Numerical Solutions of Ordinary and Partial Differential Equations
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Lecture - 34
Finite Difference Approximations to Elliptic PDEs - IV

Hello, good morning. In the last class we have been discussing about non uniform grid. For example, we have to solve Laplace equation or Poisson equation inside a circle with a said d 3 or Neumann boundary conditions. So, then when we discretized, so definitely when we construct the boundary points what would happen when we fit a standard around 5 point formula. So, the boundary points will be slightly skewed towards inside. So, you would expect non uniform mesh size. So, let us solve as problem and try to see how we can handle this.

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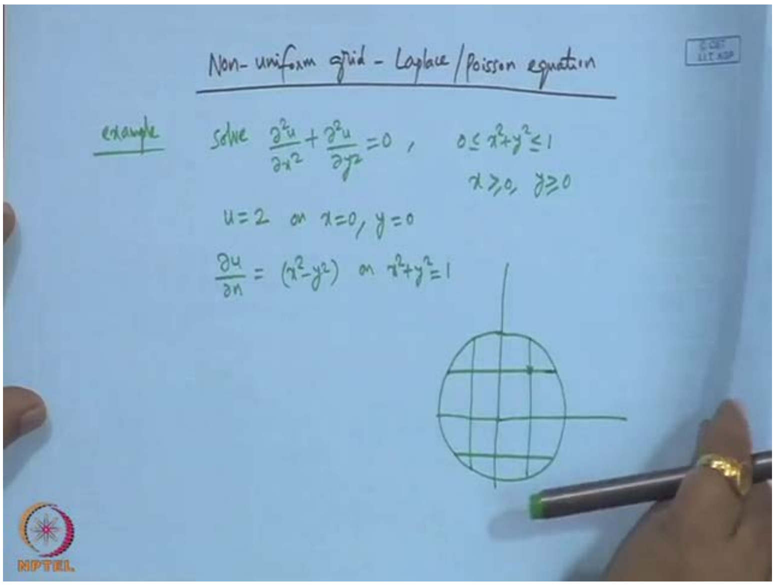

So, it is 2 d case, from the first quadrant u is 2 on x equal to 0. So, this we started in fact in the last lecture. So, when we discretized I already mentioned. So, for example, this is uniform then suppose we discretized, so then if you consider this as u i j. So, then these two are having equal length, these two are having equal length. So, the step sizes are non-uniform. So, we cannot use our regular technique, we have to come up with slightly modified technique. So, we will address this.

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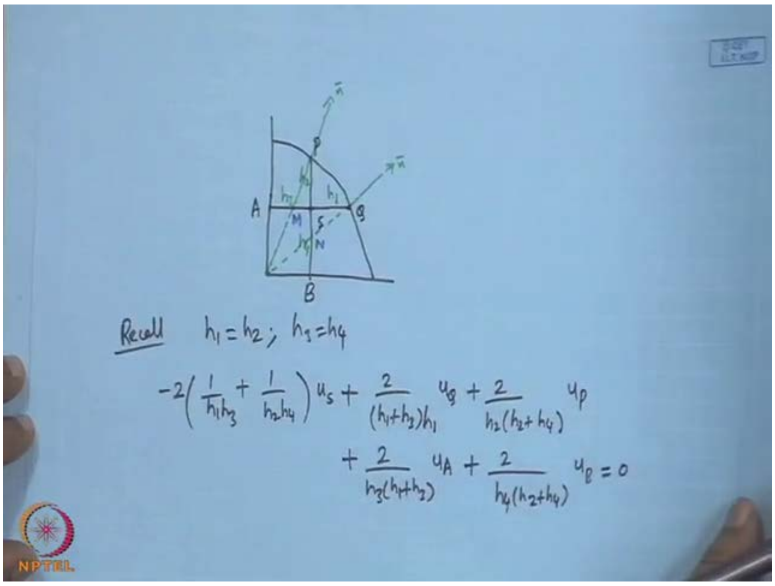

So, now I draw this and this is a point say S Q P A and B and we this is h 1 and this is h 2 and h 3, h 4 now we take the normal through Q and it goes like that and if we take the normal to P goes like that. Now these points, this point we call n and this point we call m now we have to say we can compute these points, but when we run the standard point here it expects this and this. So, how do we handle so recall the formula. So, in this case h 1 is h 2 and h 3 is h 4 and formula which we got u at. So, this u x i y j plus u Q plus u P plus u A plus ((Refer Time: 06:03)) with this irregular grid, from the general equation which we have discussed in the last class. So, we can arrive at this so u Q u P u A and u B are involved.

