Numerical Solutions of Ordinary and Partial Differential Equations
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Lecture - 28
Tutorial – IV

Hello, good morning. Today let us solve some problems displaying both explicit, implicit and also derivative boundary conditions. So, let us start with Crank Nicloson scheme, which is implicit scheme.

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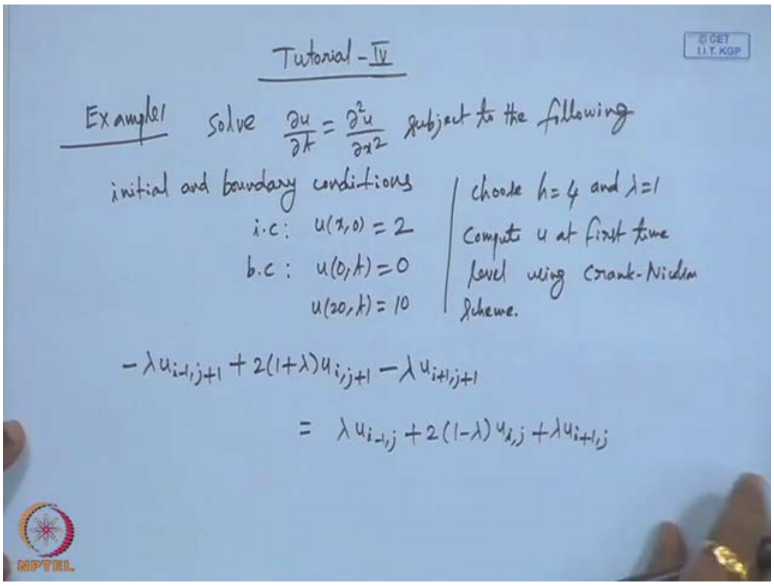

So, I start with example one. So, initial conditions, boundary condition and choose h equals to 4 and lambda equals to 1. This is for calculation sake in general. We may not choose such a huge step size. Compute u at first time level using Crank Nicloson scheme. So, this a problem. So, let us solve this using Crank Nicloson scheme, so the Crank Nicloson scheme. So, as you can see this is implicit all the time levels. Higher time levels are to the left and the past time levels are to the right. So, now in the present case we have lambda equals to 1.

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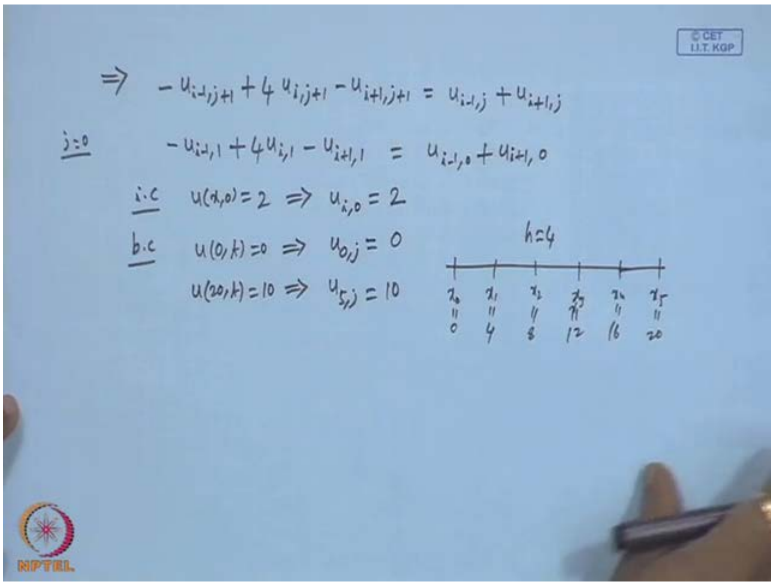

So, accordingly the scheme reduces to… So, the middle term gets skilled because of 1 minus lambda term. Now this j equals to 0. We have now our initial condition. This implies equals to 2 and boundary condition, so this implies 0 j equals to 0 and u of 20 t equals ten implies. So, our grid is x 0s 0 x 1 is 4 next is 8 x 3 is 12, 1 6. So, this implies u 5 j is 10 because we need at the boundary point. So, this is because h equals to 4. So, now let us write down the equations corresponding to i equals to 1.

