Numerical Solutions of Ordinary and Partial Differential Equations

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Lecture - 36
Finite Difference Approximations to Hyperbolic PDEs – II

Good morning. So, we had discussed yesterday about explicit methods for wave equation, and then we started implicit method. So, let us continue with the details on implicit method and also stability aspects of both explicit and implicit methods.

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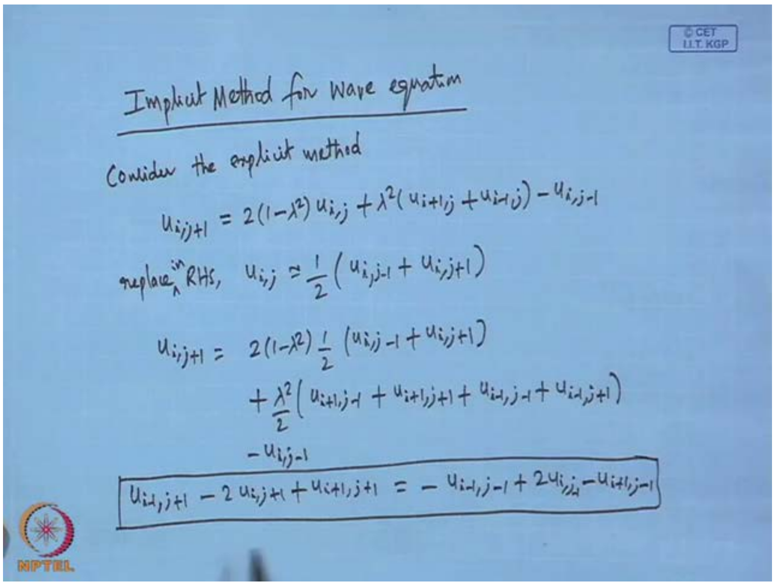

So, in order to derive implicit method for wave equation, so let us consider the explicit method. Consider the explicit method given by, so this is the explicit method. Then replace RHS. Replace in RHS, u i, j by u i, j minus 1 u i, j plus 1. So, this is a special case as I mentioned. So, then if we do that, we get that means we are expanding the scope of the levels, so j to j minus 1 and j plus 1 plus u i plus 1 j minus 1, then for this term.

Then, since this is at j minus, we keep it. Then after readjusting, we get terms, this j minus 1 j minus 1. So, this is a kind of specific method you can adjust. So, lot of terms get cancelled and then lambda square cut through across. So, hence lambda must be non zeroes. So, indeed lambda is non zero. So, we get. However, one can derive more general implicit method. So, let us see that.

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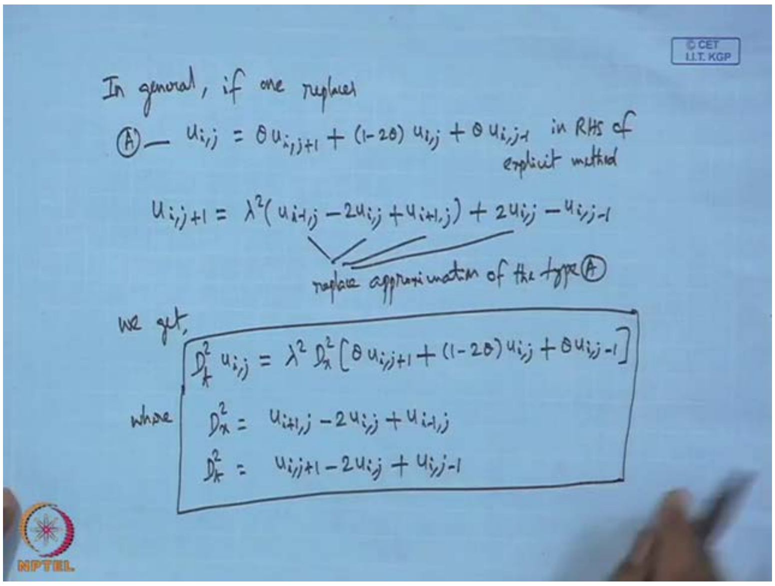

So, in general if one replaces u i, j by in RHS of explicit method, so if we consider the explicit method, so this is given by, so this is now each of them, so let us call this A, so for all this, replace approximation of the type A. So, when I said of the type A because this is for u i, j. Correspondingly, we have to adjust for i minus 1 and i plus 1. So, if we do that, we get, so different literature refers different notation.

So, in this case, theta is a kind of w8, so which can vary. Of course, when we discuss the error, theta will also come into picture. When we discuss stability, then theta will also come into picture. So, when we replace right hand side of explicit method by approximation of this type, we get this. We have to abbreviate. What is this notation? So, this notation is so this with respect x which is nothing but the central and this is t. So, that means this is clear when operation u i, j, exactly this disappears, whereas here on u i, j plus 1, so index has to be adjusted and this is exactly this. Here again, index has to be adjusted. So, this is implicit method of general implicit method. Now, let us try to fix that for a special case.