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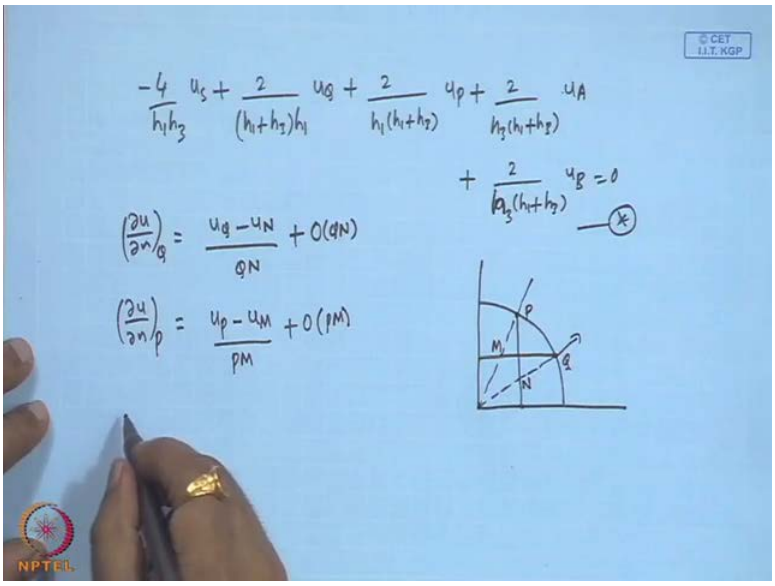

So, this now for the case of h 1 equal to h 2 h 3 equal to h 4. So, let us simplify this, this is our equation however the original problem which we defined, we have the normal prescribed. So, we have to compute, this is our Q, this is P. So, we have to compute dou u by dou n in terms of these values. So, we can do that, dou u by dou n at Q can be written as u Q minus u N by q N of course, this is up to first order. So, what is happening her, this is Q and this is N. So, dou u by dou n at Q is u Q minus u N by the distance, then P, u P minus u M by P M. So, this is P and this point is M. So, the normal is given similarly, one can compute the value u n as follows.

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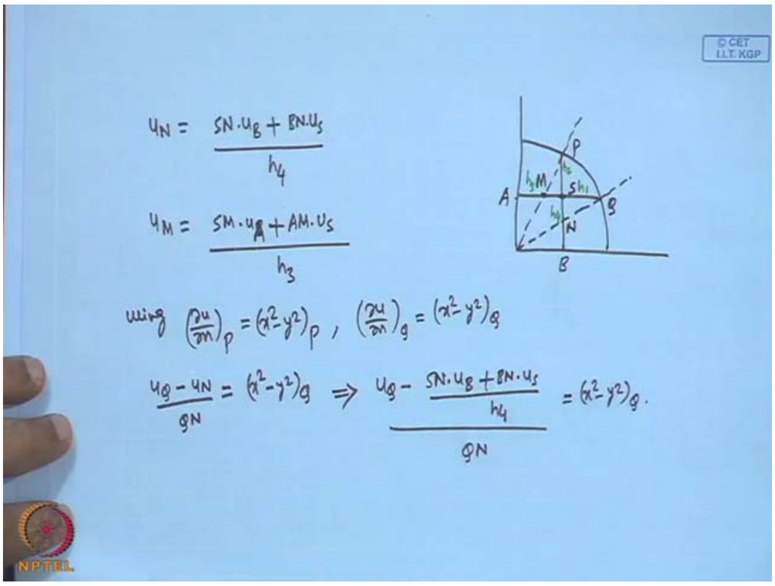

We need to draw this again and again. So, then u n is given by. So, this point is S. So, u n is S N u B plus B N u S by h 4. So, this interpolation, so let me this arm is h 1, this arm is h 2, this arm is h 3 and this arm is h 4. So, these arms are h 1 is equal to h 2, h 3 equal to h. Now, this is by usual interpolation then similarly, u M here will be S M into u B. So, this is A and B plus S M into u A plus A M into u S by h 3 because S M into u S plus A M into u S by h 3 now these values. So, we have computed in terms of the existing grid points the derivatives. So, that since the derivative boundary conditions are prescribed. So, we can explicitly get algebraic equation.

Now, using, Neumann data, when we use Neumann data we have. So, this is both P and Q and we have computed earlier the normal at P and Q moreover we have computed u n u m for our interpolation now we combine this we combine this for combining u Q minus u N by Q N this is equals to x square minus y square at Q. So, this implies u Q minus u N you have computed at Q.

