Numerical Solutions of Ordinary and Partial Differential Equations
Prof. G. P. Raja Sekhar
Department of Mathematics
Indian Institute of Technology, Kharagpur

Lecture - 24
Finite Difference Approximations to Parabolic PDEs

Hello, welcome back after introduction with the first order and second order PDEs classification etcetera. Now, let us proceed for the numerical discretization and the various techniques to compute partial differential equations. So, if you recall the classification with respect to the second order, in general we classify as parabolic hyperbolic and elliptic. So, though it is I mean nice to start with first order and then second order, but in general in particular when you deal with partial differential equations numerically, normally we start with second order. So, let us start with second order and then see how we proceed further.

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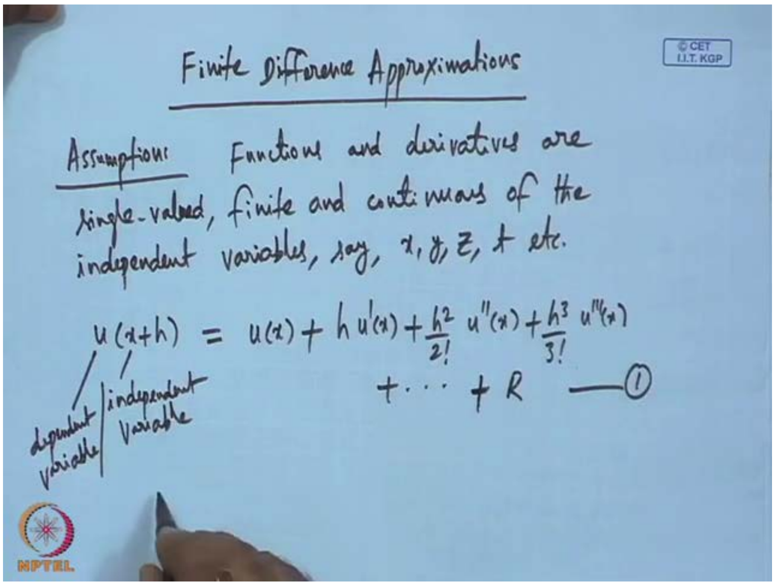

So, the title I gave is finite difference approximations. So, two parabolic because I would like start with parabolic, but before we start with parabolic, so what is the general assumption? So, the general assumption is the functions and derivatives are single valued. Single valued finite and continuous of the independent variables say x, y, z, t, etcetera. So, why this assumption is required?

So, this is a concern. So, this assumption is required because as you can see when we deal with the partial differential equations, so you get derivatives with respect to all the independent variables. Now, when we approximate these derivatives we need to expand them and tell us a reason, so we need these assumptions. So, let us start how we approximate.

So, consider say u of x plus h, so x is the independent variable and u is dependent variable. So, this we go for Taylor series expansion and h is increment, so h u dash of x plus u double dash of x plus u 3 of x plus. So, I am not writing the reminder term because in first order when we are doing O D E S, we have discussed thoroughly the Taylor series method. Therefore, I just write the remainder term. So, let us call this 1.

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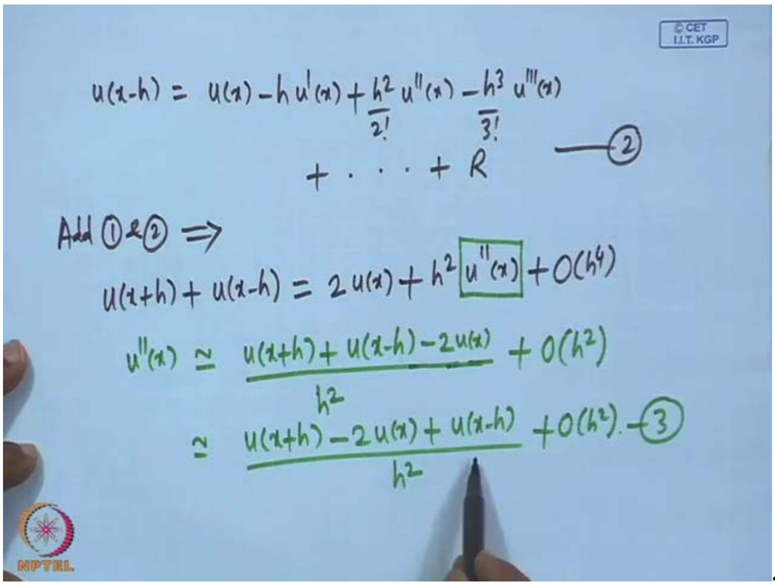

So, then similarly let us consider x minus h and expand in Taylor series. So, this will be so alternate signs so plus the remainder, now we had one this term. So, this expression then 2 is alternate signs, so say we add 1 and 2. So, this implies left hand side, then right hand side, so since it is alternate signs so u will be so we can see u and this u 2 times and this 2 times get cancelled. So, it is like that, so the first term will be and h coefficient 0, since h square by 2.

So, this will be this, so third will be cancelled, so then I am putting order h 4 which means the first non zero term which we are throwing it is starting from h 4. Now, from here we can obtain approximation for, so this u doubles of x is approximately by so this coefficient is h square and this term, this is a plus sign here and this goes to the left hand side. So, then since I have divided by h square, now this becomes so this is typical approximation. So, generally we write it as so this is an approximation for the second order derivative. So, remember we have added it.