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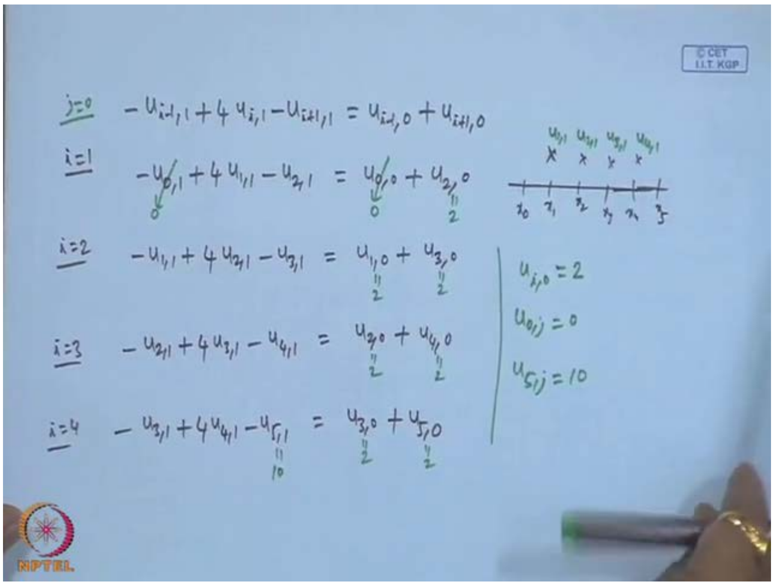

So equation is minus. So now let us write i equals to 1 so the equation will be… So, then i equals 2 because what are the unknowns are… This is x 0, x 1. x 2, x 3, x 4, x 5. Since 3 are no derivative boundary conditions. The unknowns are at this grid points. So i runs from 1, 2, 3 4. So, 1 next 1 is 2. This will be i 2, so then i 3 then i 4. Now let us look at the values u 0, 1. So, we have also discretised and obtained the… So, u i 0 is 2 u 0 j is 0 u 5 j is 10. So let us record these values. We also have u i 0 is 2. u 0 j is 0. u 5 j is 10.

Now for u 0 comes from this, so this is zero. Then this is unknowns are as I mentioned. u 1, 1 u 2, 1 u 3, 1 u 4, 1, so u 0, 0 u 0, 0. We have this is for j 0. If you recall this equation and we have written, we have written for this. Hence, we have to use this value not this value because this is from i. We are running the equation from i 1. So, this and u 2, 0 is 2. So, this is how the equation gets reduced, so one 0 and 3, 0. These all are 2, this is 2, this is 2. And here we have 5, 1 which has to be taken from here. So, accordingly the system simplifies to, system simplifies to…

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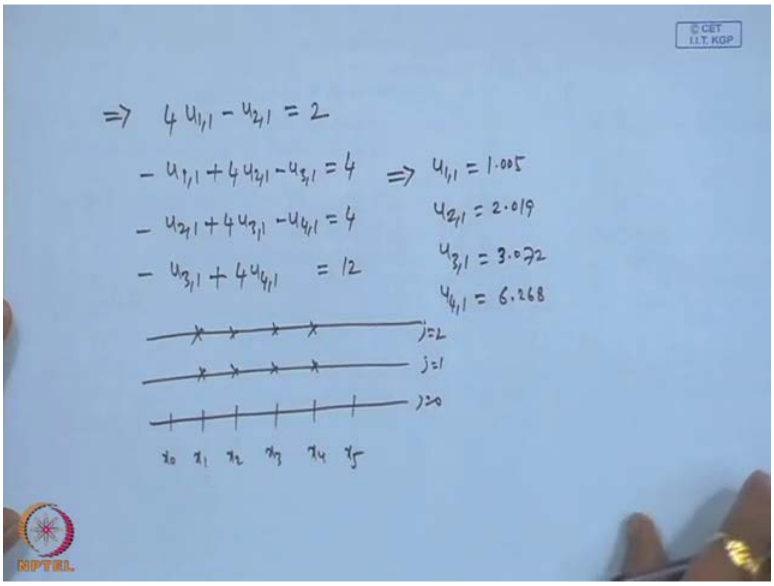

So, this will be 12. So, we can solve this. This tri diagonal system and we can solve. So, I have pre calculated values. So, I am not showing the working. So, you can compute. So, if you would like to obtain the values at higher time level. So, for example, this is whatever values we have obtained this corresponds to j equals to 1. If one wants corresponding to j equal to 2 we can use this compare values and compute at equals to 2. So, we can do the march past like this with time step. So, this is implicit scheme. So, now let us look at a scheme where the derivative boundary conditions are given means not in the scheme of course, boundary conditions involve derivatives. So, accordingly the end points are also to be considered as unknowns. So let us look at the example two.