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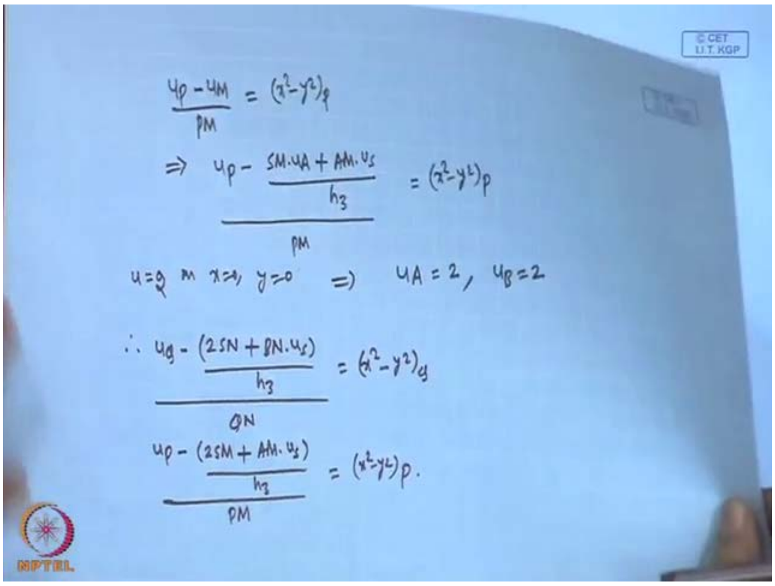

Similarly, we get u P minus u M by P M. So, this implies and we have u equals to 0 u equals to 2 on x equals to 0 y equals to 0. So, this implies u A is 2 and u B is 2 because both are on the axis therefore, u Q minus. So, from the earlier one u B is 2. So, we use the value, so u Q and the second one u P minus 2 S M. So, we have got explicit algebraic equations now we have to compute these distances, which distances Q N and P M.

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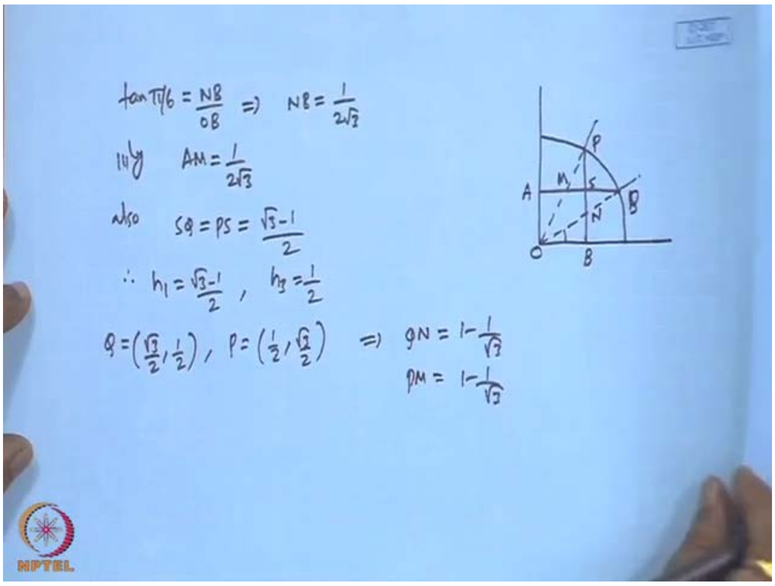

So, let us proceed as follows for this slightly algebra routine calculation. So, from figure this is Q P N m now this is A and B therefore, tan pi by 6 is N B by O B. So, this implies N B is similarly, A M so you can compute then also S Q equals to P S equals to. So, this you can compute therefore, this is our S therefore, h 1 and h 3. So, these are straightforward S Q P S these are equal. So, this is h 1 then similarly, S B and A S h 3 and h 4 they are half. So, we have points Q, Q is and P is P and Q.

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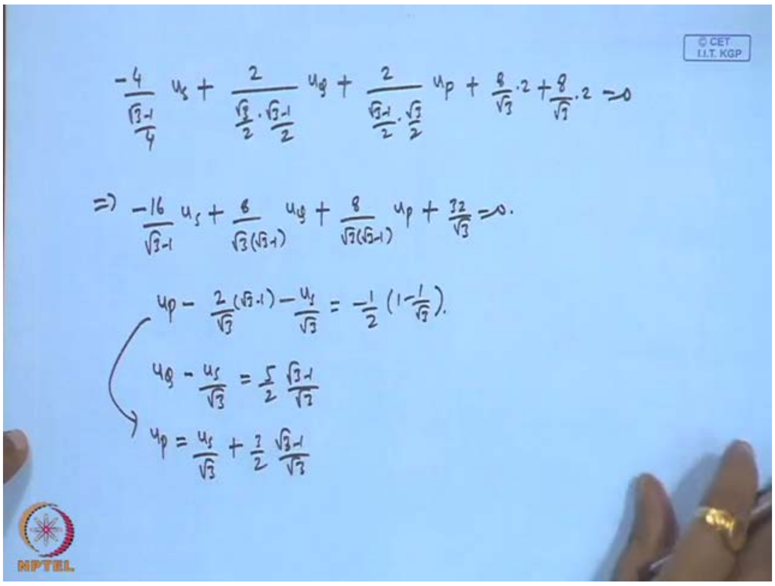

So, accordingly Q N is and P M is, now the formula the… Algebraic equation simplifies, I am giving crude calculations you can simplify. Then further this can be simplified to the other two equations u P 2. So, this is one equation and similarly, u Q. So, you can simplify and moreover this can be simplified.

