Numerical Solutions of Ordinary and Partial Differential Equations
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Lecture - 31
Finite Difference Approximations to Elliptic PDEs-1

Hello welcome back, so far we have learnt last few lectures on numerical approximations to parabolic PDE. So, today let us start another class of partial differential equations that is elliptic. So, as the title says as final difference approximations to elliptic PDEs 1. So, that whatever following lectures I will put two and three depending on the topics, now when we say elliptic PDE more general partial differential equations are Laplace equations.

So, typical of this is this is completely depending on spatial co-ordinates not on time variable. So, in this respect this is very unique and the possible applications from which this originates. For example, potential flow and also magnetic conduction, electrical current, etcetera, so let us see bit of the Laplace equations origin and then proceed to numerical approximations.

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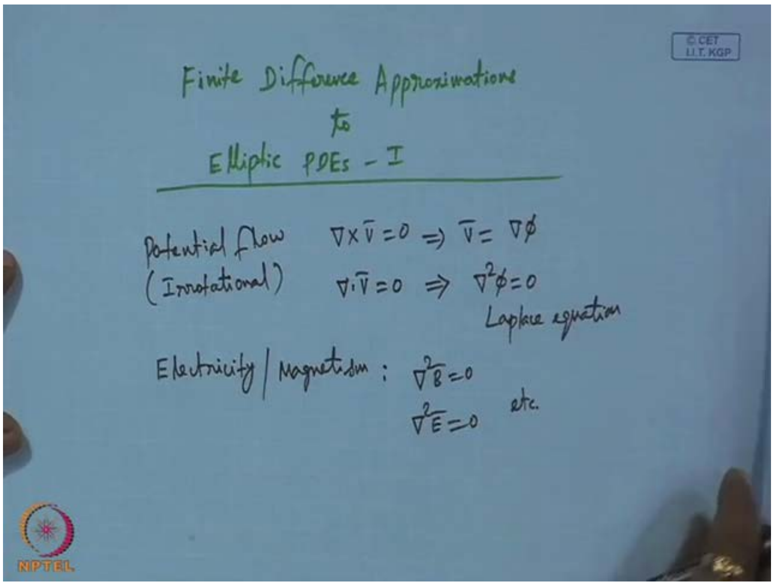

So, as I mentioned for example consider potential flow where the velocity satisfies ir-rotational. So, potential flow or ir-rotational of course there are some differences, but more or less from general point of view we can be written as gradient of a scalar, if curl is 0, then we have conservation of mass. So, if we combine this and this we end up with, so which is very popular Laplace equation. Similarly, we have in electricity or magnetism of, so we have del square b bar equal to 0. Where this is magnetic field and this is electric field. So, we can have several such examples, now what is the main feature of a Laplace equation.

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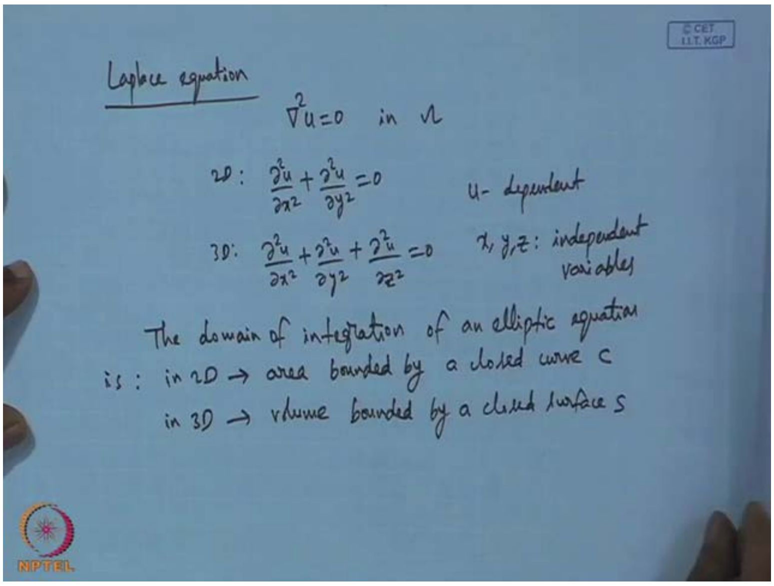