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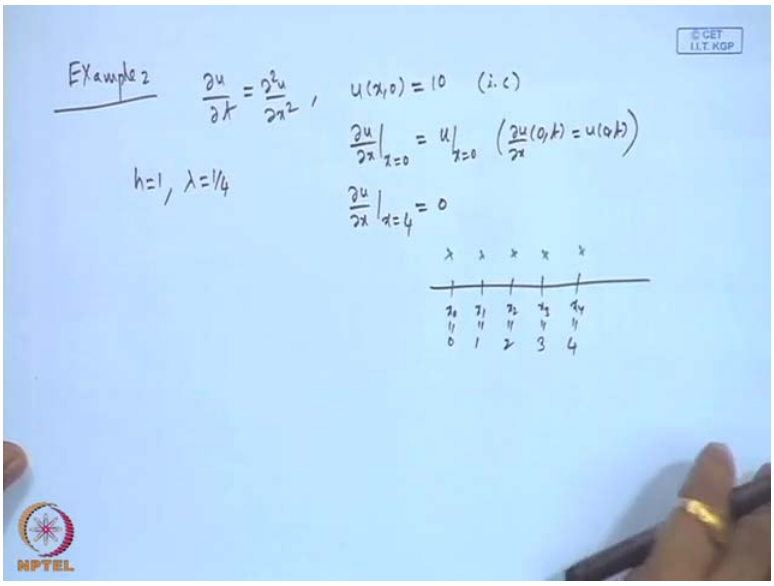

So, I am not writing problem formulae. So, this is initial condition and boundary conditions. So, variety of ways for example, earlier we were writing x comma like that. So, this is another way so this equals to u. Of course, at so the same thing we can write at 0 comma t is equals to u zero comma t. Both mean the same. So, another is x equals to four is equal to 0 and say given h equals to 1 and lambda equals to 1 by 4. So, that means accordingly we compute for the next time step, which is one, can compute delta t from these two values. So, obviously you have derivative boundary conditions hence, if you look at the grid. So, the solution needs to be obtained given at the end points because this becomes also an unknown right. So, let us look at the explicit scheme in this case.

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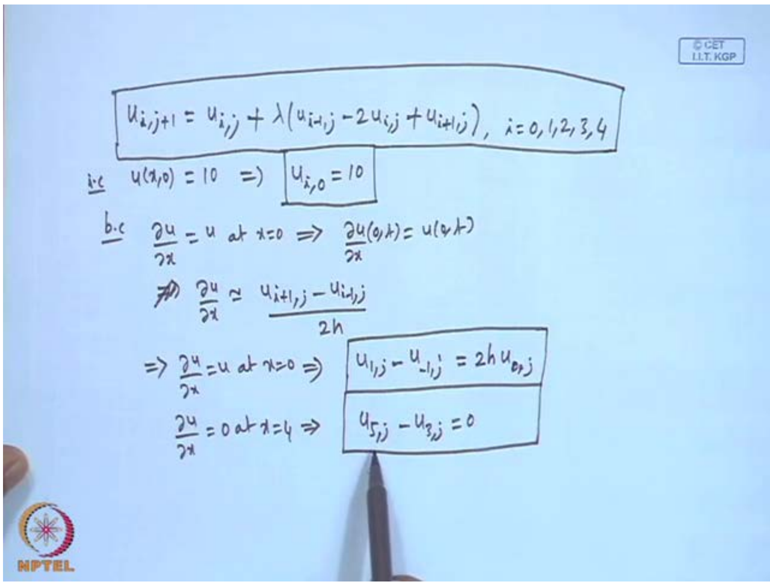

So, u, i, j plus 1. So, this is explicit scheme however in the light of end points also been unknowns. This is supposed to be valid at all the grid points. 1, 2, 3, 4. 0, 1, 2 3 4. So, now let us look at our initial condition then look at the boundary condition. So, this is initial condition so d u by d x equals to u at. So, this has to be. So, first let us discretize this. So, if you discretize. See this d u by d x at 0 comma t is. So, we have to write at all j. So, this will be let us approximate this by 2 h. So, this not implies so this is discretization we are considering accordingly the condition of u at x equals to 0 become this is x equal to zero. So, u 1 j and here we get u minus 1 j is equal to 2 h u 0 j

The other condition x equals to 4 so this corresponds to… So i is 4 u 5 j minus u 3 j equals to 0. So, we have unknowns at 0, 1, 2, 3, 4. If you recall if you run at i equal to 0 here we are going to get u minus 1. So, this is a fictitious value. However we have one equation involving this fictitious value. Similarly, if you run this equation at i equals to 4 we are going to get another fictitious value at the other end. We have an equation involving this fictitious value. So, let us A B… So, let us run equation A at i equals to 0.