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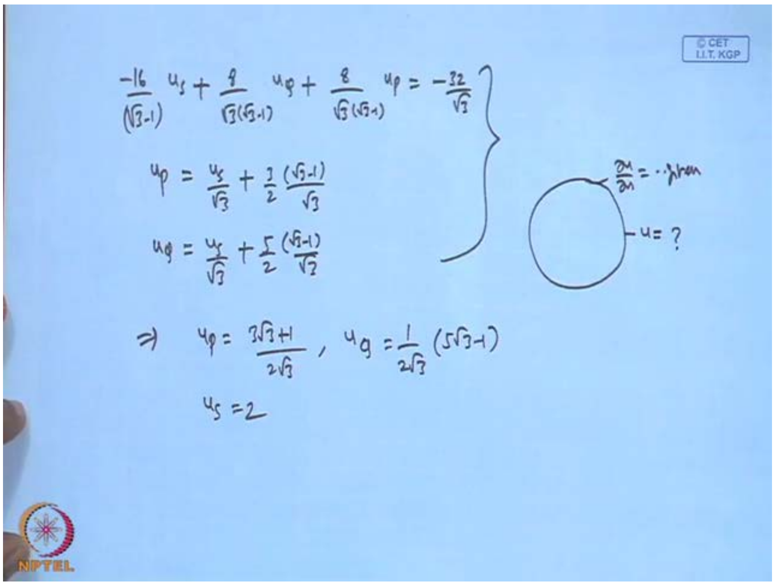

So, now we have u P u Q u S, three equations three unknowns. So, let me put them from, these one can obtain for example, solved. So, you may also try, when we have Neumann’s data equals to given what is the Cauchy data, Cauchy data is you have to be obtained and this is what we have obtained the value of u. So, this is the case where you have the grid non uniform grid and you have to interpolate and also approximate the derivative. Then obtain the corresponding boundary data because we have Neumann data given therefore, the corresponding digitized data we obtain. Now let us see cylindrical coordinates how the corresponding digitization works out. Now, let us see cylindrical coordinates how the corresponding digitization works out.

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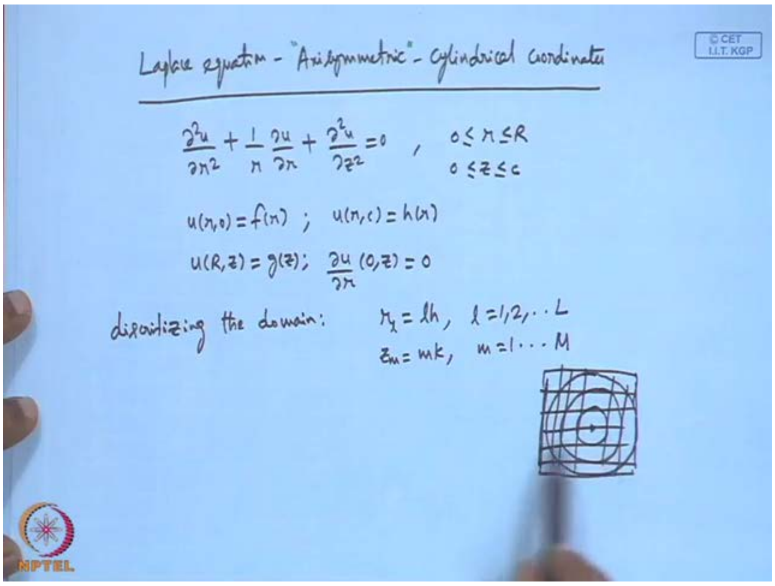

So, Laplace equation axi-symmetric cylindrical coordinates, accordingly the operator becomes I mean the equation. So, remember I have considered axi-symmetric. So, the dimension has been reduced by 1 otherwise we have the standard r 5 z cylindrical, but because it is axi-symmetric these are independent of phi. Now, say in a domain we need the corresponding boundary conditions say this is f of h of r then u r z of z, say here the derivative now we have to discretized, discretizing the domain, r l is l h then z m is m k. So, then when we have this type of discretization it is like. So, this type of discretization we, r 0 to r has been discretized as l h. So, l equals to 1, l equals to 2 in this direction. So that means l values we should literally take from minus 1 to 1 when we say l 2 minus 2 to 2.

