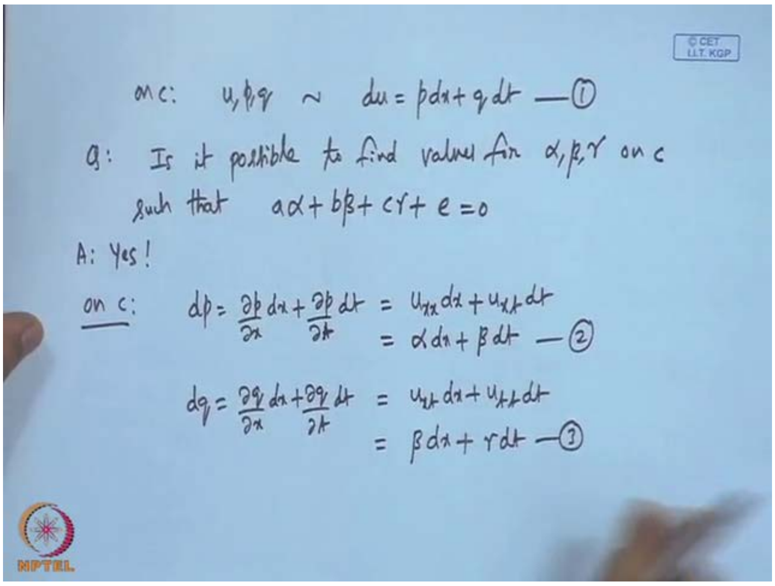

So, we know on c, u p and q are related via q d t which is one then the question is, is it possible to find values for alpha, beta, gamma on c such that a alpha plus b beta plus c gamma plus e equals 0, what is this? This is our original equation see this is our original equation which we have considered a u x plus b u x t plus c u t d 0, and this is a notation we have adopted. So, then if u p and q are known we would like to determine on c, these alpha, beta, gamma such that this is satisfied. So, that is what the question and the answer to this is yes of course, via c we do this.

So, let us see how we do now on this curve d p is since p is known on this curve it satisfies this which is now p x is u x is p so p x is u x x, then this is u x t because this is p is ux and pt is uxt dt and this is alpha dx plus beta dt call it 2, then d q so this would be u x t d x u t t d t this is equals to beta dx plus gamma dt. Now, the answer we want to see whether we can compute the that derivatives double derivatives because along the characteristics they propagate. So, in order to in order to find out the double derivatives we go back to the characteristics. So that is the concept.

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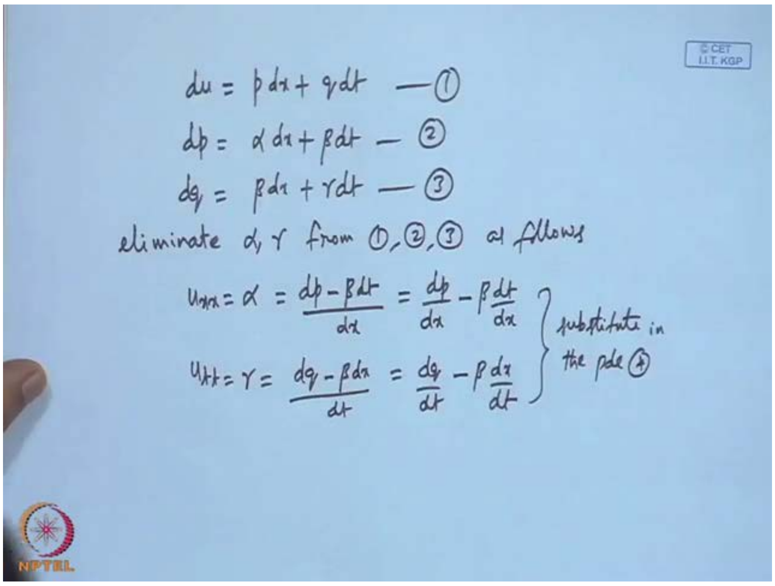

So, we have d u equals to p d x plus q d t then we have dp equals to alpha d x plus beta dt then dq is beta dx plus gamma dt. So, we have this then eliminate alpha and gamma from 1, 2, 3 as follows. So, why are we doing this we will come to know little later so we want to eliminate alpha and gamma because alpha, beta is common to both so alpha and gamma we want to eliminate so to do this so this is. So, this is equal to so from this three equations. So, we get this now let us substitute in the P D E star, so that is in this so we substitute. So, remember we are trying to substitute u xx and u tt so there will be definitely u xt remain.