So, consider Laplace equation say del square u equals to 0 in some domain. Then if it is 2D and 3D Cartesian 2D Cartesian, so when we say this here u is dependant variable x y z are independent variables. And there is no time involved in this, so this is typical of general elliptic equations moreover each with respect to each space here we have second derivative. So, of course you can have some un homogeneous terms which we will see, but this is the simplest scenario and these suggest that the domain of integration the domain of integration of an elliptic equation is for example, in 2D this could be an area bounded by bounded by a closed curve C where as in 3D this is volume bounded by a closed surface S. So, this is a the typical of a elliptic equation now let us see what are the possible boundary value problems.

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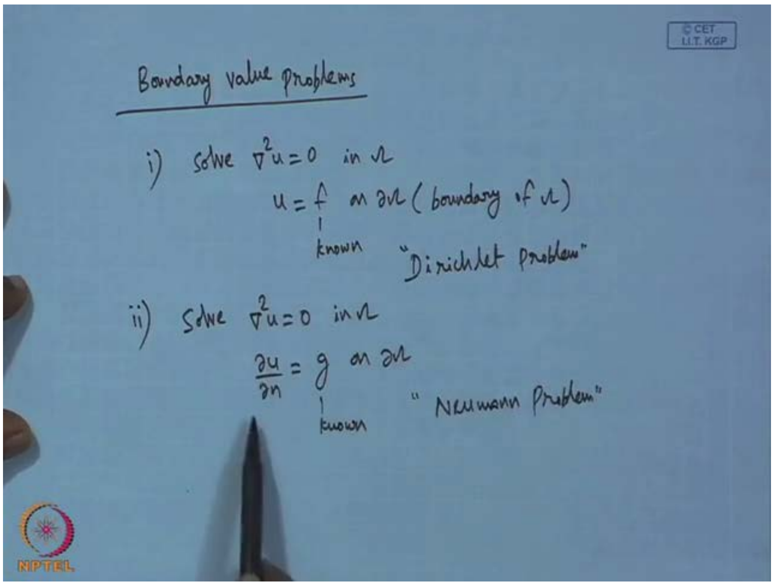

So, boundary value problems that are possible for Laplace equations is solve del square u equals to 0 in omega u equals to some f on the boundary. So, this is the boundary of omega, so I give one solve Laplacian in omega which closed and bounded then u equals to f, which is known. So, that means the function is prescribed on the boundary, so this is called yes you are right Dirichlet problem. Then solve where the the normal is given on the boundary and this is yes again you are correct. So, this is Neumann problem, so that means when you have a domain you have function prescribed on the boundary. So, this is Dirichlet problem normal prescribed on the boundary Neumann problem.

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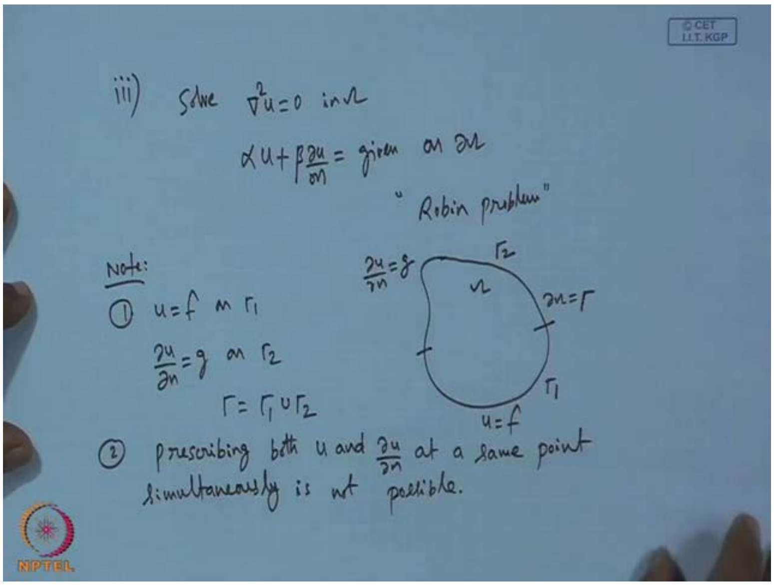

So, there is a third type as well some alpha times u plus beta times the normal is prescribed on the boundary. So, this is some of you may not know robin problem, so this is more general case however one note. For example, this is your omega then so this is a boundary.