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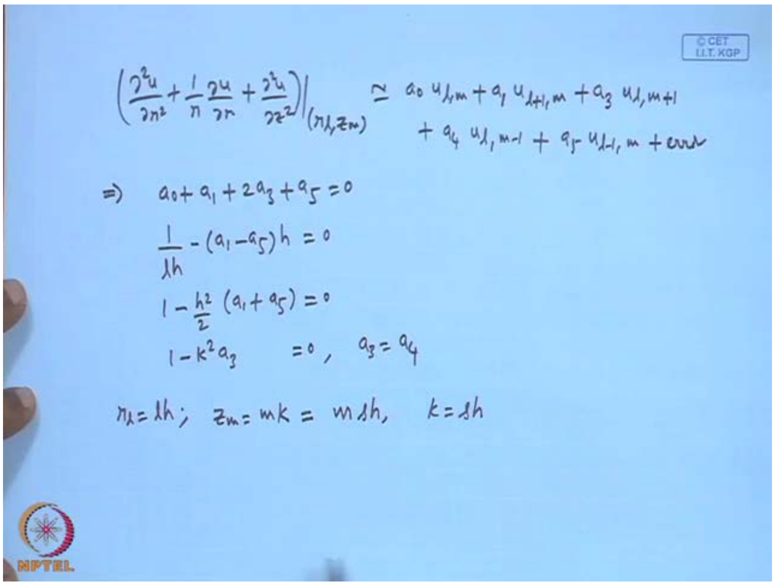

So, now what kind of approximation we use. So, this we discretized plus of course, error. So, we approximate, then expand the ternary expansion and then collect the coefficients and compare. So, by doing, we get the following system. So, please do yourself and make sure this system is obtained. So, now when you remember r l is l h and z m is m k. So, then let us have this m s h that means k is taken as s h special case.

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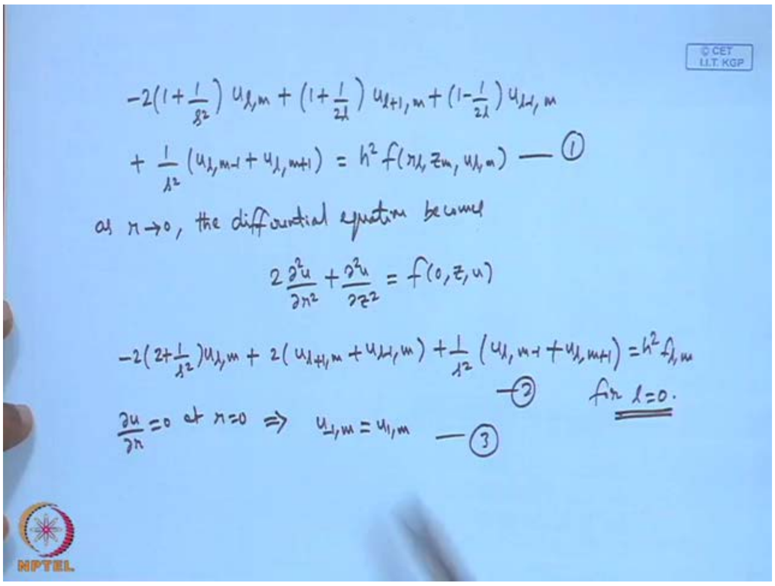

So, then the discretized equation reduces to assuming f is function of u as well now this is a equation discretized version. However as r goes to 0, so the differential equation becomes as r goes to 0, reduced to this. Now, corresponding discretized version corresponding to this the discretized equation. However, this is as r goes to 0. So, however this is for l equals to 0. So, this is important now we have dou u by dou r equals to 0 at r equals to 0. So, this implies the fictitious value can be eliminated because then l equals to 0, we have fictitious value from here. So, this can be eliminated, 1 2 3 with these one can obtain the solution.

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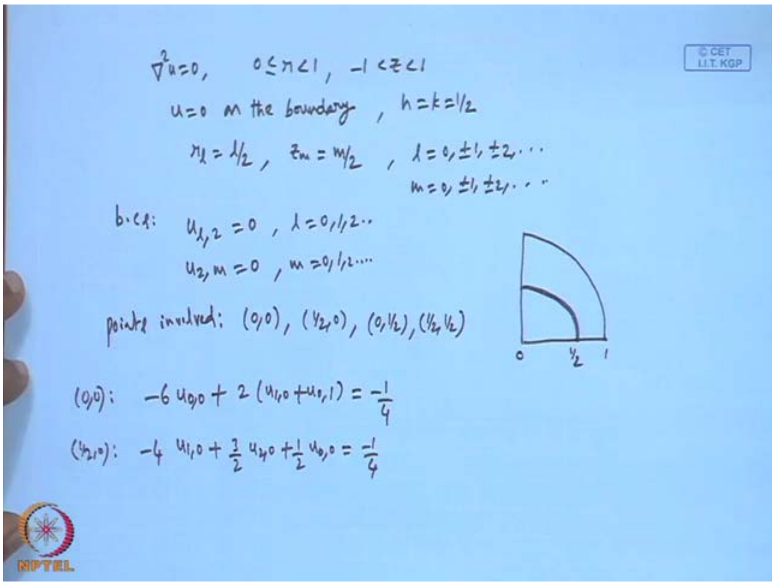

So, let us see at least for a simple case the discretization and the system of equations, say we want to solve this u is 0 M the boundary and say h equals to k equals to half. So, this gives r l equals to l by 2, z m equals to m by 2 and l therefore, the boundary conditions u l 2 equals to 0, u 2 m equals to 0. This is for l 0 1 2 because you can see h equals to k equals to half. So, from 0 to r 0 to 1 therefore, we have only r 0 to 1. So, that means we are, half this is 0 and this is 1 and we should have drawn better. So, this is half and z planes. So, z planes also k is half.