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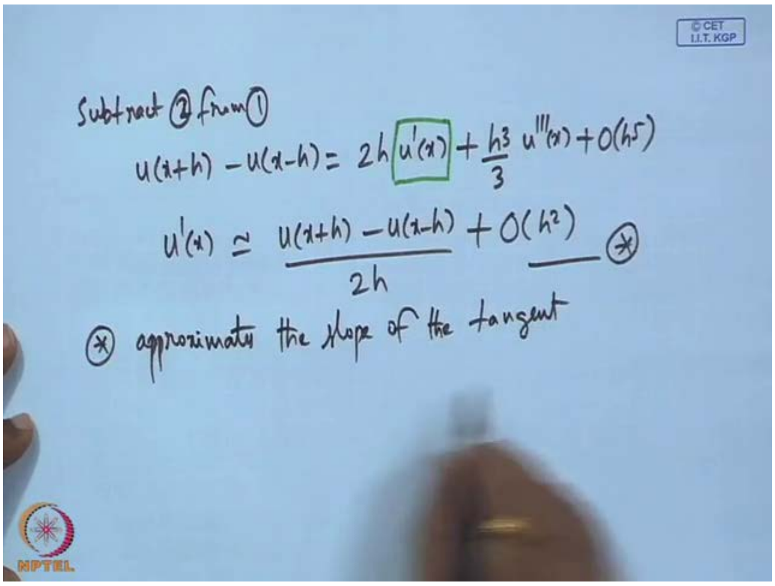

So, now let us see what would happen if we subtract, subtract 1, subtract 2 from 1. So, then we can see so if you so the first term is u, so that get cancelled and we get 2 times h and second derivative get cancelled, so we get h cube by 3. So, again from here we get approximation for the first derivative. So, since we divide by h this is so in by adding we approximated second derivative and by subtracting we approximated first derivative. So, what is happening? So, geometrically what is happening? Geometrically star approximates the slope of the tangent. So, in what sense let us see.

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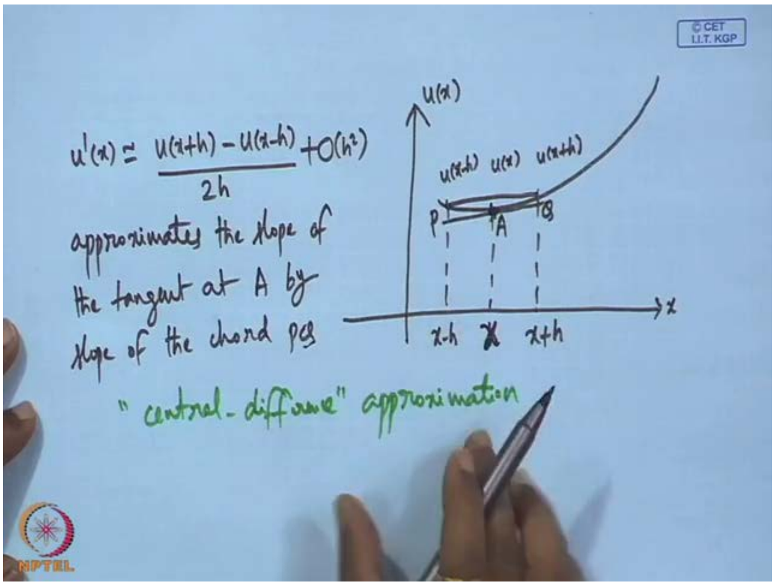

So, geometrically suppose the function like this and say this is x minus h and this is x and this is x plus h, say this is point P, say this is A, this is Q and the cord this is P Q. So, the function value here u of x minus h and the function value here, so this is u of x and the function value here u of x plus h. So, geometrically what is happening? So, the formula which we have obtained u dash of x, u of x plus h x minus h by 2 h. So, this approximates the slope of the tangent at A because there is x.

So, what is x? x is the point A therefore, approximate the slope of the tangent at A by slope of the chord P Q. So, slope of the tangent at A has been approximated by the slope of the chord P Q. So, slope of the tangent at A has been approximated by the chord slope of the chord P Q. So, this typically is known as central difference approximation, this is because you are approximating slope here by the slope of the chord. So, you are using pass point here and the previous point here. So, the difference is so this is for equidistant so the same star can be generalised.

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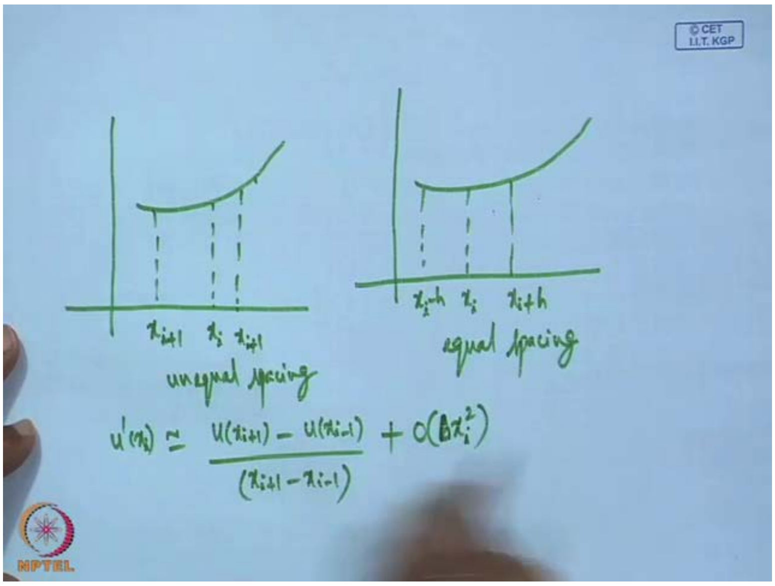