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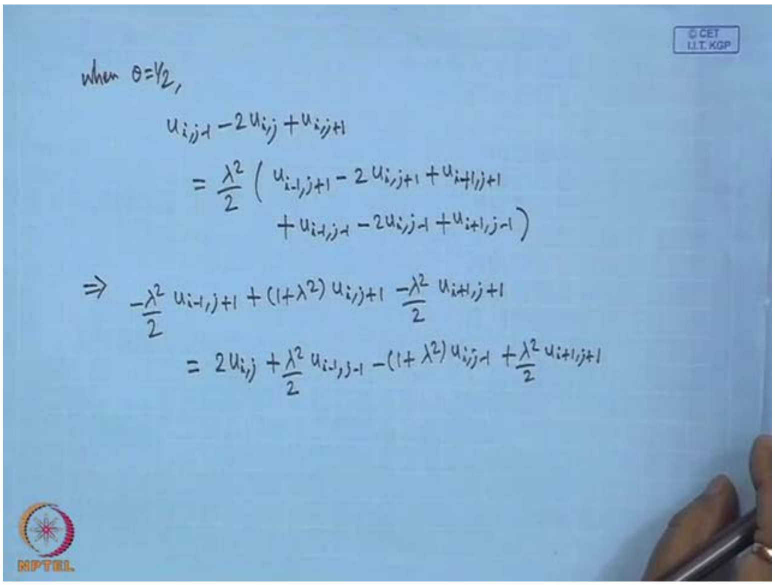

So, let us say when theta equals to half, we have u i, j minus 1, which is nothing but so this term is dependent of theta. So, exactly I have written the same. Now, when theta is half, this term goes off and we get these two. So, let us simplify. So, this is equals to lambda square by 2 because theta is half. So, that comes out. Then, D 2 x is operating on i minus 1 and i plus 1. So, this we get minus 2 i plus u i plus 1. Let me check one more time. So, this is on i, j plus 1 and the D 2 x therefore i plus 1 j plus 1 i, j plus 1 i minus j plus 1, so i minus 1 j plus 1 i, j plus 1 i minus 1 j plus 1. So, here minus i used, so this is plus. Then similarly, the next term which 1 here i, j minus 1, D 2 is acting on it. So, this would be j minus 1. So, this gets simplified. So, all higher term is to the left and lower term to the right.

So, we are just collecting, for example, u i minus 1 j plus 1 i minus 1 j plus 1 the quotient is lambda square by 2 and no other place it appears. Therefore, when it is left, we have minus. Then u i, j plus 1, we have 1 u i, j plus 1 here and we have minus lambda square. So, when it is to the left plus lambda square minus, so this is also we have i plus 1 j plus 1 the only term. So, minus is to the left hand side. So, this is equals this. We are pushing to the right hand side. Then i minus 1, j minus 1 with a lambda square by 2, then i, j minus 1 i, j minus 1, we have this. So, lambda square by 2, then i, j minus 1, then the remaining. So, this is a special case where theta is half.

Suppose, if you vary theta is 1 by 4, then we get more terms because when theta is 1 by 4, these terms are wise. When theta is half, this term got vanished. So, correspondingly, we get more terms. However, they get adjusted in terms of the coefficients. Now, this is for theta half. Let us say we consider this and try to solve a specific problem.

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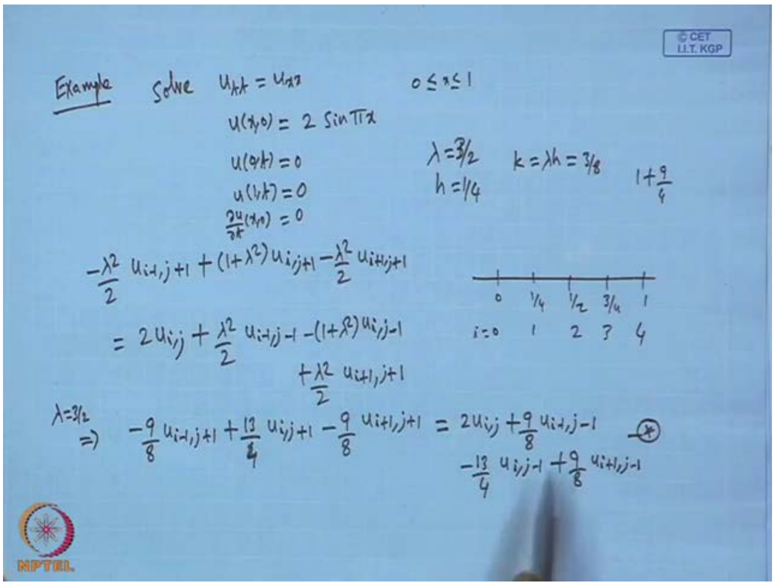