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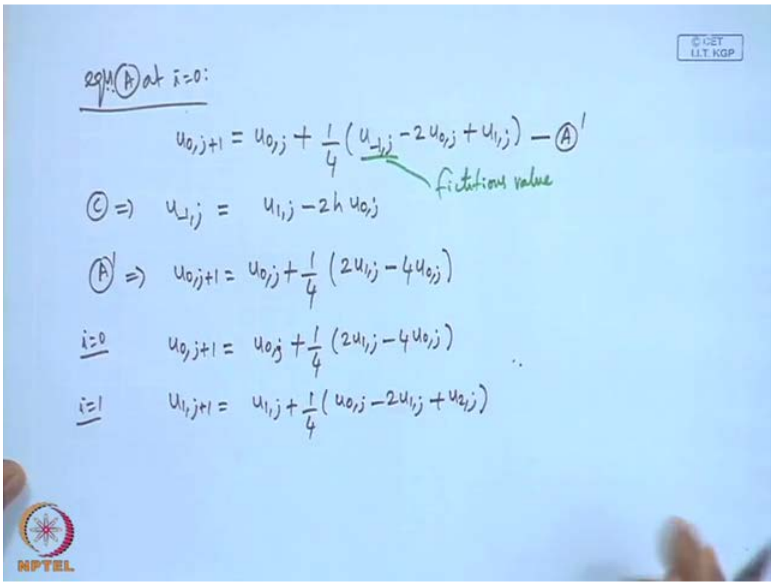

So, equation a at i equals to 0. This will be u minus 1 j this is i equals 0 minus 2 u 0 j plus u 1 j. However c implies u minus 1 j is equal to u 1 j minus 2 h ((Refer Time: 20.46)). So, the above equation becomes say this is a prime, so for u minus 1 j so we are eliminating the fictitious value. So, this is a fictitious value and we were eliminating is in this equation.

So, 1 u 1 j and we get and our equation h is 1 so this is minus 2 so this will be minus… So this is equation, which we have to run at each grid point. So, accordingly i equal to 0 we have remember we have others and also we have fictitious value u 5. So, that needs to be eliminated when we write the equation for i equals to 4. So, this i equals to 0 will be, i equals to 0 will be j plus 1 by 4 then i equals to 1, i equals to 1. So, we write down all the equations.

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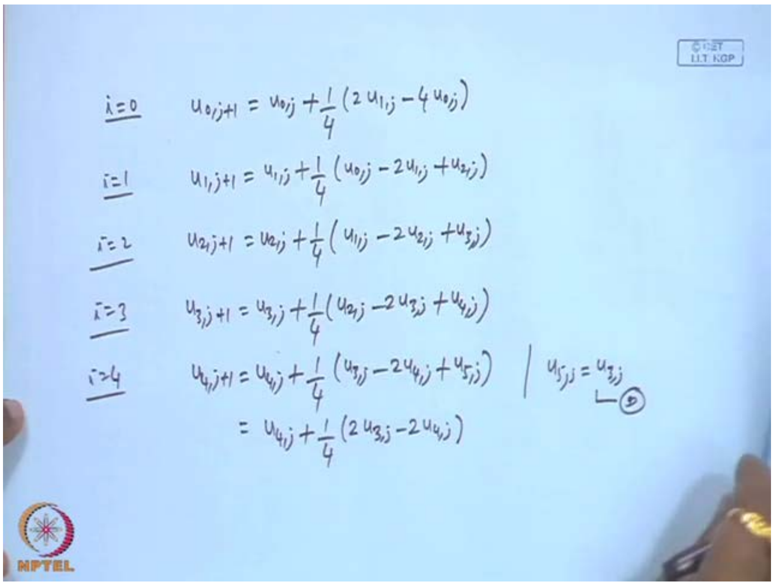

Then this is routine. Now comes other end point so here when we write down, here when we write down we have the other fictitious point, which is this. However we have u 5 j equals to u 3 j. So, this is our equation number is d. So, this implies now if you see the system of equations this five they do not involve any fictitious value. So, they do not involve any fictitious value. Now we have to start feeding the initial values to compute at the next time level. So, that means we fix j equals 0 and compute the values j equals to 1. So, if we do that.

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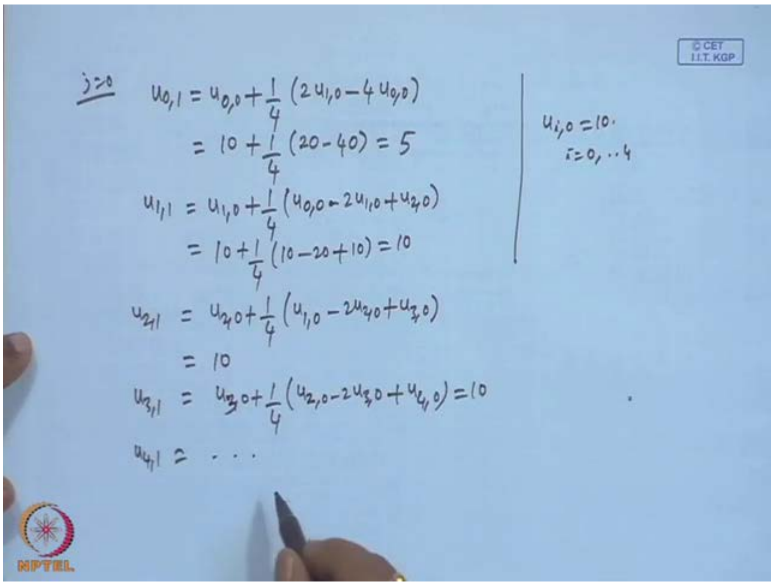