So, the star can be generalised. So, say this is x, some x i minus h, then this is x i. So, this is x i plus h so this is for equal spacing. Suppose, we denote it by x i plus 1 and this need not be equal spacing, unequal spacing. So, this approximation can be generalised u at x i is so this is so delta x i increment square. So, this is a generalisation, okay? So, now let us see how the general approximations are done for first derivatives etcetera.

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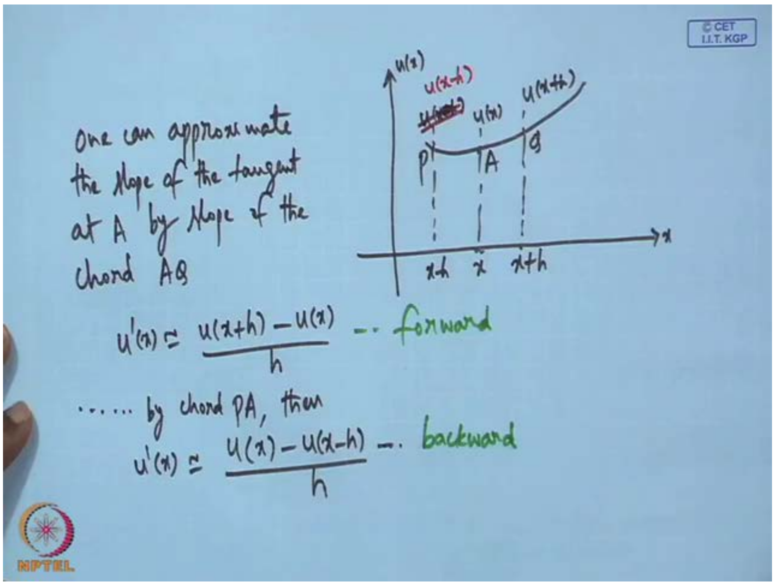

So, let me consider this, so this is so generally we consider equispacing for the simplicity. However, for more complex problems one has to go with unequal spacing.

Now, this is x, this is u, now one can approximate the slope, slope of the tangent at so P A Q, let us call at A by slope of the chord here P Q. So, instead of that A Q so this is u of x plus h u of x and u of, sorry this is x minus h x plus h. So, this is so if we do this then the corresponding approximation which we are going to get is what will be the distance this here to here. It is just h, so x plus h by h so this is called forward approximation, okay?

Similarly, if you approximate so dot dot dot by chord P A, then we have u. So, this is backward, so this is a general technique of approximation. So, having done the approximation, let us see consider maybe a first order P D E where you have a simple situation, one two independent variables and one independent variable and see how the corresponding discretization leads to the corresponding difference equation, okay?

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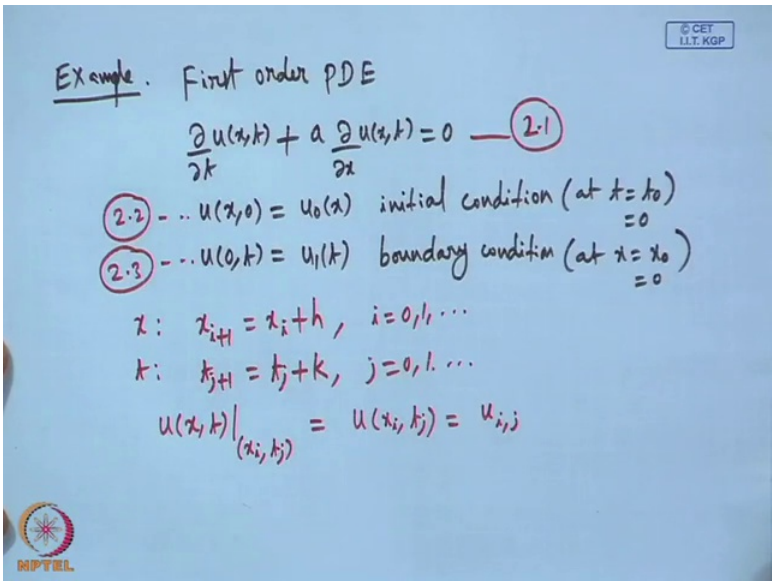

So, let us start with a simple example, so we are considering a first order P D E, say so x and t are the independent variables, u is the dependent variable, a is some constant and when we define a P D E it would be better to completely define it by prescribing the corresponding boundary conditions. If it is only space dependent boundary and initial conditions, if it is both space and time dependent. So, since in this present case it is dependent on both the space and time so it would be better to completely define the problem. So, with respect to space, it is one derivative so we need initial condition say u 0 of x. So, this is our initial condition, so this is nothing but at t equal to t naught.

So, generally this is taken as 0 because you can see here, so then it is one dimensional with respect to space and the derivative is also only first order involved. So, here with respect to time first order involved so we need to supply only one boundary condition, say u of t. So, this is boundary condition, so this is at x equals to some x naught, in this case taken as 0, okay?