So, if you want to prescribe part of it say u equals to f here where as, so call this gamma 1 call this gamma 2 and g there. So, this is quite possible so that means u equals to f on gamma 1 g on gamma 2 where gamma is. So, this is very much possible however note two is very important, this these two prescribing both u and the the normal at a same point simultaneously is not possible because we cannot satisfy simultaneously these two conditions.

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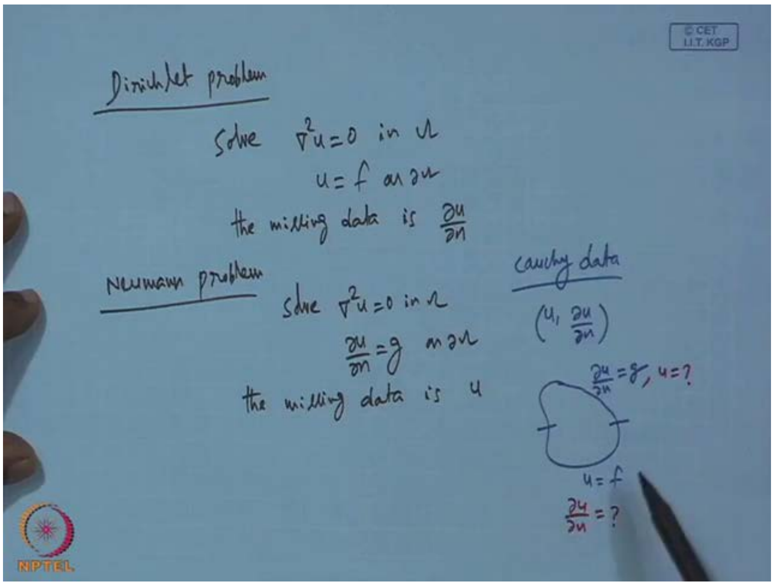

So, with respect to Dirichlet problem, so you have to solve del square u equals to 0 in omega then say u equals to f on this then the missing data is dou u dou n. Similarly, Neumann problem suppose and this is g on this then in this case the missing data is u. So, this has a name Cauchy data is u. So, this is Cauchy data, so that means if for example here u equals to f. Then here we need to determine, so suppose here g then we need to determine, so this the missing Cauchy data. Now, let us so this is general motivation of Laplace equation and the corresponding boundary value problems. So, let us start with the corresponding finite difference schemes, so since we have. So, let us consider two dimensional case, so we have second order derivatives in space with respect to both the variables. So, let us use central difference scheme usual what we have been using, so then let us see how the method proceeds.

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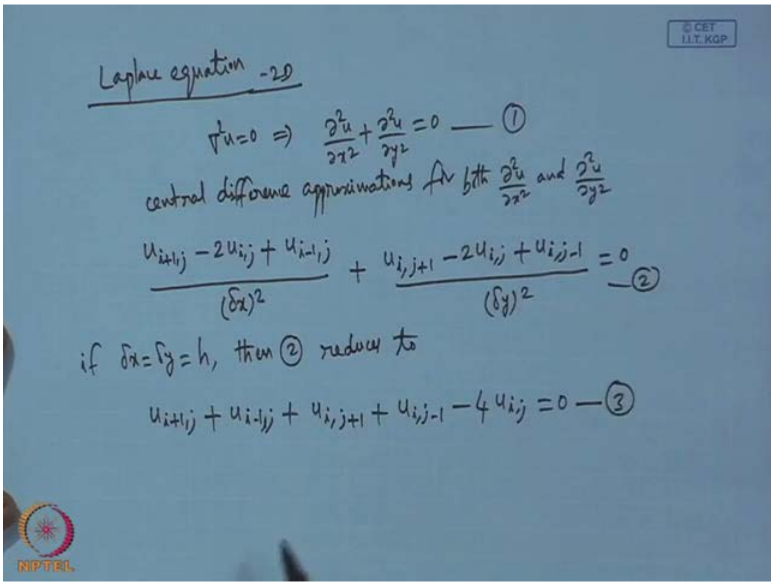

So, we consider Laplace equation 2D, so this implies now using central difference approximations for both of them we get. So, you can recall u by delta x square plus equals to 0. So, where delta x is corresponding x grid length and this is corresponding y grid length. So, typically if equals to h then 2 reduces to 2 reduces to you can see, if this is h square and this is h square. Now, we can combine so this h square is gone then we have reduces to this term this term this term. This term we have minus 2 and minus 2. So, this is a corresponding approximation and this approximation.