So, we have to correspondingly we have to discretized therefore, u l 2 is 0 u 2 m is 0. So, these are the boundary conditions. So, then if we recall, we have these equations. So, we have to run these equations at points. So, since z is minus 1 to 1. So, the points involved are 0 0, half 0, 0 half, half half. So, at these points we have to run the equation, this equation we have to run.

So, for example, at 0 0 then half 0, so again for the remaining points these points we run and then we get a algebraic system. So, this gives some idea of discretizing the cylindrical polar coordinates of course, not arbitrary case we have considered axisymmetric case. So, let us try on stability aspects of these discretizations.

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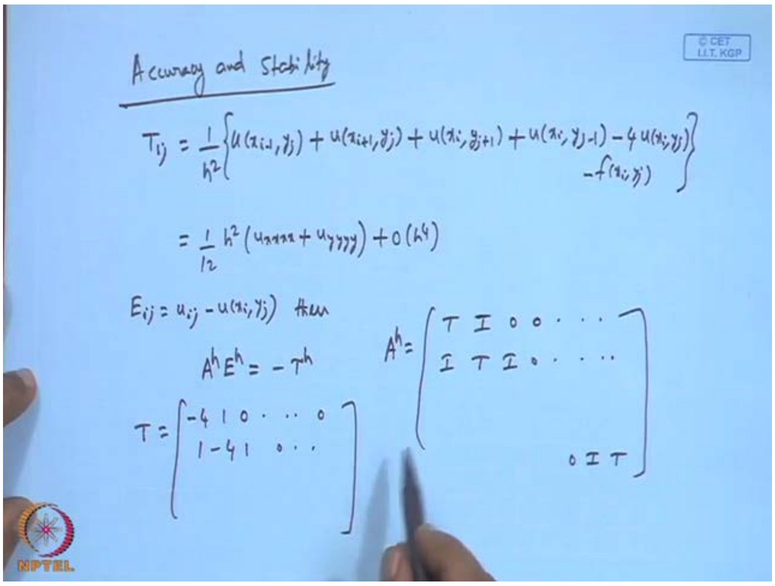

So, let us check the stability aspects accuracy and stability. So, these standard 5 point methods are discretized and the error is of the form u of x i minus 1, y j plus u of x i plus 1 y j, plus u of x i y j plus 1 plus u of x i y j minus 1 minus 4. If it is laplacian and then if you have a non-homogeneous. So, this is the error, so when we expand and see this reduces to and correspondingly, if I introduce then error satisfies the following equation error satisfies A h for a particular step size. So, that A has specific structure, so A h has the specific structure, where T also has a specific structure. So, this we have written before. So, I am not putting completely.

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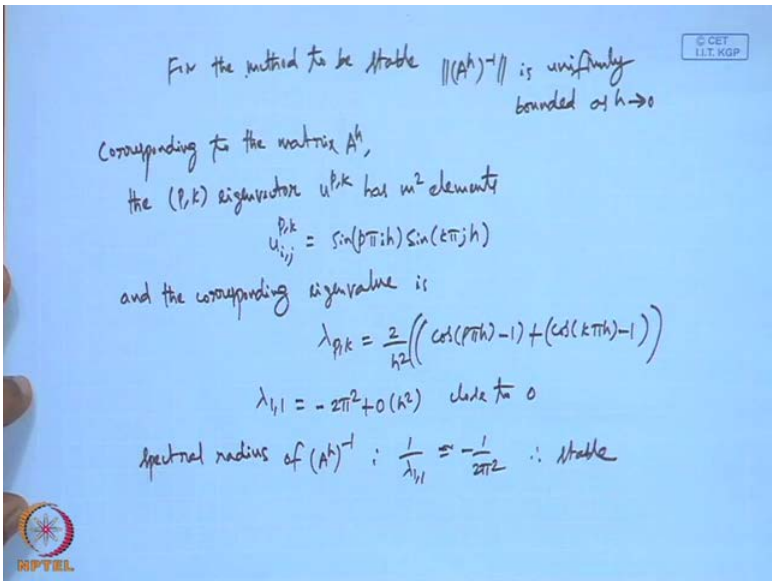

So, now for stability of the method, for stability of the method we need the following for the method to be stable with respect to any norm suitable norm. So, this is uniformly bounded as h goes to 0, now let us see corresponding to the matrix A, just now we have defined the p k Eigen vector u p k, this has m square elements given by. So, correspond to this matrix the p k Eigen vector this has m square elements.