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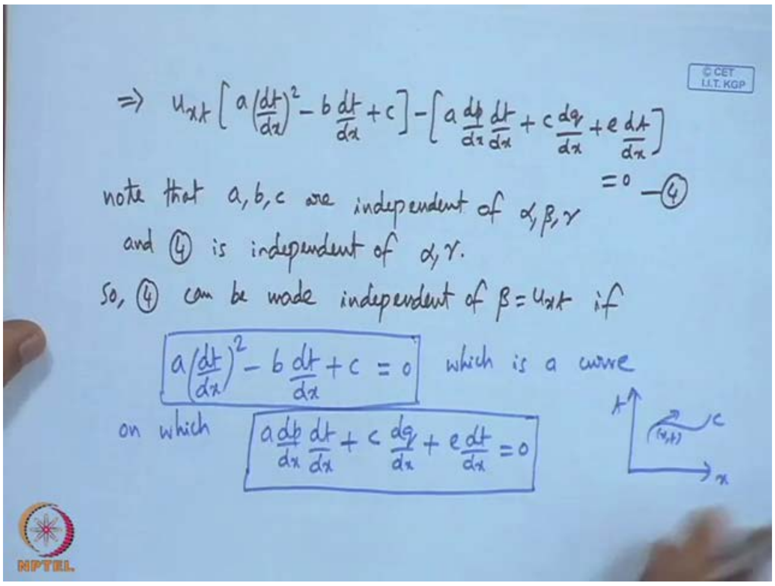

So, if you do that we get u xt so we get this so as I mentioned we substitute in here. So, when we do that we get u xx will be replaced u tt will be replaced, and we collect the coefficients and we get this. Now, note that a, b, c are independent of alpha, beta, gamma and 4 is independent of alpha and gamma because we have eliminated them. So, this can be made independent of this is nothing but beta so this can be made independent of beta if this is 0, if this is 0 whole thing is independent of beta and this relation is satisfied.

So, 4 can be made independent of beta which is u x t if this is equals to 0, which is a curve on which see this is 0 a dp dx dt dx plus c dq dx plus e. So, what we are trying is that this is also independent of beta if this is 0. So, that means which is a curve on which, so this relation holds so this is very important for us. Now, what does it mean? Slope of the tangent at see, because dt by dx is slope of the tangent, this is c x and t so if you take any point x t dt by dx is slope of the tangent at xt. So, slope of the tangent is route of this equation so that is first difference, so let us write down.

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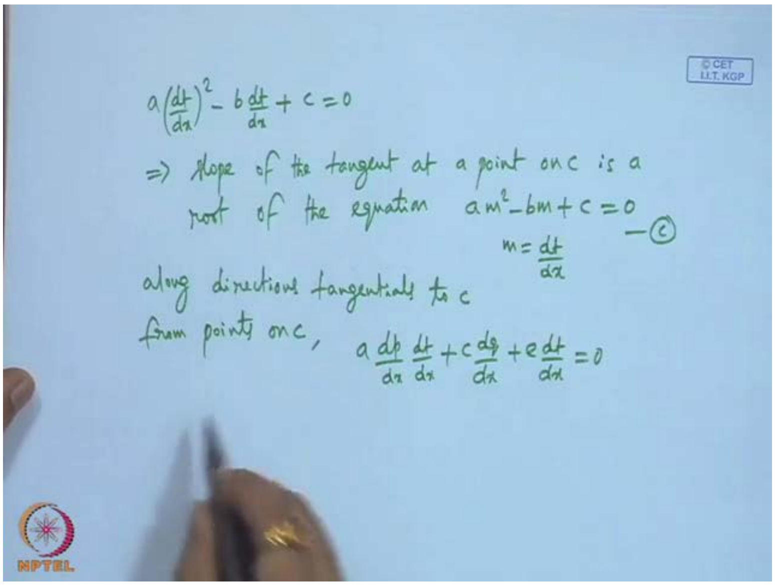

So, let us write down a dt by dx minus dt by dx plus c equals 0 implies slope of the tangent at the point on c is a route of the equation a m square minus b m plus c equals 0. Of course, m is dt by dx so this is say c right then next inference, along directions tangential to c from points on c we have this relation. So, these are this gives that slopes satisfies this relation, then in that case this relation holds. So, it's a combined so slope satisfies this equation, then along direction tangential to c from points on c we have a dp dx dt dx dq dx e dt dx equals to 0. Now, we have two routes for this equation, so what will happen in that case?