So, example, solve subject to the initial conditions and boundary conditions, this equals to 0, say lambda is 3 by 2, h is 1 by 4. Accordingly, k is lambda, h 2 by 8. So, h is 1 by 4 and let us say we are solving this. So, then we have, so i 0, 1, 2, 3, 4. Now, consider our implicit equation, so which is given by, so this is our implicit equation when theta is half. Now, lambda square is 8 by 4. So, lambda is 3 by 2. This gets simplified to and then 1 plus lambda square, so 13 by 2.

So, this will be 1 plus lambda square, so 13 by 4, 13 by 4 u i, j plus 1 minus 8 by i plus 1. So, this is equal to these terms 2 u i, j plus 8 by 8 minus 13 by 4 u i, j minus 1 plus 8 by 8 i plus 1 j minus 1. Now, please remember that we did not write one more initial condition. We did not write one more initial condition. So, this initial condition would be on say equals to 0. Now, when we run this set, j is equal to 0, we get terms including minus 1. So, let us see what will happen. Now, I just keep it so that you can see.

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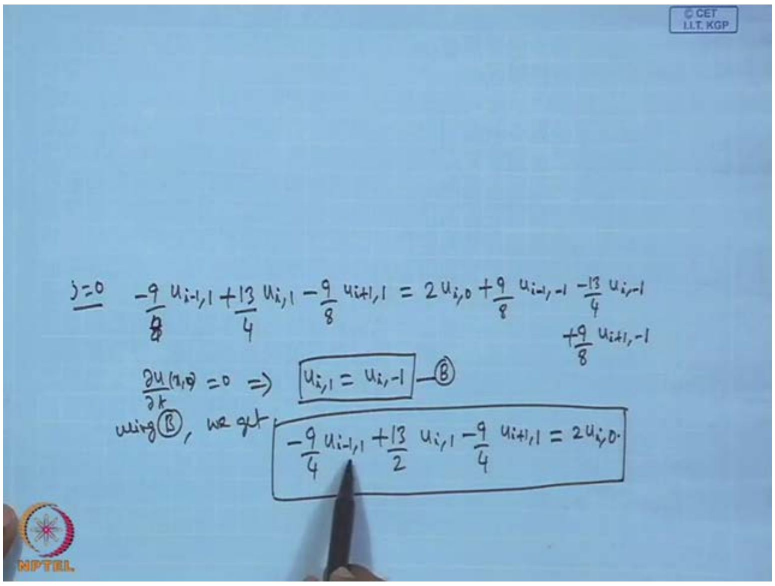

So, this when j equals to 0, we get 1, then 13 by, so this is so 13 by 4 u i 1 u i plus 1 1 equals to u i 0 plus 8 by 8 u i minus 1 minus 1. So, these are the fictitious values. However, we have our initial condition. So, let us discretize that. So, this would give us u i 1 equals to u i minus 1. So, if we use this that right hand side, we get you see u i minus 1 1 and we have already u i minus 1 1 with a minus 8 by 8. Then this goes. Using B, we get these quantities will be doubled. Similarly, u i minus 1 is u i 1, so plus 13 by 4, two times 13 by 4. So, all the terms on the left hand side get doubled. So, we get minus 8 by 4 plus 13 by 2 minus 8 by 4 equals to just, so this is for j equals to 0. Now, we get at each grid point at time level one. So, we use this.

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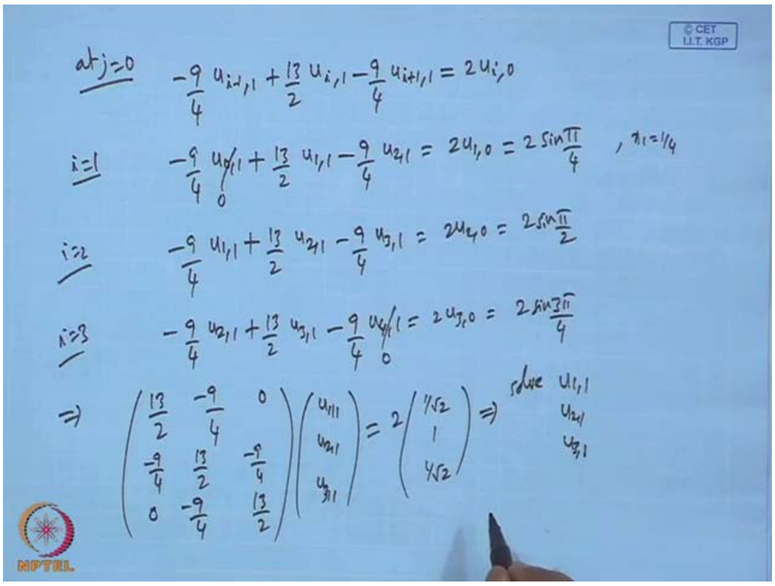