So, now the problem is defined. So, let us say we would like to start discretizing say 2, say 2.1, so 2.1 and give 2.2 2.3, now if we discretize 2.1. So, first we have to discretize this, that means first order with respect to time. We would like to discretize so since this is a P D E which involves two dependent variables. So, we have to define how x is being discretized. So, x is being discretized as x i plus 1 equals to x i plus h equispaced. So, i equals to 0, 1, etcetera. So then t t j plus 1 t j plus k so we discretized, this is the notation. Accordingly, u of x t at a particular point x i t j we are going to denote it by u of x i t j, this is u i j, so this is our standard notation, okay?

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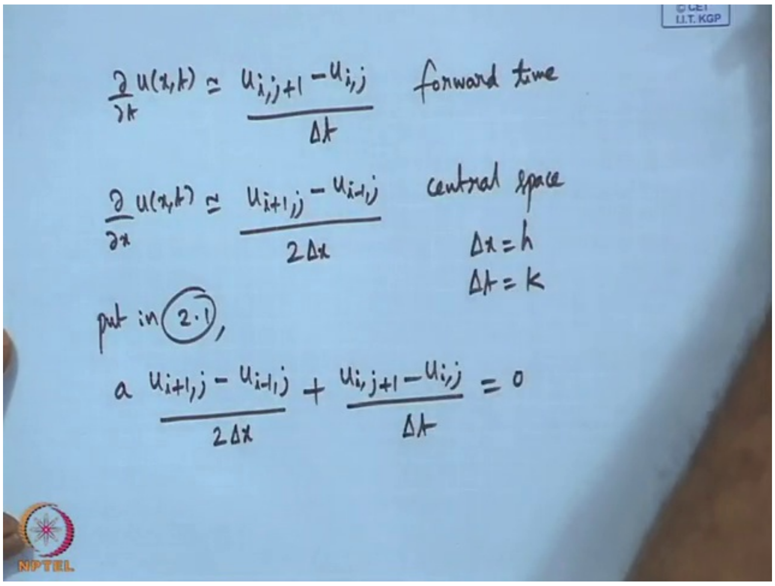

So, now with this notation let us approximate the first derivative as follows, dou by dou t u x t. Of course, at a point x i t j so this will be forward time by delta t. So, this is forward time, then minus by 2 delta x. So, this is the central space, so I have used earlier h and delta t as k. Now, if you substitute this in so we get what is you can see this is with time plus constant times with space. So, what you get u i plus 1 j minus u i minus 1 j by

2 delta x, this is a times plus u i j plus 1 u i j by delta t. So, I just swapped the terms, so I put this first and then this.

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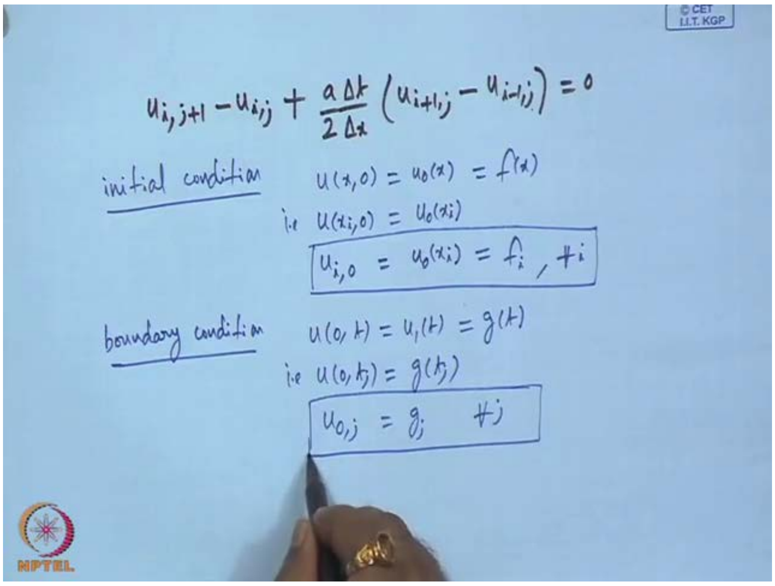

So, this is the corresponding discretized version. So, this further can be simplified u i j plus 1 j plus a delta t by delta x by 2 u i plus 1 j minus u i minus 1 j. So, this equals 0 so this is the corresponding discretized equation. However, if you recall the original equation, we have initial condition and boundary condition. So, it is our duty to discretize these as well, so let us consider the initial condition. So, initial condition u of x 0 is u 0 of x. So, this correspondingly it will be discretized as u of x i 0 u 0 of x i. So, further in our notation u i and this is for all t j t j equal to 0, right? For all j t j is 0, so we this is usual notation. So, u 0 of x i so we should not get confused with this notation, so i retain this, okay?

So for example, if we say this is a function of x then this would be f i. So, this is for all i which depends on the increment similarly boundary condition. So, the boundary condition we have u 1 of t, so say some g of t. So, this is 0 t j equals to g of t j, so this will be the corresponding discretization of initial and now if you look at carefully given the initial and boundary conditions, given the initial and boundary conditions we have to solve. So, it depends on the boundary and depending on at which time step, okay?