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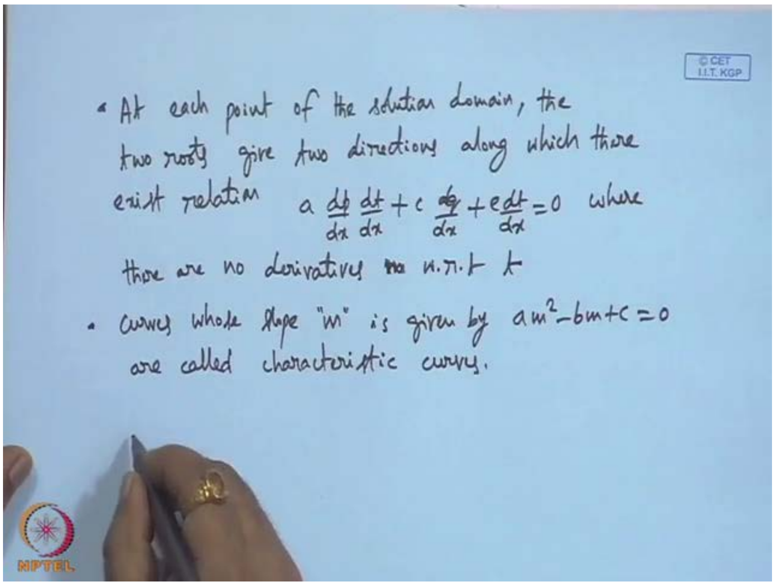

At each point of the solution domain the two roots give two directions along which, there exists relation a dp dx this dq dx dt dx equals to 0, where there are no derivatives with respect to t, why we are making this conclusion because we forced from here, we forced this becomes independent of t if this is 0. So, hence this is 0 so this equations satisfied means, there no derivatives with respect to t. Now, we have to talk about these two roots.

So, the curves so you can give some bullets, so curves whose slope m is given by a m squared minus b m plus c equals 0, these are called characteristics curves. So, that is where we are relying on the characteristics curves. So, that means what did we do we have tried to capture the curves along which the derivatives are vanishing with respect to t. So, we have eliminated alpha and gamma before and them we made it independent of beta, and we have captured the corresponding curves it has two roots and then it corresponding to each curve each root we get characteristics curve.

Now, what information this characteristics curves gives, and more over is it true that always we get these characteristics curves? So, it is a simple question to answer because we have quadratic with respect to the slope now definitely it depends on the kind of roots we have. So, we have to answer this question based on the roots that is where the second order classification has been done based on the behavior of the roots.

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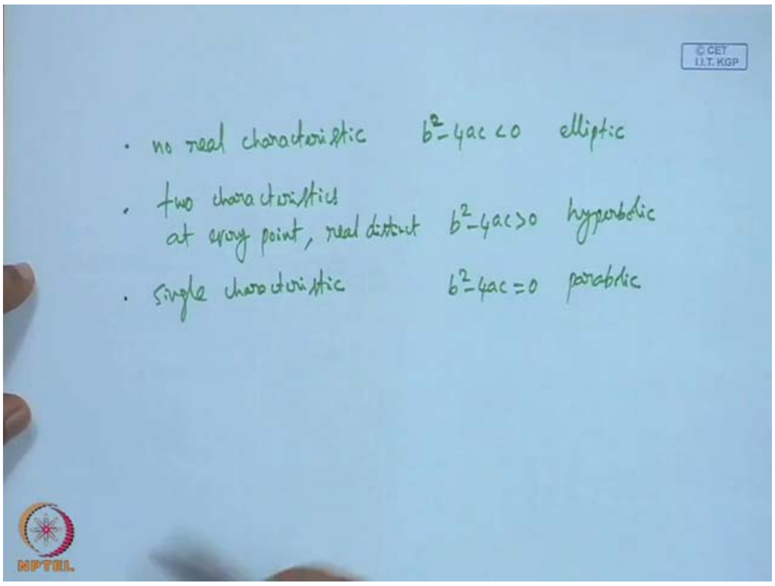

Now, in this case, case one say no real characteristics, so this means b squared minus 4 a c is less than 0 and according to second order classification, we call this elliptic. And then another case two characteristics at every point and they are real distinct roots, this is the case of real distinct roots. So, b squared minus 4 a c greater than 0 and this is the case of hyperbolic then single characteristics and this is the case of so only one root. So, this is the case of parabolic, so which means for example, if since the information passes through this characteristics.

In case of parabolic and hyperbolic at least there is somebody who can percolate the behavior. So, if the derivatives are discontinuous then the derivates are definitely discontinues along those, but in case of elliptic. Since, there are no real characteristics so one cannot rely on the characteristics, so the entire solution domain will be responsible for the for computing the next one. So, in case of elliptic that is the reason we have complete closed boundary. So, we do not have any real characteristics on whom we depend and try to compute the solution, where as in case of parabolic and hyperbolic we have somebody on whom we can rely and try to compute the solution. So, we are talking right now of the hyperbolic, so let us see how we can rely on the characteristics and compute the solution.