Therefore, at j equals to 0, we have, now we have to run at each grid point. So, this is nothing but 2 sin pi by 4. Our initial condition pi x, so 1 x, so this is due to x 1 is 1 by 4. So, then i is 2, then i is 3, so this can be put it in matrix form. So, you can simplify. So, this will be 1 by root 2, 1. So, why, how did we get here u 0, 1? So, we have dicretized the equations. So, this becomes 0, then 4, 1, so this becomes 0. Therefore, in the first equation, u 1, 1, u 2, 1 and in the second equation, we have 3 and in the third equation, we have only u 2, 1 and u 3, 1.

So, this implies we can solve u 1, 1. So, this is explicit method, implicit, we get system of equations from the explicit method by considering the weighted approximation of all the values at jth time level. We spread our j minus 1 and j plus 1 and then obtain implicit method. This is the solution for a specific case. Now, let us see stability axis of both explicit and implicit methods.

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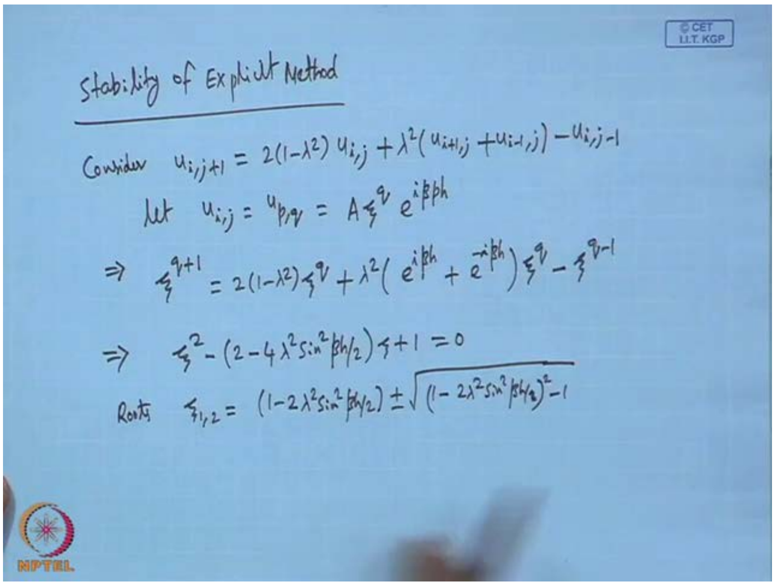

This is stability of explicit method. So, we consider, so this is our explicit method, then let u i, j, so for convenience, we are changing. So, this is Fourier stability analysis. Then when we substitute this and above equation, we get. So, this is because of j plus 1, we get zeta of q plus 1 so and so forth. Then we are left with one term here with positive and one term with negative. Then we have because of this, so this further can be simplified. So, this is quadratic. So, these are the roots. Now, for stability, we have to consider the stability conditions. Roots must be within the unit circle so less than equal to 1, so including the boundary.

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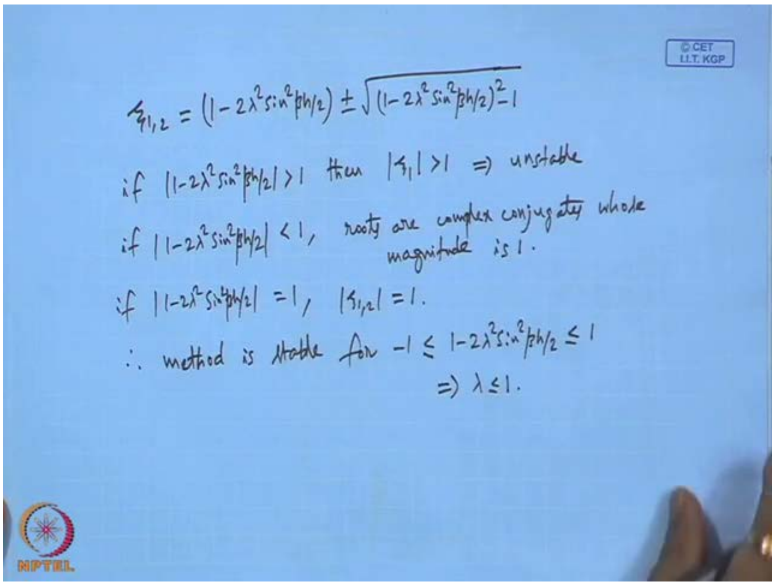

So, zeta 1, 2 are given by this. Now, if mod 1 minus 2 lambda square, if this is greater than 1, then is greater than 1. So, this implies unstable, then if this is less than 1, so let us write if less than 1, then the roots are complex conjugates, where whose magnitude is 1. Then if it is equals to 1, then it is 1. Therefore, this is correct, method is stable for, so this implies, so this is a stability criterion for explicit method. Now, let us see further implicit method.