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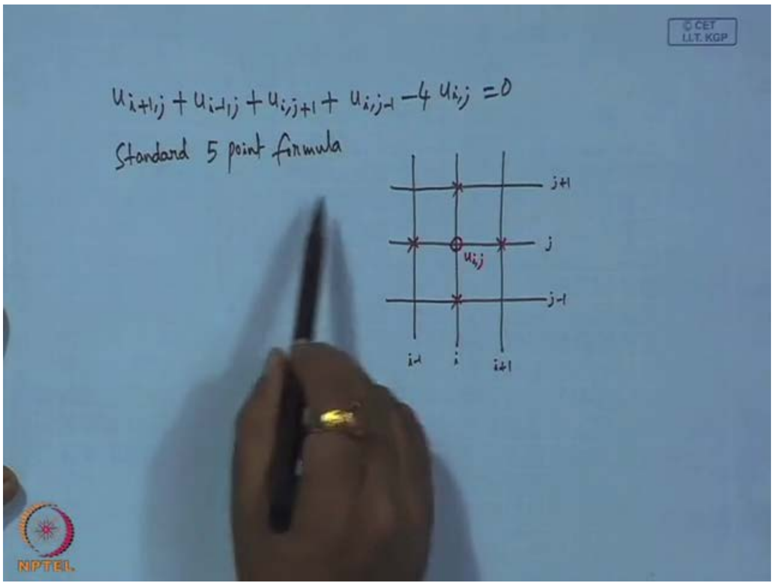

So, I have to rewrite here, so this is called standard 5 point formula. Why you can see so these are the 5 points i plus 1, j i minus 1,j i,j plus 1 i,j minus 1 and i,j. So, this it is like this suppose this is i-i minus 1 i plus 1 j j plus 1 j minus 1 then we have. So, this is u i j, so then we have u i minus 1 j u i plus 1 j u i j plus 1 u i j minus 1. So, this is a structure of standard 5 point formula, so you can see it to compute the value here. It demands 4 values at these corners like this. Now, suppose you have further the grid is proceeding then when we ask for the value here it ask this value this value this value and the one next to it. So, we have to compute like that. So, let us see with an example how this standard 5 point formula works out.

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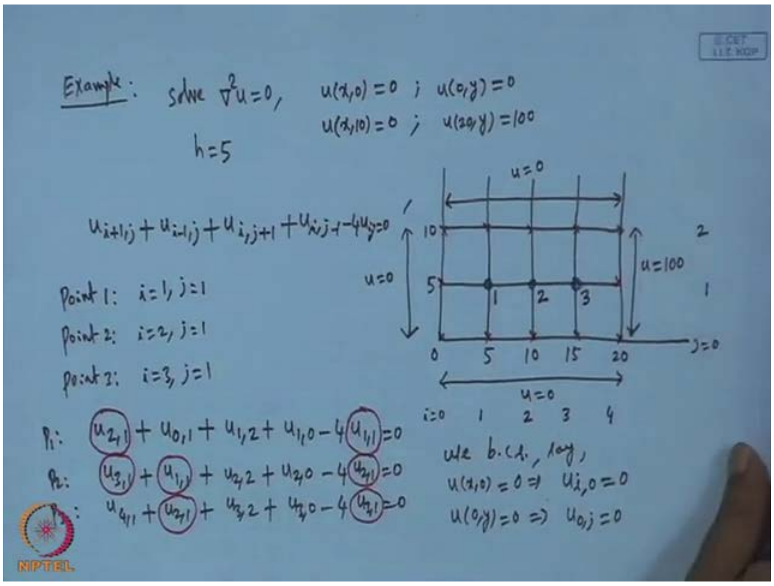

So, let us start an example solve del square equals to 0 subject to u of and say h is 5. So, let us see the grid first, so the boundary x extends from 0 to 20 and h is 5. So 5, 10, 15, 20 so we make it. So 0, 5, 10, 15, 20 and y is going from 0 to 10. So, this is our grid now u of x 0 is 0. So, on y equals to 0 it is 0. So, here then x equals to 10 is also 0, then u of 0 y. Whereas, here it is u equals to 100, now what are the unknown are so we have grid points. So, these are boundary points these are boundary points and these are the internal points. Now, we have to compute the value at these points so say we call them 1, 2, 3 right.