j equals to 0, j equals zero will be…So, this can be computed… Now it would be better to write the corresponding initial and boundary conditions. We have u i 0 is 10 so accordingly we have 10. This is valid for i 0 to 4 because end points are also unknowns. So this 10 so this would be minus 20 then u 1 1, we have written values. So, this will be u this is a corresponding to i 1 and j 0. So, this will be 0. So, this will be u, so we can compute this comes for to be 10 u 3 1, u 3 1 will be u 3 0, this will be 10 and finally, u for 1, you can compute.

So this the derivative boundary condition case of course. Here just to make the calculation simple I have considered explicit scheme but if it is in implicit then you further get a system of equations. And then you have to handle the system. So two issues one is if the scheme is implicit you have to handle the system. And if you have the derivative boundary conditions then you get fictitious values due to the, due to the end points. If you run the corresponding discretized equation at the end points you get the fictitious values, which one has to remove using the derivative boundary condition discretization. So, now let us discuss an example where consistency can be understood.

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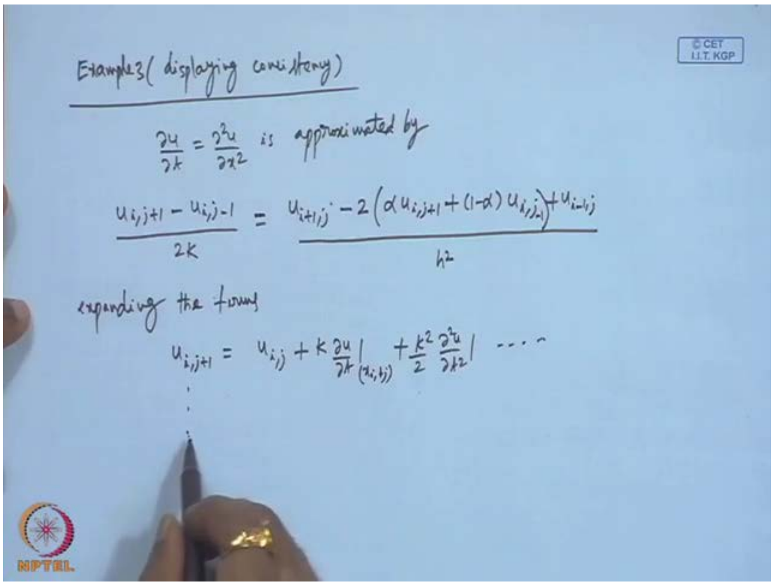

So, this example is displaying consistency. Suppose this is approximated by this j minus one, suppose this is approximated by… So, this i plus 1 j, i j plus 1 and j minus 1 j. Now we would like to based on the error expansion we would like to check the consistency of the, of this approximation. So, expanding the terms we have usual expansion. So, I am just doing one so this is with respect to t so u i j plus k so this is it. So similarly, other terms… Now if we do that…

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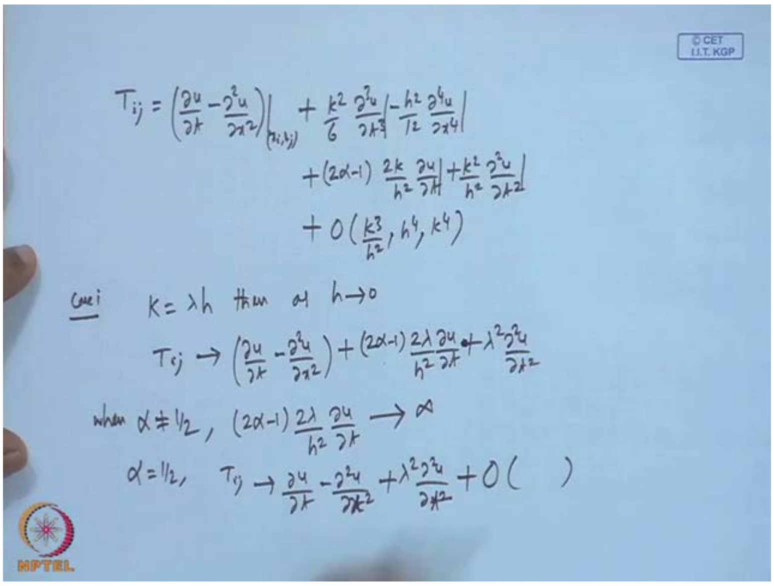

So, it is an exercise you must try. T i j is all of them are evaluated this point. So, you have to expand and make sure that we get this. Then let us look at case one were k is lambda h. Then as h goes to zero T i j goes to you can see as h goes to zero T i j goes to this is the first term plus 2 lambda by h square plus lambda square.