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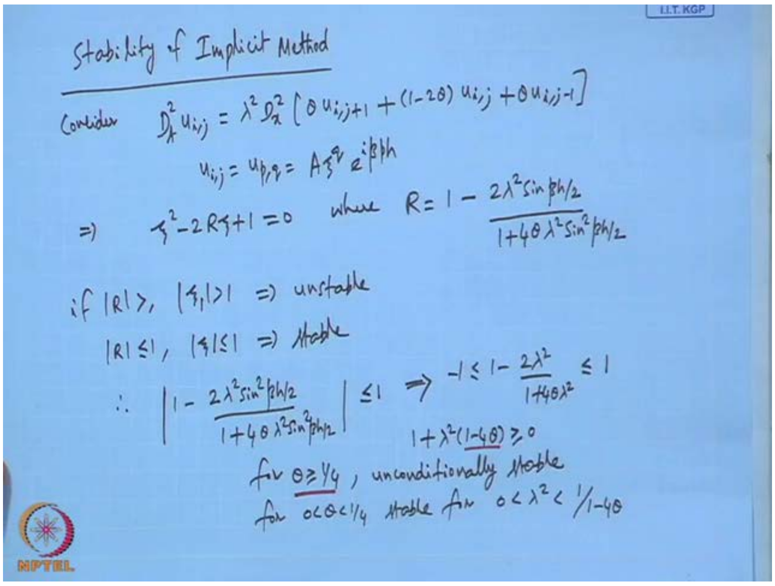

Of course, these are Fourier series stability analysis. Consider, now if we do similar, then substitute, we get equal to 0 where R is this. So, this is just algebra. So, I am not giving the details. You can work it out. Now, if mod R greater than 1, then we have theta 1 greater than 1. So, then the method is unstable. Then if mod R less than 1, less than or equals to 1, then the method is stable. So, you can compute the roots. So, from there, I am concluding, therefore mod 1 minus 2 lambda square.

So, this is less than or equals to 1. So, this implies minus 1 less than 1 minus 2 lambda square 4 theta lambda square less than 1 because these are sin square beta h by 2 sin square beta h by 2, therefore 1 plus 1 minus 4 theta. So, this means from here, see this must greater than equal to 0. So, if we pick up theta greater than or equals to 1 over 4 so that this stays positive, then we get remark. For theta greater than 1 over 4, unconditionally stable because of this. Then if theta is then stable for, you can obtain. So, this is the stability aspect of implicit method. So, stability method that we have discussed is based on one among stability that is the Fourier series stability. So, let us see the matrix stability as well.

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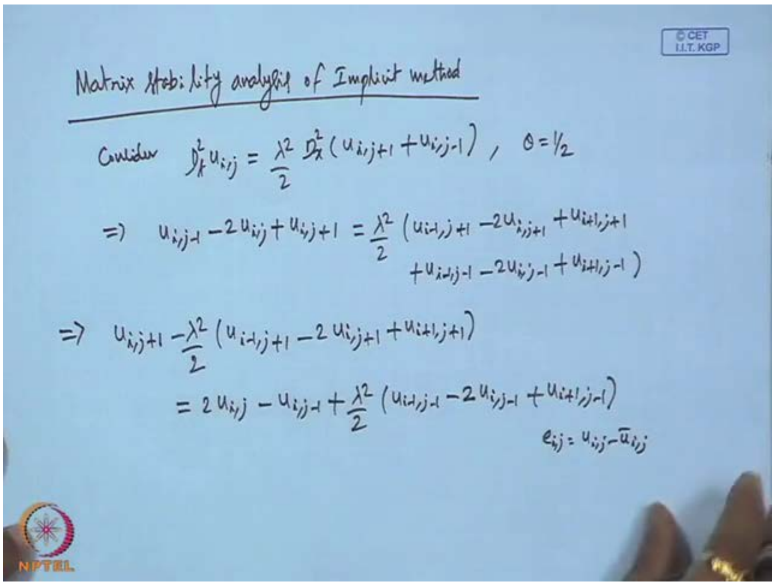

So, this is of the implicit method. So, consider this is the case of theta equals to half. So, this will be, so I am expanding this operator, then this one. So, this on slight rearranging u i, j plus 1 minus lambda square by 2 u i minus 1 j plus 1 u i, j plus 1. Solve the quantities j plus 1 up to the left. Now, if we introduce the error, it means if we denote e i, j by u i, j minus u bar i, j. So, the exact equation is satisfied by the difference equation minus the u i, j numerical. So, that is the local truncation error. Now, with this notation, if we calculate, so then let us look at the coefficients u i, j plus 1. So, we have 1 plus lambda square. So, let us do little minor.