So, let us go to higher dimensions, not higher dimensions, the higher order derivatives. So, because as I mentioned this is just to give you an idea, first order we have taken and then we have shown the discretization. But now let us proceed to the first class of the second order P D E which is the parabolic case. So, let us start with the parabolic case, so this involves several variables. Therefore, functions of several variables so before we proceed to the parabolic.

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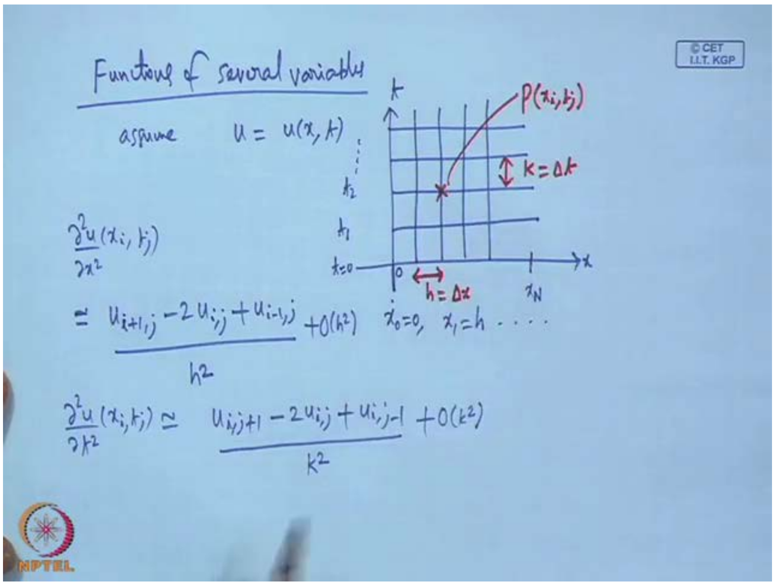

So, here assume u is u of x t 2 variable which I have already shown. Now, we can have a little better picture, so you can see so this is x and this is t. So, do not get confused, earlier I have plotted graph x with u, but here we are trying only the discretization of the domain. What is our domain depends on the x range and depends on the t range, so this is our domain.

So, now this is the discretization of the domain it goes like this, so you can see this is a h which is our delta x. So, then this one is k which is our delta t. So, if you take any point, so this p point this is at some x i t j. Now, depending on the boundary this depending on the step size the i will be decided. So, let us say this is x naught typically 0. So, if you start x naught is 0 so then depending on the step size x 1 is h, so on, right?

So, let us go for the second order discretization, so I have already shown in the beginning we would like to approximate. So, this is given u of i plus 1 j, so this I have shown in the beginning, how do we get it to consider the u of x plus h. So, then consider u of x minus h expansions then add them we get this. So, this is remembered this is the derivative with respect to x. Suppose, the derivative is with respect to t. So, this is approximate so this will be u i j plus 1 by k square plus order of k square. So, this is typical second order approximation. Why we are calling second order?

So, we will discuss more on this little later, the leading term which we are throwing starts at h square, that means h square will be the coefficient and the coefficient evaluated at x i t j that will be multiplying h square. Similarly, here some other coefficient evaluating at x i t j will be multiplying k square. So, this is for the general case. So, we are proceeding along this, so how long we proceed? Until we reach the boundary. Similarly, at different time levels, so this time level corresponds to say t equals to 0, then this is t 1, t 2.

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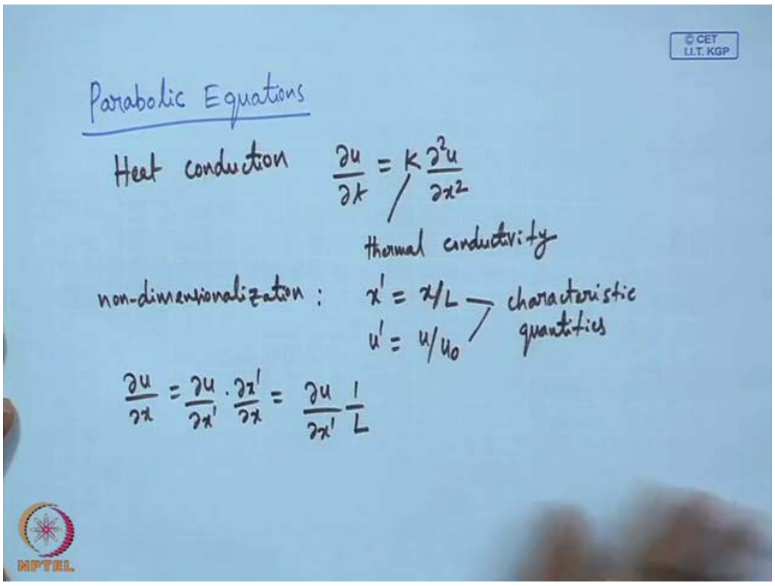

So, let us parabolic equations, so the first one I am going to consider is heat conduction given by, so where k is thermal conductivity, k is thermal conductivity which is supposed to be a constant. Now, we can go ahead with this, but typically from applied math point of view when you solve such problems, this will be non dimensionalised. So, we can do that even though this is not part of numerical solutions, I would like to just mention. That means see each k has dimensions, assume this is temperature; this is time and then thermal conductivity. So, it would be better to non-dimensionalise so that the solutions are uniformly valid, universally valid, okay?