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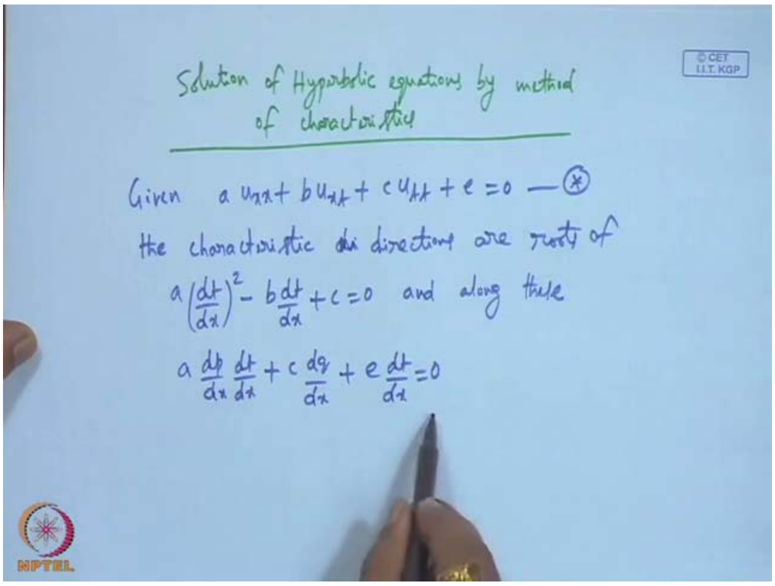

So, solution of hyperbolic equations by method of characteristics. So let us given our second order PDE. So, the characteristics roots the characteristics directions are roots of equals to 0, and along these we have a dp dx. So, this, what we just now informed now this can be recast in a slightly different way.

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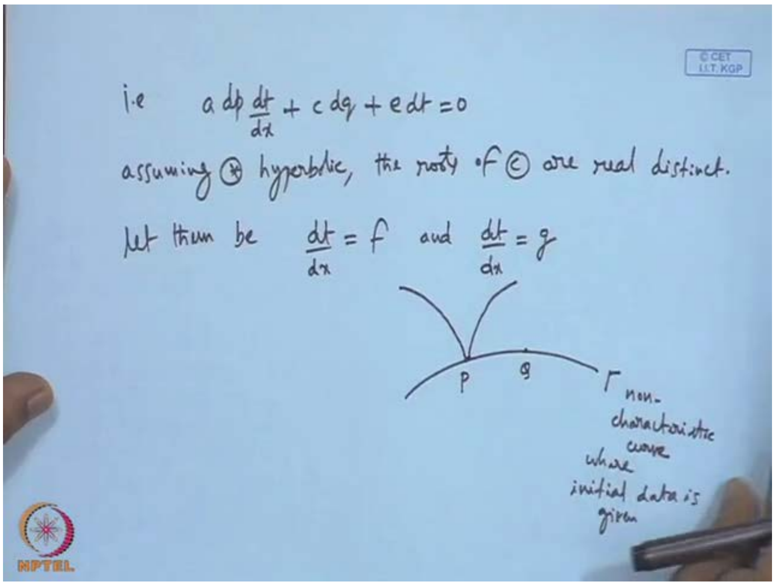

So, that is a d p into the d x d q d t 0 than since we are discussing hyperbolic, assuming star is hyperbolic the roots of so this is c the roots of c are real distinct, let them be that this roots be d t by d x equals to f and d t by d x equals to g. Suppose, if we take so what is the idea we would like to make use of these characteristics and solve the value because these characteristics are spread over the entire domain.

Now, instead of making use of grid by dispatching the domain we would like to compute the solution along these characteristics. So, if we consider gamma which is non characteristic curve, maybe along which the initial so where initial data is given so this is a gamma. Now, at each point we have two characteristics and we call them f characteristic and the g characteristic say this point P, similarly, at point Q we will have, right?