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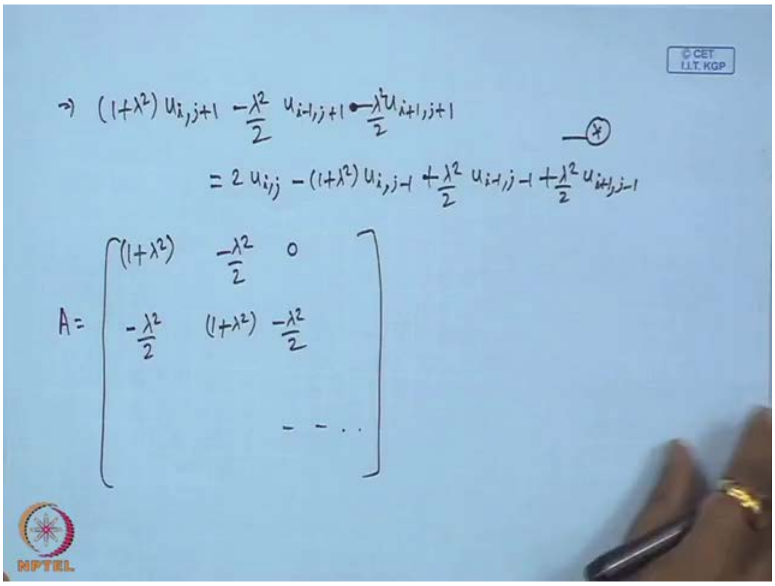

So, this will be 1 plus lambda square u i, j plus 1 minus lambda square by 1 u i minus 1 j plus 1 minus u i plus 1 j plus 1 and right hand side 2 u i, j we have this term, so minus 1 and minus lambda square. So, this will be minus u i, j minus 1 plus lambda square by 2. So, if we introduce the error, then if look at it, i minus 1 j plus 1 here, so then i, so the coefficients are coefficients are, so in this form. This is the standard form we have. So, this is our matrix A. So, why this is because if j is 0 j is 0, we have boundary points.

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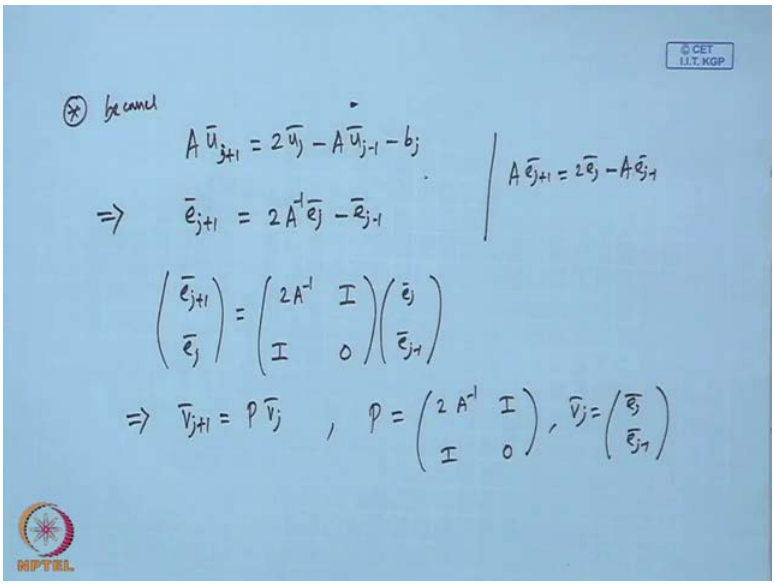

So, with this notation, the equation star becomes A u j plus 1 2 u j bar. So, what is the notation? We have this equation. So, this produces matrix A on j plus 1. So, we are using the notation u i, j is u j. So, with notation, this contributes to matrix A and the left hand side they are at j plus 1. Now, here similar matrix, however on j minus 1, therefore reduces to A u j plus 1, then 2 u j, then the same coefficients with the negative sign, therefore minus a u j and the boundary terms. So, this is what this is. Now, this suggests that e j plus 1 is 2. How did you get that?