So, if k is lambda h, then so this term goes to 0 this is 0 and we get these are the terms, which we have to address, then when alpha is not equal to half 2 alpha minus 1, 2 lambda by h square. This goes to these blows up right. Then when alpha is half T i j goes to so this term goes to 0 so the non 0 terms this is x square plus lambda square. So, therefore is d square. Therefore, what is net conclusion? So, you expand then if this case k is lambda h so you substitute then take limit h goes to 0. So, you can see this has to be taken in two different cases when lambda is half and lambda is not equals to half. When lambda is not half this blows up whereas, when lambda equals to half T goes to this plus the other terms.

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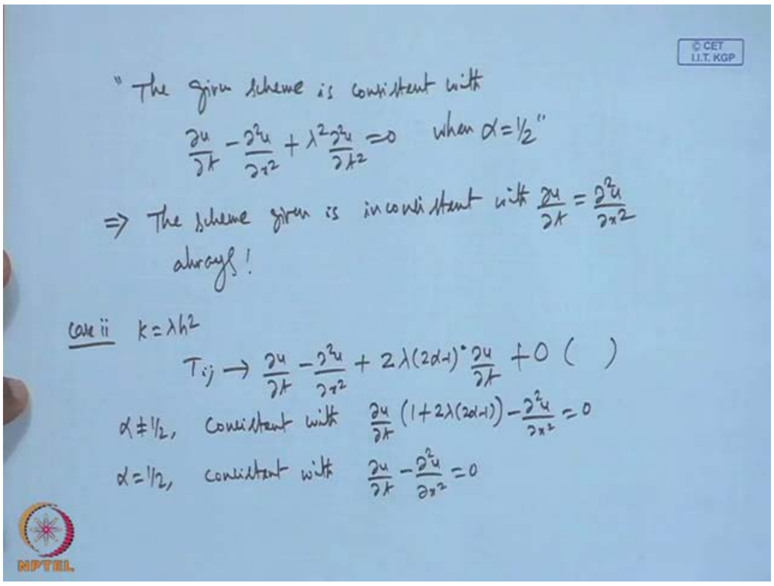

So, which means for us, which means the given scheme is consistent with. So, this is the conclusion. So, this implies the scheme given is in consistent with this is our given equation always, but this is the given equation. However in one case when alpha is not equals to half one term blows up whereas, when alpha equals half the error is going to this term. So, if this is 0 then there is corresponding error, which is contributing from the next non 0 term. So, that means the sense the given discretization, when alpha equals to half is consistent with this equation and not this equation.

Now, let us see this case, so in this case T i j goes to then alpha not equal to half consistent with because you have other terms. So, when alpha not equals to half the equation is consistent with some other the disc ray scheme is consistent with some other equation. What is that equation you club d u by d t 1 plus 2 lambda 2 alpha minus 1, whereas an alpha is half consistent with… So this tells the verification of a consistency. So, in case one when k is lambda h we have seen the one of the non 0 term blows up the limit h goes to 0. So, that means definitely error is not bounded, so consistency does not arise. Whereas, its no longer consistent whereas, in the second one when k equals to even when k equals lambda h if alpha is half then it is consistent with some other scheme.

So, whereas, in k equals lambda is square so for alpha not equals of it is constant with some other scheme. Whereas when alpha equals to half it is consistent with the given parabolic equation. So, that is how one verifies consistency check. So, now let us look at another example where a slightly different equation. I mean you have standard equation every time we are considering d u by d t is d square d u by d x square but let us consider equation, where you have first derivative involved with respective the space variable. And then also derivative condition. So, let us look at it.