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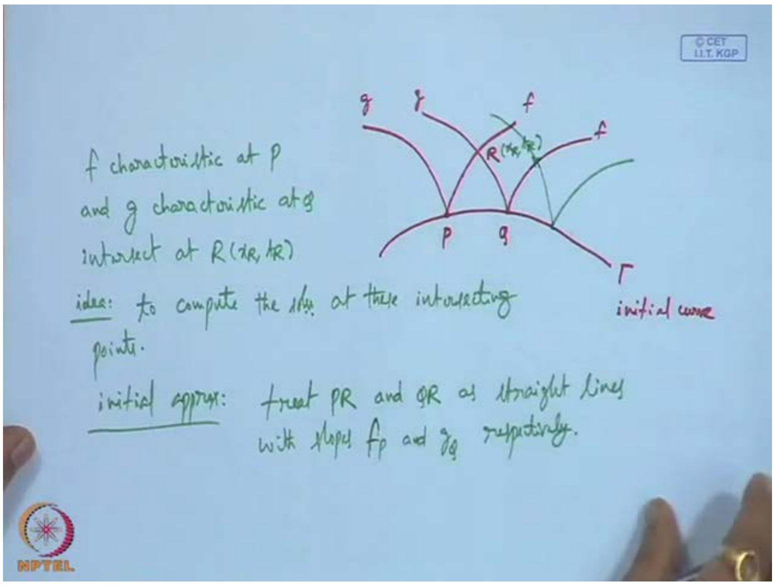

If gamma is such a initial curve so I have written previous non characteristic curve, so then at P we have f characteristic and g characteristic, then again the point Q this is f characteristic of P and g characteristic so what you could see the f characteristic of p and g characteristic of Q they are intersecting at R and what is this point R is x R t R. So, f characteristic at P and g characteristic at Q intersect at R which is x R t R. So, the idea is we would like to compute the solution at this point, then you can see. Suppose if we consider another point then I will have so again next point of intersection so then higher levels similarly, so that is how numerically we would like to compute.

So, the idea is to compute the solution at these intersecting points, now you can see these are curves so how do we compute. So, we treat them as initial initially to start with treat them as straight lines and then interpolate, so then we can refine so to start with initial approximation treat P R and Q R as straight lines with slopes of a obviously f P and g Q respectively. So, the idea is we would like to make use of this as straight line approximations with the slope f P and this is g Q and interpolate and compute the solution.

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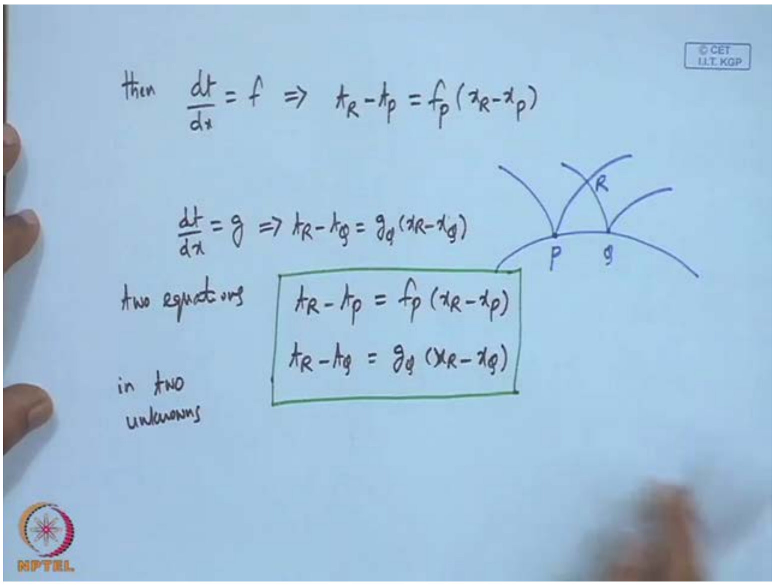

So, if we do that then what would happen d t by d x equals to f so this implies so this a interpolations because on the characteristics curve this is true on f characteristic. So, t R minus t P equals f P x R minus x P so this P means capital P point. So, it this P I am talking we get another which is a first derivative later on. So, this is R now this is a approximated right now along g, so this implies t R minus t Q is g Q x R minus x Q, right? So, that means we have two equations in two unknowns what are they our first task is to compute the point R. So, when we compute the point or then we next task is to compute the solution at R. So, this interpolation gives the point R, so once we have the point we get the solution there. So, how do we do that so we have to make use of the relation the second relation.