Now, consider our bisection this is minus 4 equal to 0. So, this is our bisection, now let us say this corresponds to point 1. So, point one corresponds to so i 0, 1, 2, 3, 4 and j 0, 1, 2, so this point 1corresponds to i 1 j 1 right. So, point 2 corresponds to so all of them corresponds to j equals to 1. So, if we start writing this equation, so this equation u 2 1 u 0 1u 1 2 u 1 0 1 equal to 0, so at each of the points, so P 1 P 2, so P 2 then this is u 3 1 plus u.

So, this is i 2 and j 1 i 2 then this is and then point 3. Now, out of this we have to use these data. So, for example if you consider this u i 0 is 0 accordingly this will be 0. This will be 0 this will 0, so we have to use that and then. So, use boundary conditions say u of x 0 equals to 0 implies u i 0 equals to 0. So, for example I will show you u of 0 y equals to 0 implies u 0 j equals to 0. So, when we substitute what are the unknowns the unknowns must be at these points. So, 1, 1, 2, 1, 3, 1 must be the unknowns, so this is an unknown so there are three unknowns its repeating 1, 1, 2, 1, 3, 1. So, these unknowns can be solved using this boundary conditions, so the whole cuts of the problem is to solve this.

Now, I am not solving this system because we have to discuss few concepts before we do, so I will just proceed further to explain those concepts before we attempt any specific example. Otherwise at this stage you must understand given the Laplace equation we discretized which is standard 5 point formula. Then we go for the discretization of the domain and identify the points which are to be solved and then compute the corresponding discretized boundary values. And run the system at each grid point the internal grid points. So, then we get a system of equations eliminate the boundary values and you get a system of equations in unknowns and that that needs to be solved. So far, if you see the Laplace equations it is homogeneous there is nothing to the right hand side.

So, the story remain the same even if you have something on the right hand side. So, let us see case where we have right hand side non zero. So, as you know if right hand side if you have a non zero term typically say it is also a function of the special co-ordinates. So, then we say the corresponding equation is Poisson equation. So, having discussed Laplace equation let us put Poisson equation and execute the discretization. Then when we realize both would lead to the same situation which means. We end up with a system of equation, then we can discuss the concepts involved in order to solve the system.

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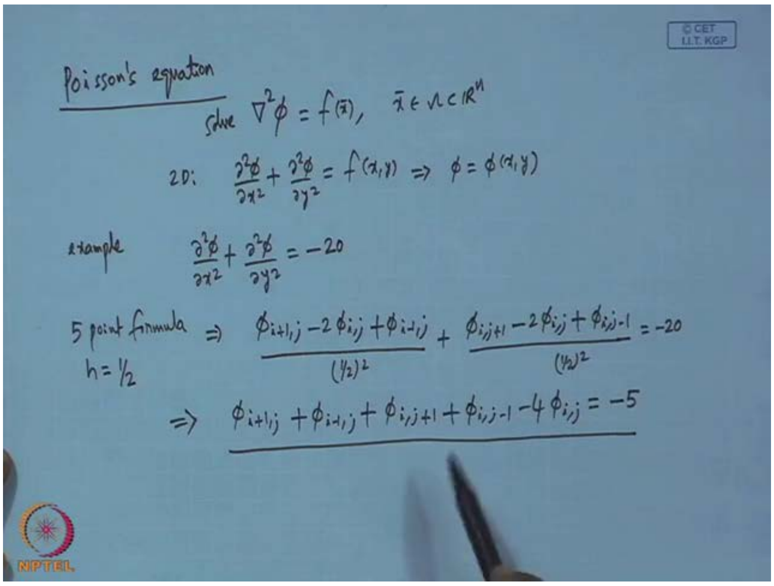

So, let us now proceed to Poisson equation, so x can be any point in a set in Rn. So, this is more general so for example 2D, so when I say x if you want you can put some bar to distinguish. So, this means we are expecting what is this right, so example say minus 20. For example, now if we go for 5 point formula if you go for 5 point formula.