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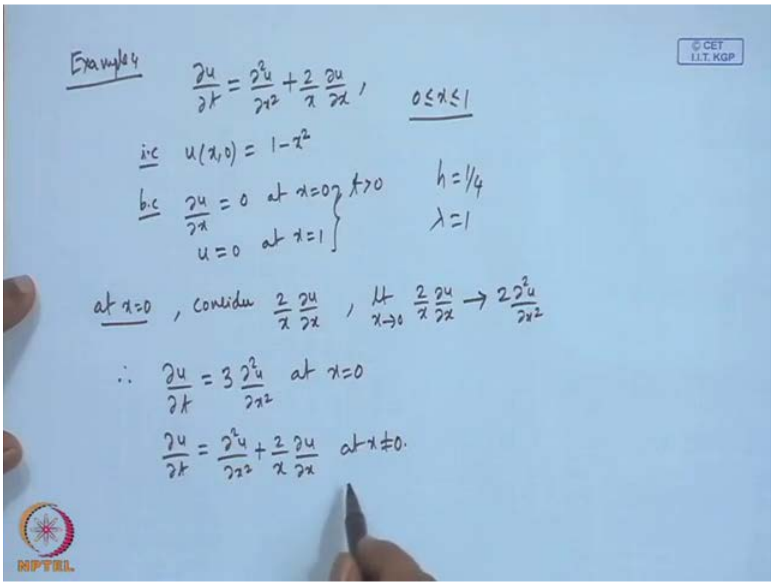

So, this is our equation. This is initial condition boundary condition. So, it is slightly different form what we have been considering because of presence of this term. Now this presence of this term creates slight trouble because our domain is… So as you can see at x equals 0 that is a slight difficulty this term. So these are for simple cases we are choosing such lambda. So, that the calculation becomes simpler now x equals to 0. Consider this term. So there is a similarity so we have to take limit. So, this goes to… Therefore, the equation reduces to… So there is one and two so this is… So, this is at x equals to 0. So, this is a set we have to use for discretization. So, let us look at it.

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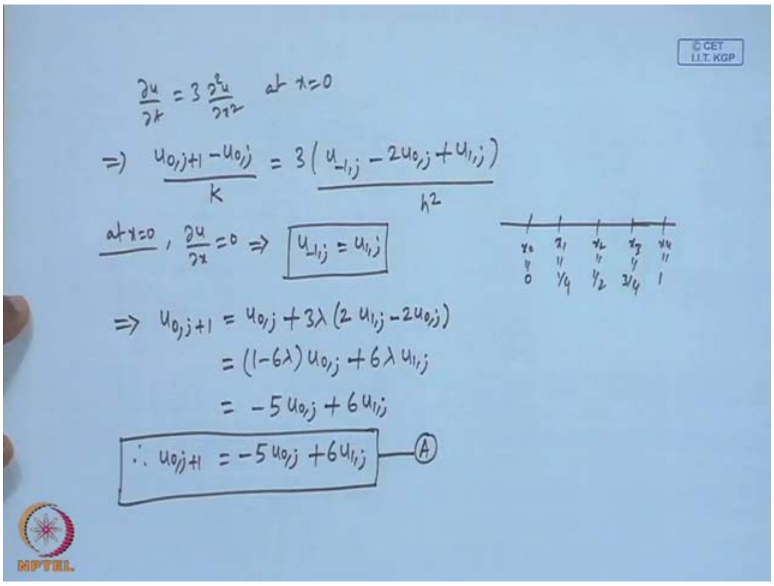

So, this implies. So, this is forward time, this is center second order. So, at x equal to 0 because of x equals to 0 discretization we got this fictitious value. This is similar to… Right so since we had to discretize equation at x equal to 0 we got a fictitious value right. Now we have at x equals 0 d u by d x is 0 so this is our given boundary condition. So, this implies. Right so this gives our equation to be… So, take in this right 3 lambda so u minus 1. I am going to eliminate. So, we get… So, this is… and since our in a present case lambda is 1 therefore, so this is say A.

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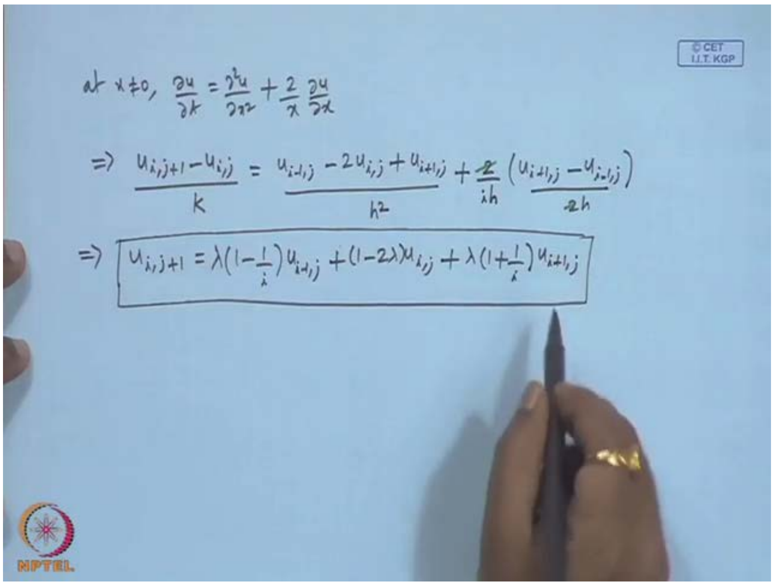

Now we have to discretize at x equal to, x naught equals to 0. so our equation given equation is… So we discretize… 2 by x, so this will be i h. So, we have to arrange the terms, so here there is 2 get cancelled. So, we have when k goes there this becomes lambda. So, we have terms taking common. So, the given problem contains if you see, only one derivative boundary condition.