So, you can write at other time level and then take the difference. So, then we get A e j plus 1, so here we get e j plus 1 is 2 e j minus A e j minus 1 and the error b j boundary points, they get cancelled. So, from here, we get this. So, this can be written as, so we adjusted and supplied one more equation so that we can put it in a matrix form. So, this implies this. So, now in order to discuss the stability, we have to do discuss the Eigen values of P.

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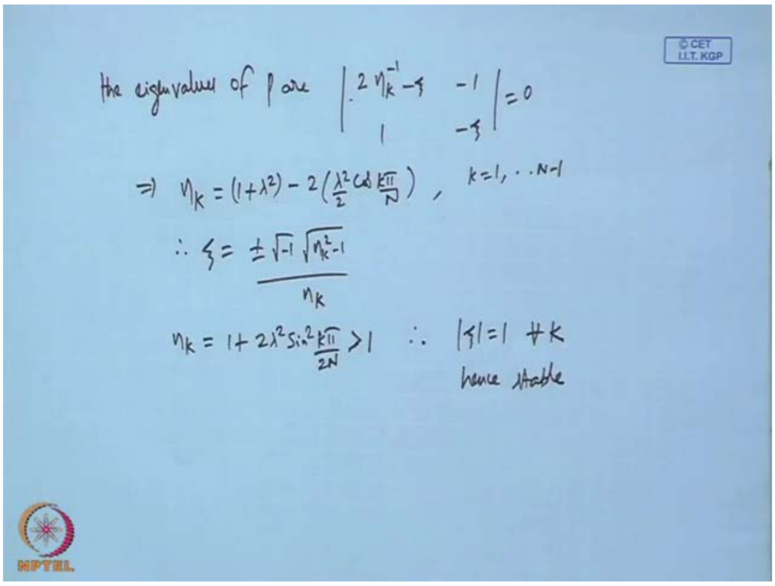

So, Eigen values of P are, so this implies, so these are the Eigen values. Therefore, zeta is, so this is just an algebra, so which is greater than 1. Therefore, mod zeta is 1 for all k, hence stable. So, this is stability of implicit method. So, we have seen, both general cases we have seen using one among stability of the implicit method. When that equals to half, then we have seen the corresponding matrix stability. So, let us see couple of problems using these methods.

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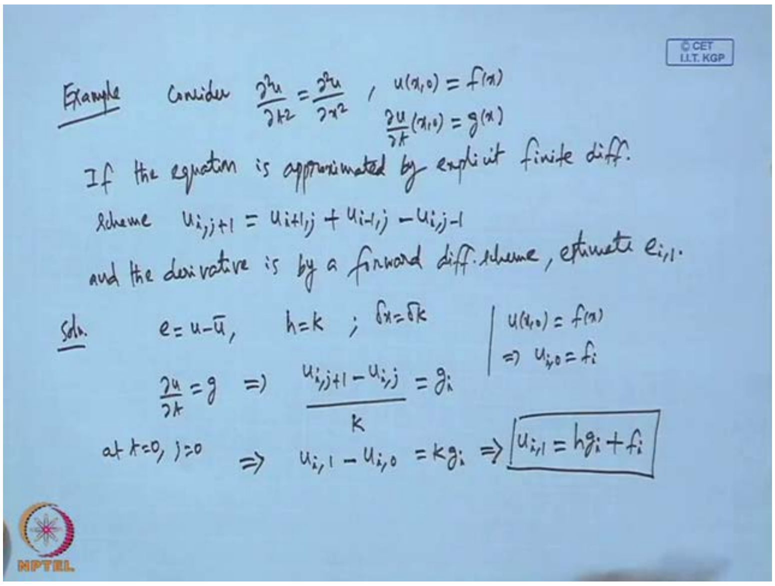

So, consider, so the problem is if the equation is approximated by explicit finite difference scheme given by and the derivative is by forward difference scheme, estimate e i, 1. So, that is the error. So, I will briefly discuss how to approach this problem. So, we have e is this notation that we are following. Consider h equals to k that is delta x delta k. So, then we want this.

So, this is the explicit scheme and derivative to be approximated by forward. So, this implies by k equals to g i. However, this is true at t equals to 0 j equals to 0. So, this implies u i, 1 minus u i, 0 equals to k g i. So, this implies u i, 1 equals to this. So, k is equal to h. So, I am giving h notation plus f i because u i, 0 I have used.