The corresponding Eigen value is lambda p k lambda 11 for example is which is close to 0. If you go for the spectral radius of this we have. So, this condition is satisfied therefore, stable. The method is stable. So, this general check for stability of this method. So, in the last class we started a problem on discretization with respect to polar coordinates. So, let us because we are almost concluding the elliptic equations.

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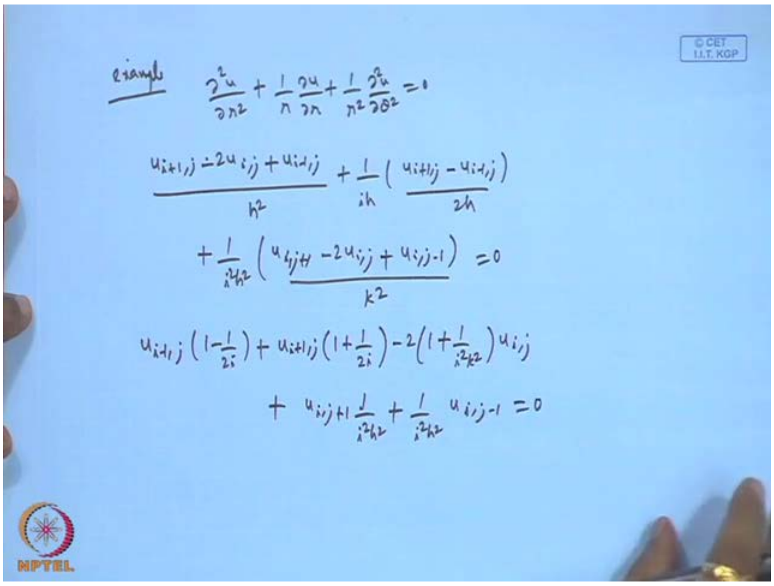

So, let us see this problem and try to finish this. So, in the circular case, we are solving this and the corresponding approximated. This term then this gets simplified. So, let us introduce some notation.

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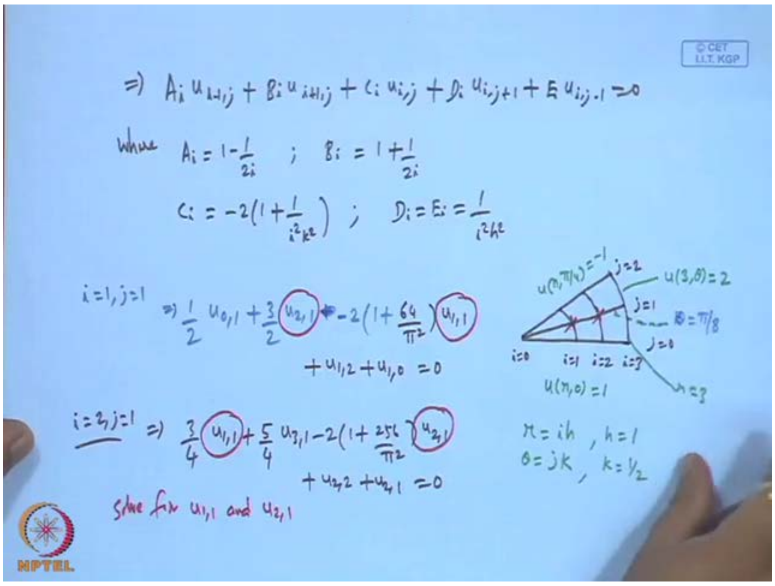

So, this is of the form where now let us say we consider a problem. So, we would like to solve. So, this is i equals to 0 i is 1 i is 2 i is 3 then j 0 j 1 j 2. So, let us say we have u of r 0 is 1 and say this pi by 4. So, u of r pi by 4 is minus 1 and say this boundary is u of 3 theta equals to 2. So, that means our domain is this, if this is the domain and this boundary data is given. So, we have unknowns only two of them. So, let us say we want to run i equals to 1 j equals to 1 this equation.

So, this if we run equation here i equals to 1 j equals to 1 it will ask you see i equals to 1 j equals to 1. So, it will ask 2 points here 2 points there right. So, we get of course, corresponding to. So, i equals to 1 so a i is half a i is half u 0 1 corresponding to this plus B i is half i is 1. So, this is 3 by 2 and u 2 1 plus c with a minus 2 1 plus. So, for these let us, if this is pi by 4 corresponding to j equals to 1 what is k. So, this gives theta equals to pi by 8. So, assuming k is half, so the discretization.