So, how do we do it? We have to introduce new variables x is space, so we have divided by a length scale and the u is temperature. So, by something and these are called L and u naught some characteristic quantities which are supposed to be known. So, if you introduce this definitely with respect to x prime and u prime, these are non dimensional, right? So, that is how we do it, okay?

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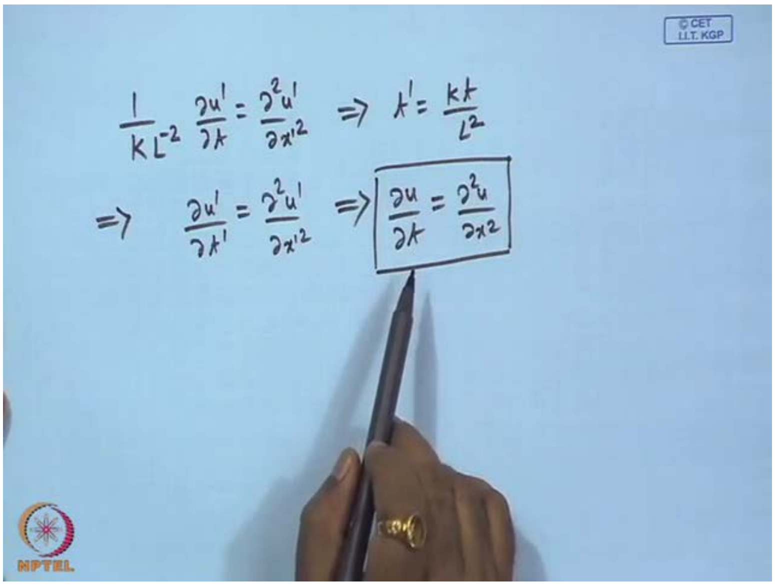

So, if we proceed like this, we get remember I have not non dimensionalised t. So, in order this to be non dimensionalised, we go for this the corresponding non dimensional time. So, when we go for this by dropping the primes, so you would have seen in many books with k, without k. So, that means you can arrive at such situation in with respect to non dimension variables, okay?

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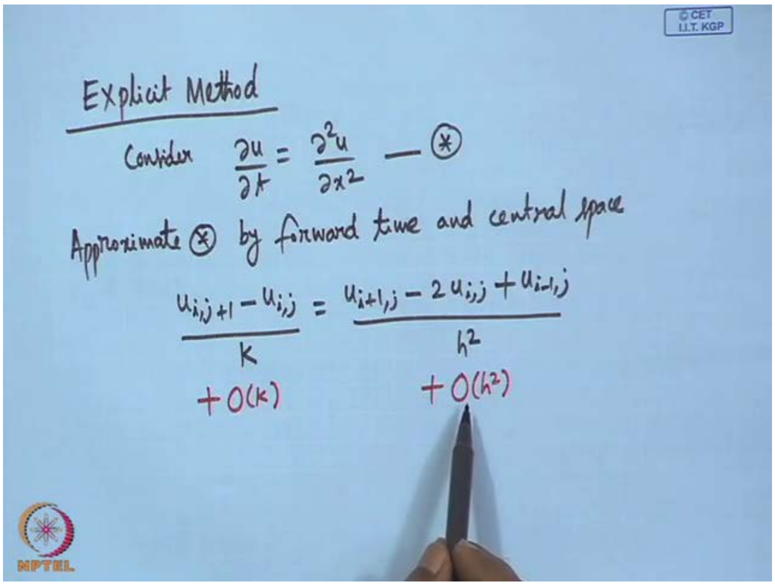

Now, we are ready to discretize this. So, let me give the heading explicit method. So, I want to give the heading and proceed, so consider the heat conduction equation, say this is star, then let us approximate star by forward time and central space. So, forward time central space by doing so u i j plus 1 forward time minus u i j equals this is central space. for the second derivative, h square. So, if I do not mention anything, it is not nice. So, better we also mention this, what are these for this approximation of the first derivative. This is the leading non zero coefficient and correspondingly for the second order space, okay?

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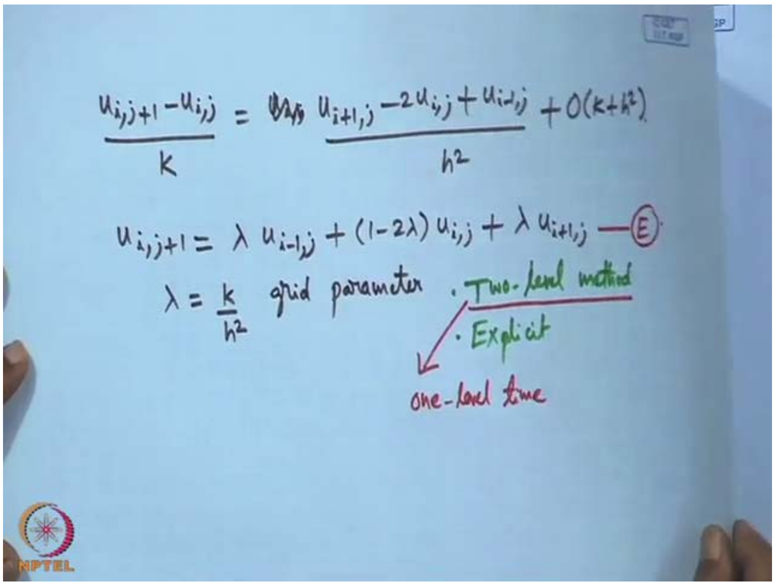

So, now this leads to j, so I can take it k plus h square. So, we will talk about this definitely, so I do not want to use the term, right? Now, what is this? So, by doing so we get this however we further can simplify, so u i j plus 1 equals to lambda u i minus 1 j plus 1 minus 2 lambda u i j plus lambda, remember? I am not doing any trick, just algebra. Look carefully why did we pull j plus 1 to the left hand side, the reason is straight forward. So, the first order approximation we have used for the time it is forward, so which involves two time levels that is j plus 1 and j and the second order approximation for the space involves only one time level, that is j.