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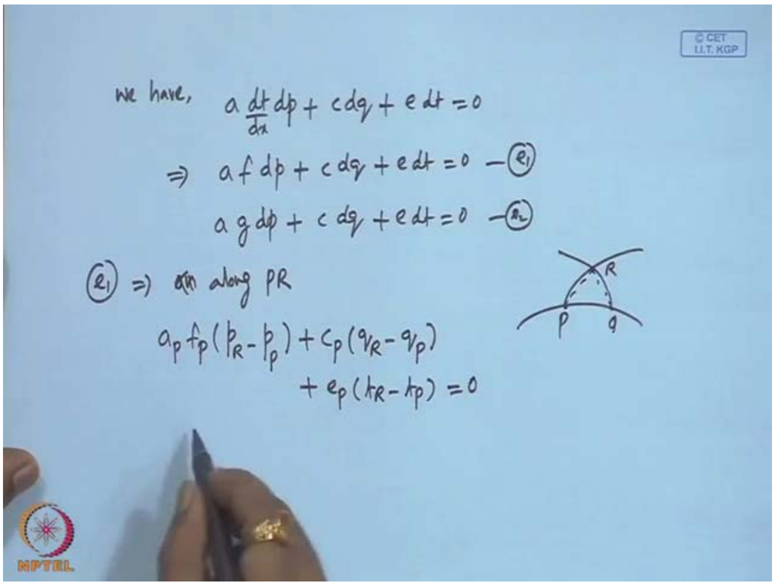

So, we have a d t by d x d p so these are small p please do not get confuse the point. So, these are first derivatives, so this the relations which we had f d p because this is f characteristics I have substituted. And similarly, on g then consider so we have to give some numbers so e 1 implies on not on may be along P R. Now, we are ready to so this is P, P Q R and we have approximate like, now consider this equation along pr this can be approximated a P at the point P then f P then so this we are considering e 1 so d p so d p along P R will be P R minus P P P R minus P P plus c P then d q Q R minus Q P then e P d t t R minus t P equals to 0. Then similarly, either do it.

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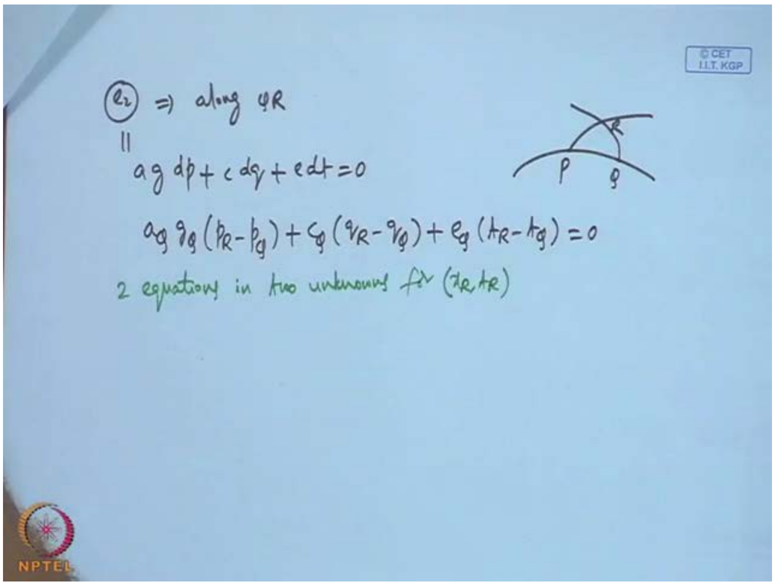

Consider e 2 this implies along Q R, so along Q R what is e 2? e 2 a g d p plus c d q e d t equal to 0. So, we would like to approximate this so this will be a Q g Q p R minus p Q plus c Q q R minus q Q plus e q t r minus t q equals to 0. So, we have two equations in two unknowns or x R t R this was the earlier an scenario, which are these.

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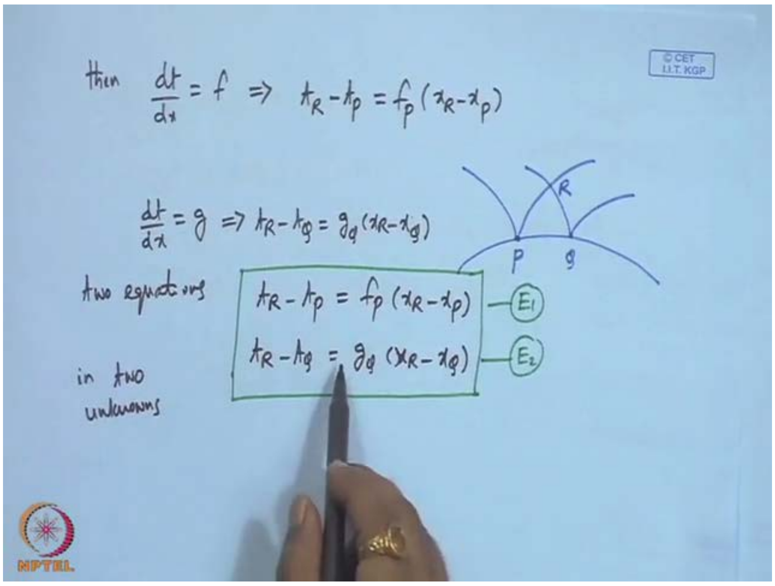

So call this E 1, E 2. So we had this two equations in two unknowns.