So, maybe we can have quick recall, r equals to i h then theta equals to j k and what is step size. So, here h is taken as 1 because I have given this is r equals to 3. So, this line r equals to 3 accordingly I have given, but whereas this line, theta equals pi by 8 that means k is half pi by 8. So, is now we get wait let me just recall this is. So, this is pi by 4 half of it, this will be 64 by pi square u 1 1 plus u 1 2 plus u 1 0 because d i e i. So, this coming to be i is 1 h is 1. So, we get then i is 2 j is 1. So, again we compute the coefficients, this will be 3 by 4 u 1 1 plus i is 2. So, 5 by 4 u 3 1 and this will be u 2 1.

We know these are the points unknowns u 1 1 and u 2 1. So, what are the unknowns u 1 1 u 2 1, u 1 1, u 2 1 and remaining we substitute from the boundary data and solve for. So, this gives some idea how to discretized and obtain the solution. So, you can take maybe two points if you take one more line.

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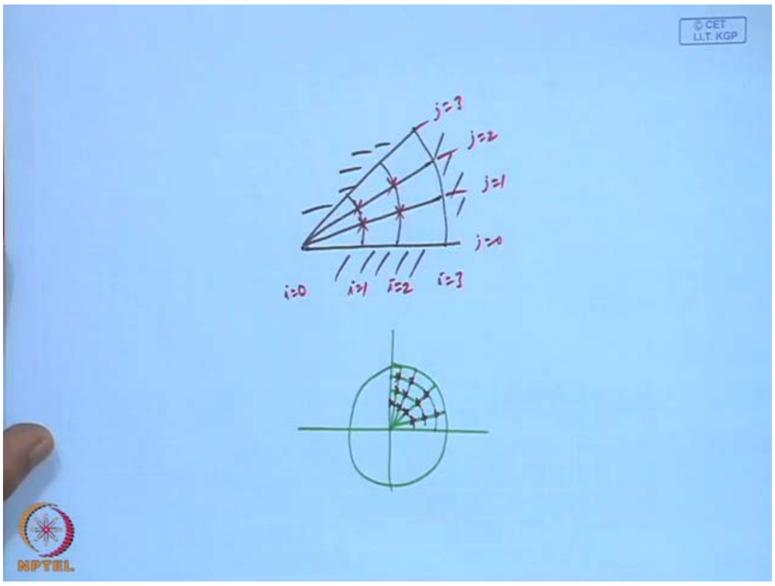

So, for example, the number of grid points increase suppose somebody takes the domain suppose you restrict your domain to this. So, your solving that means this is given u is given. So, what will be the grid points where you have to seek the solution. So, these are the four points where you have to seek the solution. So, i is 0 and this i 1, i 2 and i 3 and here j 0, j 1, j 2, j 3. So, these are the grid points, we can extend suppose if we solve it depends how many. So, accordingly the grid points will be increased see these are the grid points.

So, this is general idea how Laplace and Poisson equations are solved. So, in general for any elliptic equations, if you want to approximate using finite difference methods, we adopt this. If you consider the standard 5 point formula, you can refer some literature where if you look at you Laplace equation, it is invariate under rotation. So, if you rotate, then you get a slightly different version of A.

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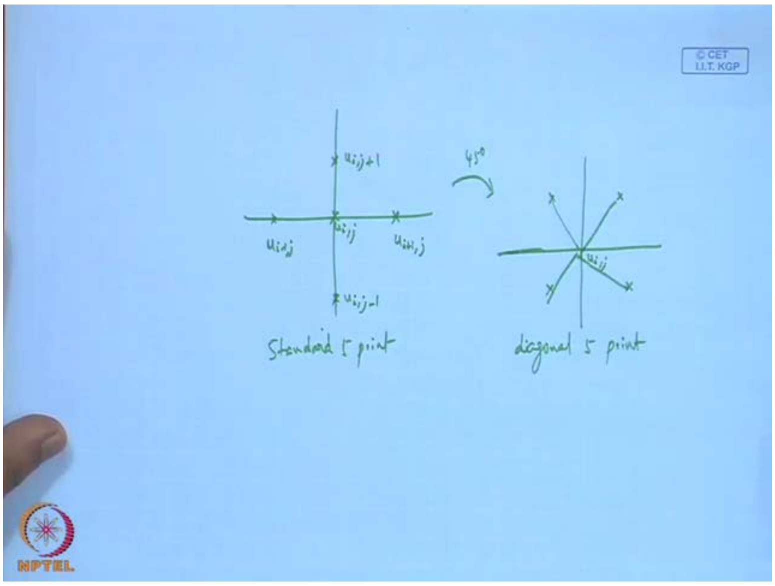

So, I would like to show you because many books refer that. So, this is i j and your diagonal is asking i plus 1 j, i minus 1 j, i j plus 1, i j minus 1. So, this is u i j, so if you rotate by 45 degree, then it should expect something here. So, this formula is diagonal 5 point formula. So, this is very much exists in literature. So, you can refer and try to solve the particular problem using this method, this method and compare how this method works out. So, with this we are almost done with elliptic PEDs. So, from the next lecture we move on to hyperbolic PEDs until then. Bye.