Now, what is our aim? Given initial condition we want to compute the solution by marching past right, because this is in initial boundary value problem. So, in this case what we have to compute? We have to compute for all times the solution at various grid points. So, that means what are the time levels involved is an important issue because your discretized equation contains, let us say 5 time levels then we are lost completely, because you unless you know at least 4 of them, you cannot compute. So, with respect to time level we are trying to classify this method whether it is explicit or implicit, this is very much important, okay?

So, if you see this particular case this entire discretized formula contains two time levels only. What are they? j plus 1 and j which means essentially if you know the values at jth time level you can compute at j plus 1 time level. So, hence we call this a two level method and explicit because of this nature. So, what is remark this is a two level method and this is explicit in nature because of this reason.

So, two level means two time levels are involved, but in general in literature this may be because this is an important remark. So, when I say two level method means this involves two levels, otherwise strictly speaking the formula as it stands say this is E. So, this E is in some sense one level time, so because two level means the formula contains two levels only, but otherwise it just one level. That means is required at one level to compute at the other level. So, that is it in that sense you have to take it, okay?

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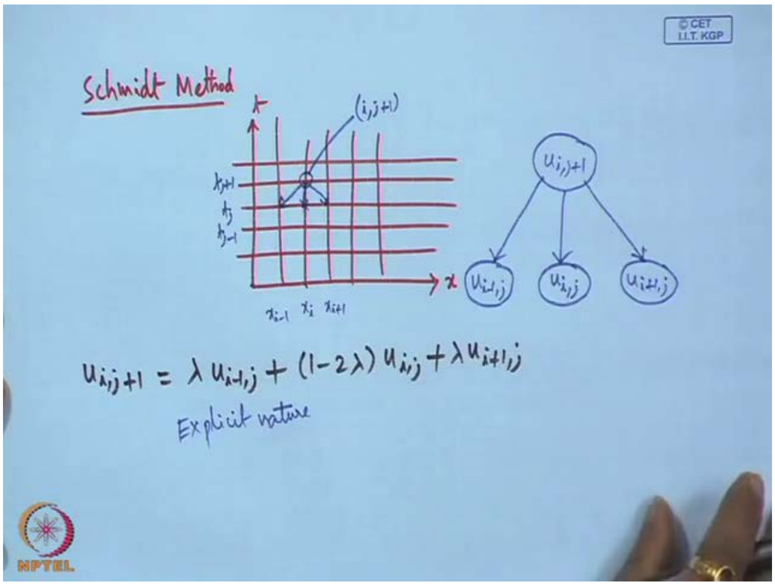

So, now this particular formula sits, let us see so this name of the method literature used is Schmidt method whatever formula we have obtained. So, this is called Schmidt method. So, it shows like this so you take any grid point, so this is x, this is t now. So, you have to mark so this is say x I, x i plus 1, x i minus 1 then say this is t j, then this is t j minus 1, this is t j plus 1. So, this is at i j this is a point at i j and this is a point i j plus 1. So, if you recall the Schmidt method, the formula is u i j plus 1 lambda times plus so this involves these points u i j plus 1.

So, let me mark and this is asking the function values at past time tap at which grid points at i plus 1 and i minus 1 so u i j, then u i minus 1 j and u i plus 1 j. So, this is what is happening here. So, this one this one and this one so this is explicit, because if you know the values at jth time level we can compute at j plus 1 time level. So, this is the explicit nature of this method, okay?

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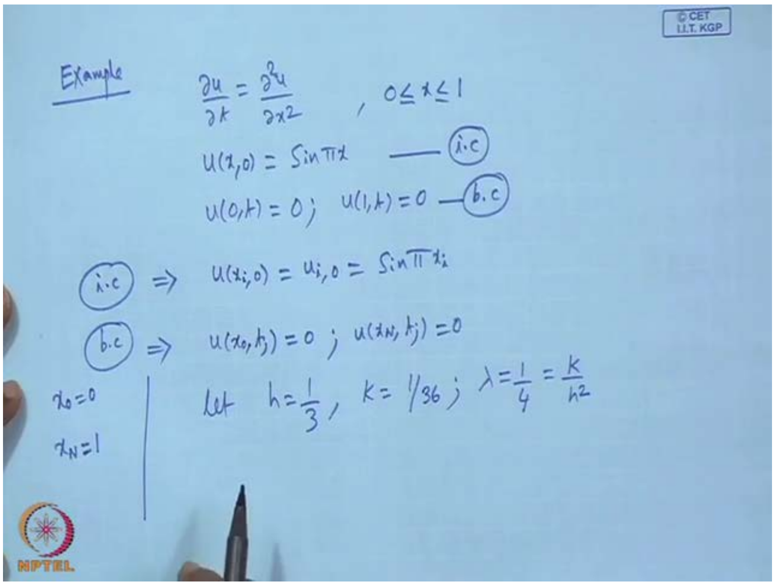

So, let us see with an example so u of x zero is sin pi x, u of zero t equals to 0. Now, see this is a onetime derivative and two space. So, u of we need boundary so this is two boundary conditions, why because this is second order with respect to space. So, 1 t this is also g 0 and the problem domain is now before we discretize so this is initial condition, this is boundary condition. So, let us discretize, so this goes to u of x i 0 u of i 0 in the present case pi x i and boundary condition u of you should be careful.