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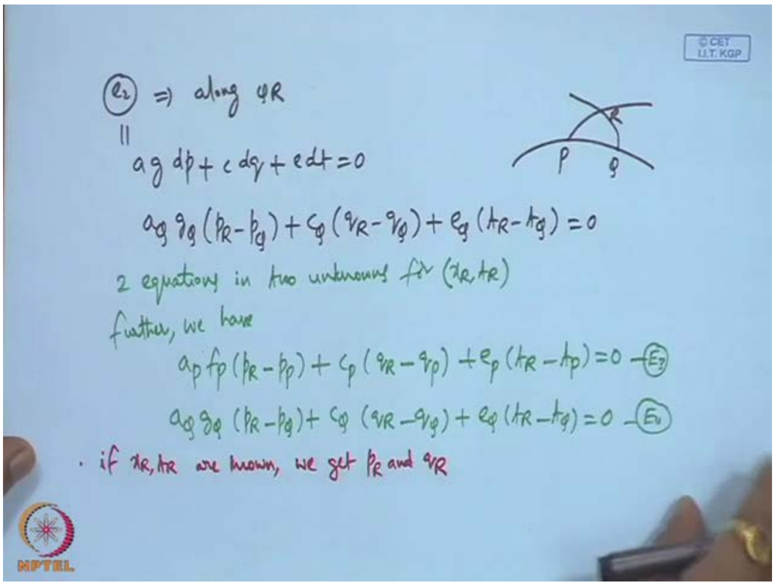

And right know we have obtained another two equations. So, maybe we have to write down further we have a p f p this is one and the other one is this. So, if so these are so that m R k's if x R t R are known then we get what do we get we get p R and q R using E 3 and E 4 and hence u can be obtain at R.

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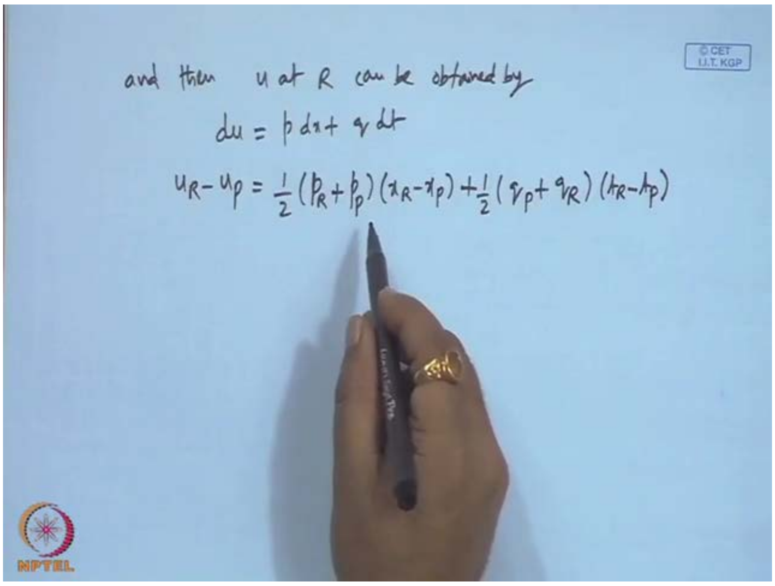

So, once we get p R and then u at R can be obtained by du equals p d x plus q d t. However, the better approximation is replace p and q along p R by average, so that means u R minus up is so let me have a quick brief because this is slightly complicated. So, we have at each point two characteristics so then we would like to make use of characteristics and get the solution. So, before we arrive at the characteristics then we have important tool on which we are aliening is this.

So, these are the characteristics directions on which this relation is satisfied, so then making use of this we compute the grid as an intersection of the characteristics. And at each grid point we have to calculate the solution, however to calculate we need the grid points. Now, in order to get the grid points we treat them as straight lines and compute the intersection point R. How do you do that?