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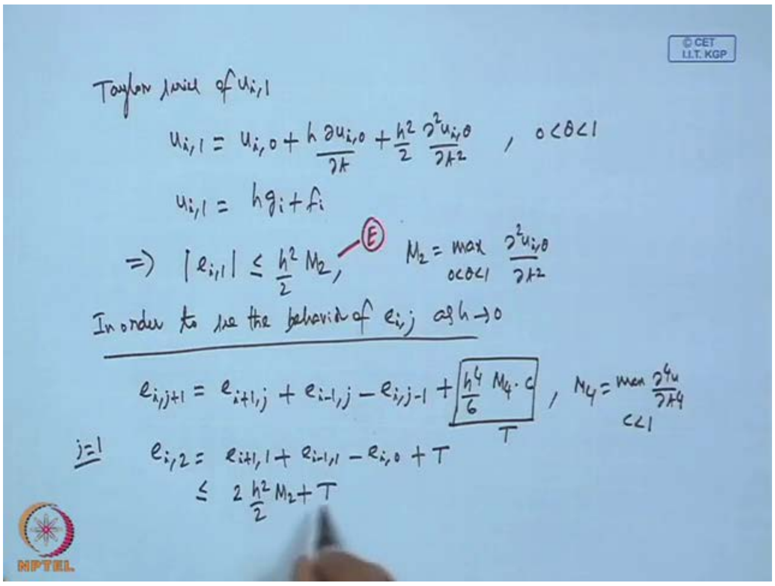

Then, expand Taylor series of this u i, 1. So, from that u i, 1 equals u i, 0, so this 1 means j equals to 1. So, that is t j so 1 means first increment. Therefore, that is expanded with respect to t 0, where this is error term. So, from this and we have h g i plus f i, so this implies is less than or equals to h square by 2 M2 where M2 is maxima of this. So, this is the estimate. Further, we can compute how the e i, j. So, this is the estimate. This solves the given question. However, in order to see the behaviour of e i, j as h go to 0, what we do?

We compute e i, j plus 1 is e i plus 1 j plus e i minus 1 j minus e i, j minus 1 plus h 4 by 6 M4 into constant because what was the method approximation. You can see this. So, if we expand Taylor series and then try to obtain the equation satisfied by the error, we have e i, j equals e i plus 1 j plus e i minus 1 j minus e i, j minus 1. That is what I have written, e i plus 1 j e i minus 1 j e i minus j minus 1 and this is the error term. So, MR is max of t 4 and c is a constant less than 1. So, when j is 1, e i, 2 is plus, call this call this some T. So, however using the estimate, this can be mad. See this is one term from here and similar estimate from here. This is 0 because t equals to 0, j equals to 0. So, we get two times h square by 2. So, this is rough idea.

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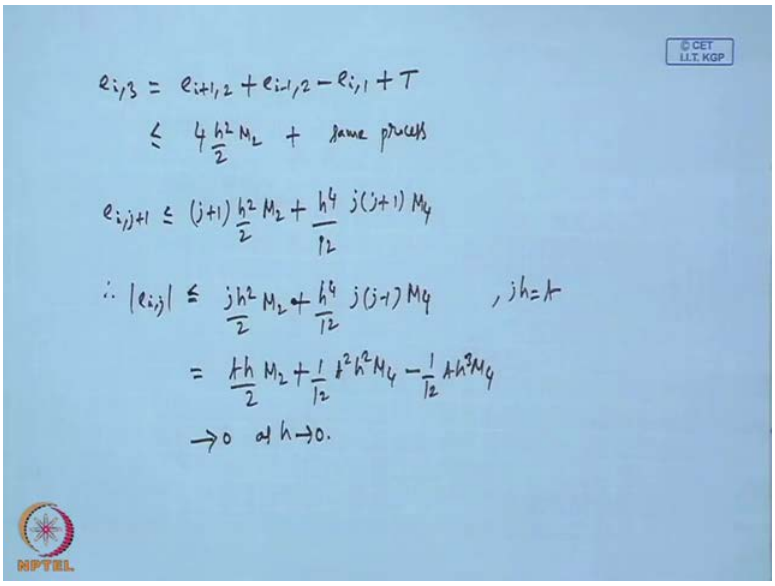

Similarly, if we proceed for each, you can make it bounded by from here to here, so we may get 4. Then similarly, for the remaining terms, so one can generalize j plus 1 h square by 2 M2 and for t repeatedly for each one, two, it will be magnified of this order. So, same process if you follow, we can get this. Therefore, mod e i, j is equals to j h square by 2 M2 because when we are replacing j by j minus 1 plus, so this can be because we have j h equals to t. So, we use that. So, this goes to 0 as h goes to 0.

So, this is a simple example where the error has been iteratively used and then the estimate at first time level that we have computed has been used iteratively. Then we have shown that the error is bonded. So, this gives idea of both explicit and implicit method for hyperbolic second order. Now, there is a special feature of hyperbolic p d a's. This is using finite differences. However, there is other concept called method of characteristics. Using numerical methods, one can compute solution of hyperbolic p d a's. So, in the next lecture, we discuss on the method of characteristic. Until then bye.