So, in this case what 0 corresponds to so remark x 0 is 0 and x n is 1 and what time step etcetera depends on the specifications. So, here x 0 and this is for any t therefore, t j is equals to 0 and this 1 x n for any t j equals to 0. So, this is the discretization, now it depends on the discretization the step size corresponding to x and then t the index will be decided. So, let us say h is given to be one third and k is given to be 1 by 36, then the lambda turned out to be, what is this? This is k by h square, the grid parameter, this is the grid parameter.

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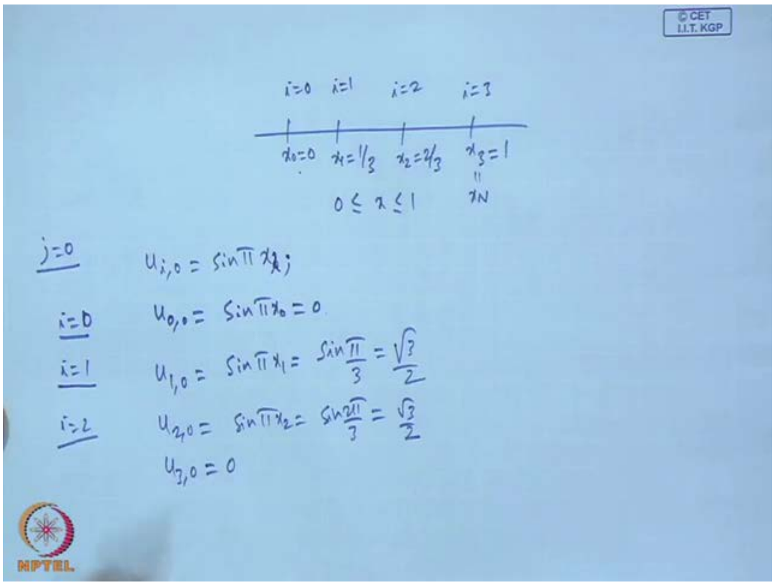

So, accordingly x 0 is 0, x 1 is one third, x 2 is two third, x 3 is 1. So, since our domain is this so this is x 0 and this is x n, right? So, consider j equals to 0, so then u i 0 is sin pi x 0, x i sin pi x i u i 0. Now, you start i 1, start with i 0, so u 0 0 then i 1 here. So, this is i 0, i 1, i 2, so i 1 so that means we are computing the initial values at each grid point, then i equal to 2 similarly 3 0.

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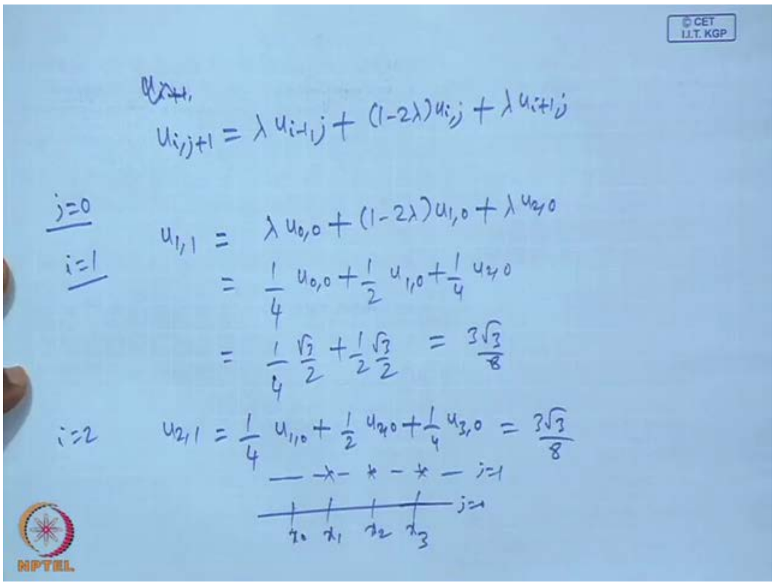

So, now if you consider your discretized equation u, the discretized equation u i plus 1, u i j plus 1 is lambda u i minus one j i j, then put j equals to 0, i equals to 1 because we want to compute the solution. The first value will be this lambda u 0 0, u 1 0, u 2 0, so this can be computed so this will be so similarly at next grid point u 2 1. So, this is 1 by 4 u 1 0, u 2 0, u 3 0. So, this also you can compute.

So, what is happening x 0, x 1, x 2, x 3 so using j equals to 0 values here, we have computed at j equals to 1 at each grid point. So, using the values at first time level we can compute that second time level because it is explicit. Once we know the values at each grid point, at a particular time level you can march past and then compute at the higher time level. So, we will discuss more problems so that we get complete idea until then good bye.

Thank you.