We get say with h equals to half, so then we get one our h squares. So, h square will not get cancel may be I will discretize then you understand. So, this means phi i plus 1 j by h square equals to minus 20. So, this would lead to this would lead to we have 4 there and that goes of, so you we get minus 4 phi i j equals minus 5. So, this the corresponding discretized equation, now given the corresponding domain and the boundary we do the similar process similar to the Laplace case. So, hence the net story is we end up with system of equations given in this in this case as well. So, in either of the cases let us look at the computations involved because now we have focus on either Laplace equation, which we have discussed here we end up with this system, so either in this case or such Poisson equation. So, the main aim is we have to solve the system and that involves some number of computations ok.

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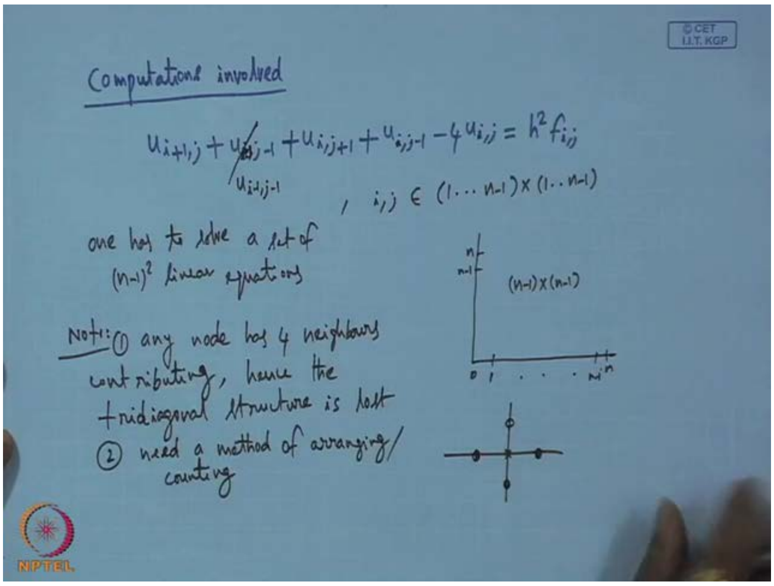

Now, let us focus our attention to the computations, so computations involved right. So, if you consider our general equation. This must be i minus 1 i j minus 1 minus 4 equals to h square f i j. So, this is more general formula, so make a pardon. So, this i minus 1 u i minus 1 j minus 1. So, this where i, j belongs to goes to n minus 1 cross, so if you have a grid goes to n. So, then what would happen 1 to n minus 1 and similarly here, so this is n minus 1. So, we have how many internal points internal points, so one has to solve set of n minus 1 square linear equations. So, this is because i is varying form this and j is varying from n to 1, so the note is, so one any node for example, if you if you take any node it is asking for these 4 other nodes right.

So, any node has 4 neighbours contributing and 4 neighbours are, so this and this the distance is more than one step and the from here to here the distance is more than one step, so any node has 4 neighbours contributing. Hence the tri diagonal structure is lost, so earlier case if you recall parabolic etc we use to have a simple structure. Where to compute a particular value you need past few values explicit scheme, so kind of so that suggest a tri diagonal system, how are in this case. So, this is a deviation two once you have a particular a grid. And internal nodes we may expect need a method of arranging i would not say arranging its it is it is like a counting means from where we call 1 2 3 etc, so you will be comfortable if we proceed further and see.