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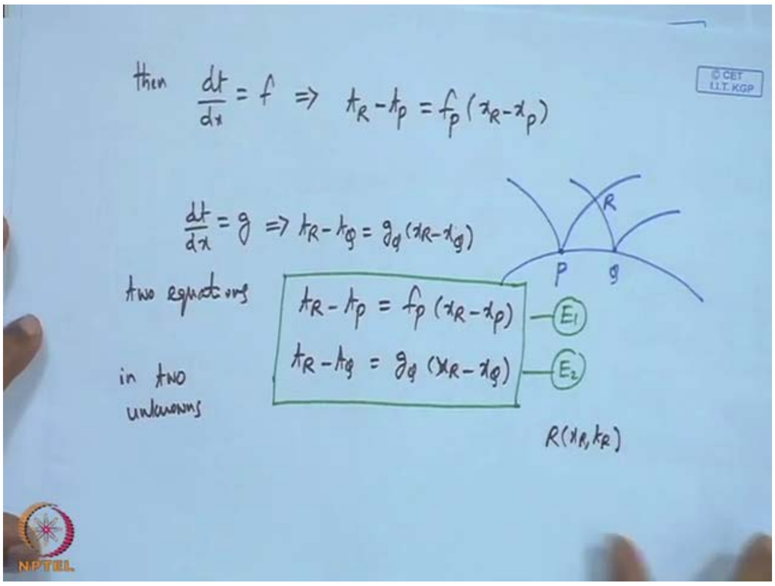

First the characteristic lines have been linearized with interpulation and we get two relations using which one can obtain x R and t R, that means we have obtained the point R. Once we get this point we would like to get the derivatives right solution, solution u is obtained in terms of the first derivatives. Now, in order to get first derivatives we approximate.

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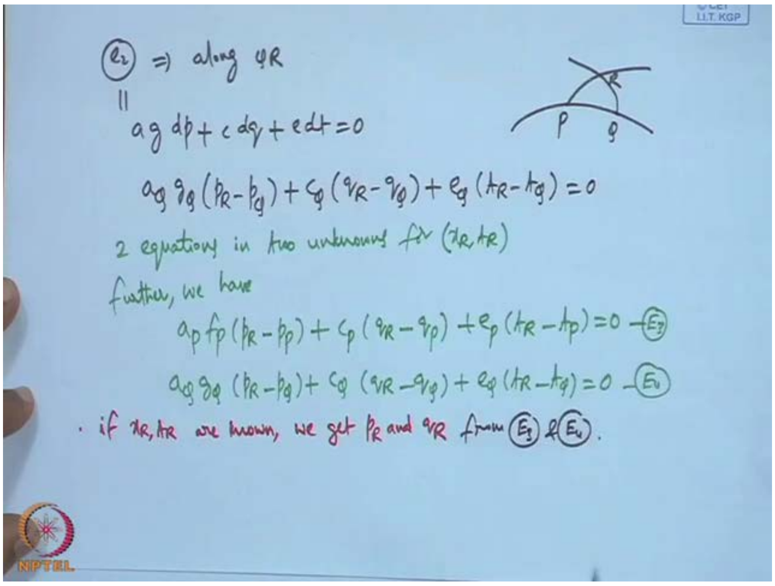

Then we get these two equations, now the remark is if x R t R are known then we get p R and q R from these from E 3 and E 4.

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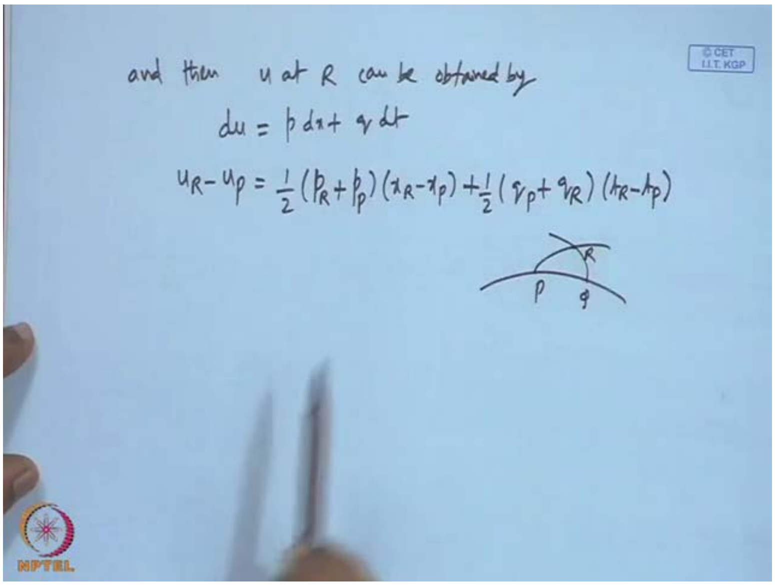

Once we know this our ultimate time is to get the solution u at R. However, what we did we have replaced here by average at p R and p P for better approximation. So, unless we solve the problem it is a little hard to follow, but however it is very beautiful concept this has originated from the analytical treatment because in analytical treatment, we capture the solution based on characteristics curves. So, via discritizing the characteristic grid we compute the solution and March past, so may we do some problems, so that we understand this better.

Thank you.