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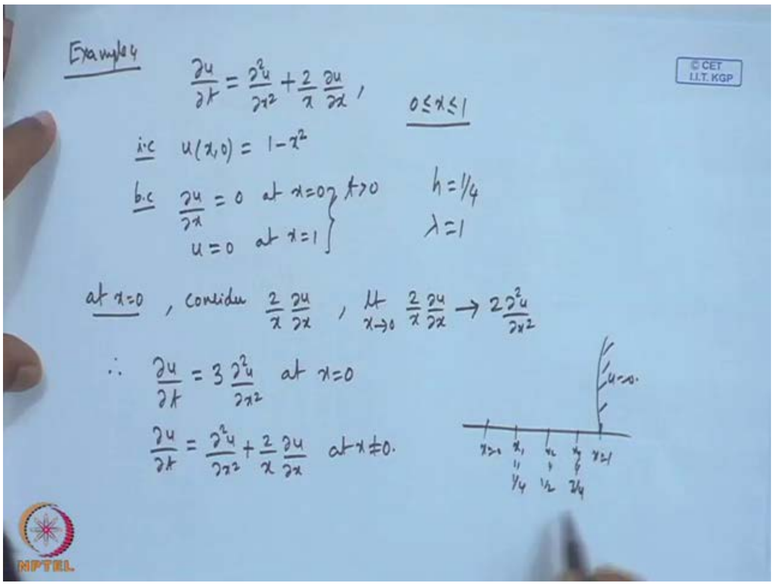

So, that means only at x equals 0 whereas, at x equals to one so this is known so for all time level so this is u is 0. Therefore, the number of unknowns are x 1 is 1 by 4 x two is half x 3 is by 4. So, the number of unknowns are 1, 2, 3, 4.

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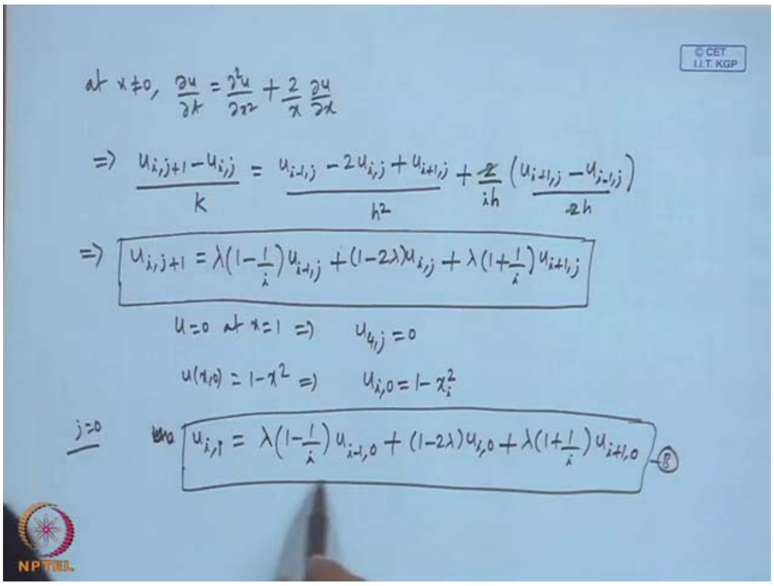

So, we have this then u equals to 0 at x equals to 1. This implies u for j equals to 0 and u of x 0 is 1 minus x square this implies u i 0 is… Now we can run the equation j equals 0, sorry u i 1 in fact lambda is one we can drop. So, this is explicit. So, this one and we can call this is B. So, this two we have to… This is A this is due to at x equal to 0 and this is x non 0 that means i 1, 2, 3. So, this is valid for 1, 2, 3, so may be you can run the equations and get explicit values which is not very difficult.

So, as you can see this equation contains the term, which is singular at x equal to 0s we have taken limit. The limit is well behaved because we use ((Refer Time: 54.29)) and then we got the derivatives, second derivative. So, at x equal to 0 the equation behaves in some sense and for other value it behaves in other sense. And then also if the derivative boundary condition is given at only one end definitely the corresponding fictitious value would appear only on one side. And for the other side we do not have any other fictitious value. So, I think more or less you got some idea how to use derivative boundary condition and also how to check consistency of given discretize scheme. Now let us see how we proceed remaining lectures until then.

Thank you, bye.