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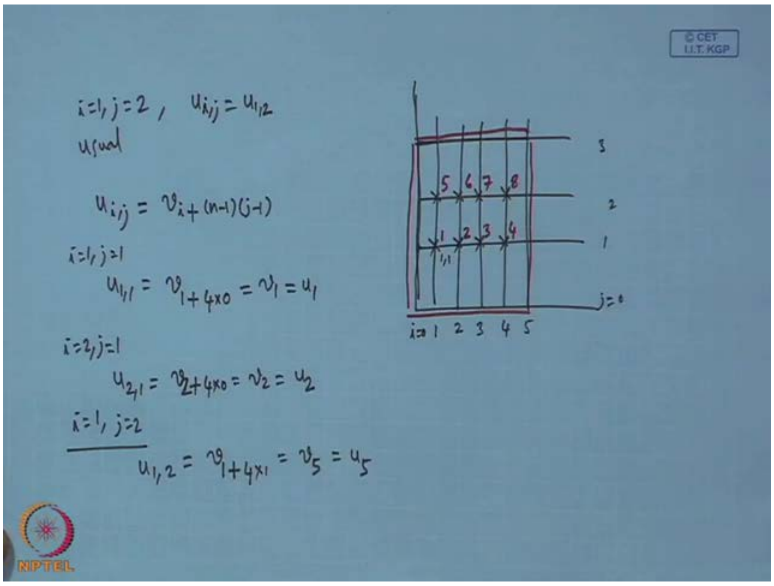

So, for example what I mean is as follows. So, for example this is a your grid say further and suppose this your grid and let us say this is the boundary. So, the boundary so this is the boundary then these are the internal nodal points. Now, we have to obtain the solution at this nodal points, now how do we order them suppose somebody wants run here of course we have original 1, 2, 3, 4, 5 and similarly for j. So, as usual we can have our own notation which we have been following, so that is for a i equals to 1 and j equals to u i j is u 1 2. So, this is our usual notation, so this is usual so. But, here we want to just make a convenient method may be we count it 1, 2, 3, 4, 5, 6 like this or.

So, there are several possibilities for example you can count like this you can count like this you can count like this, so one typical method this one, so which means so for example this point corresponds to 1 1right. So, this corresponds to 1 1, so here our n is 5 so you will see this corresponds to i is 1 and j is 1 so v 1 1 equals to v 1 plus so 4 times 0. So, you may u 1 as well so this is our 1, now let us consider point i is 2 j is to 1. So, u 2 1 will be v 2 plus 4 into 0.

So, this can be u 2, so that means we are it is 1, 2, 3, 4 we adding like this. Now, let us take this point, so that corresponds to i 1 and j 2 suppose i 1 and j 2, so that will be u 1 2 so v i plus 4 times 1 so v 5. So, this can be u 5, so that means this is our next point, so then 6, 7, 8 so this just a simple enumeration. So, with respect to this enumeration we get.

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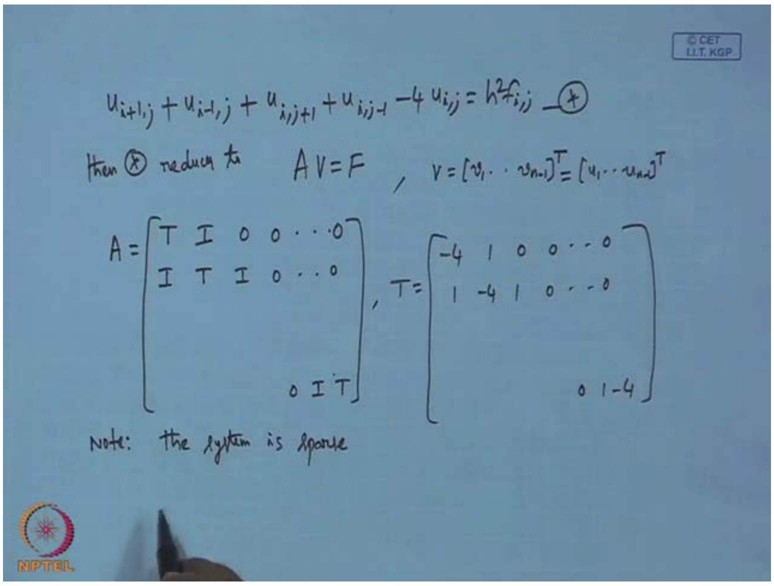

So, you consider this call it star and you propose the corresponding enumeration then star reduces to A V equals to F. Where A has the following structure and T and V, so this is typically we are calling u 1. So, this is the general structure, but note if you see the corresponding system the system is sparse. So, if you have any system sparse then definitely if you generally go for Gaussian elimination. Then see when it is sparse there lot of a zero entries appear in the in the matrix.

So, then when you try solve using Gaussian elimination it will produce at several intermediate stages it will produce several non zero entries. That means it is not really making use of the fact, that lot of zero entries in the sparse system. So, we need to consider this factor and come up with some efficient solver. So, let us see to address this particular task ok.

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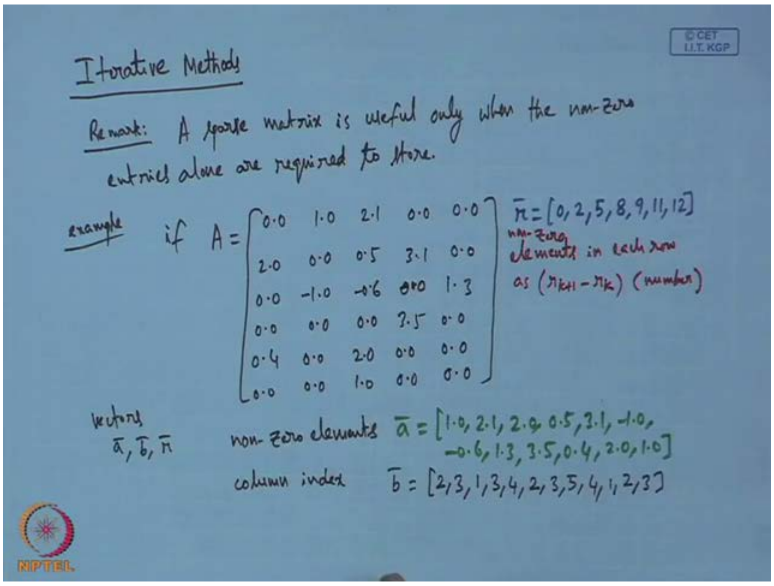

So, the next we try to think of addressing this, so hence I am giving Iterative methods Iterative methods, so a remark here. So, what is a remark a sparse matrix is useful only when non zero entries alone are required to store I mean this may sound I mean I am not putting it like a book definition. When I say see, having so many zero entries if you are still saving those zero entries and then trying to medal with it is no longer useful right. So, what I am trying to say is sparse matrix is useful when only the non zero entries are handled, so that is idea. So, for example if consider a matrix some matrix consider it little large, so we know the concept. Suppose, this is a matrix A, so you can see lot of zero entries and we want to represent it, so that only non zero entries are handled.

So, what are the possible methods, so for example we have three vectors a b and r. So, where a is say non zero elements non zero elements are given in a bar. How is this 1.0, 2.1, than 2.0 then 0.5, 3.1 then so this vector a. What did we do all that we have just stored non zero entries. Then column index, so what is the column index column index is see this is non zero entries appears in which column second column this appears in third column this appears in first column. So, we just list that 2, 3, 1, 3, 4, 2, 3, 5, 4, 1, 2, 3, so corresponding non zero element in the corresponding column this is a index. Then we have r bar, so we have to define r bar properly.

So, what we are going to store what we have going to store in r bar we have going to store in r bar. Suppose how many non zero row goes how many non zero elements in each row, so in first row two elements in second row three elements. So, we could have put 2, 3, but we are storing a slightly in a different way. So, consider r k plus 1 minus r k, so the difference is giving two non zero elements. Then here how many 1, 2, 3 so three non zero, so we have to show it as a difference so 5 must come here, then similarly here 1, 2, 3, so we must have here 8 then here how many only one.

So, 9 and here 2 so 11, then one see we want to have maintain the same dimension. So, that is the reason we are, so what is the definition the definition of r bar is elements in each row, which elements non zero elements in each row as. So, this means number of non zero elements in each row as, so you can see the number of non zero elements here two, so the difference of r 1 minus r 2 minus r 1. Then similarly here 3, so r 3 minus r 2 so this an efficient way of storing and then one can see all the non zero entries are only handled.

So, the zero entries we did not, so this is worth learning. Because the crux of Laplace solving Laplace Poisson equation reduces to solving systems more effectively, so we should know little bit of tricks. How to solve the system, so storing is one of the important parameter important concern. So, we have learnt this now we proceed to the corresponding iterative methods in the in next lecture until then bye.