Numerical Solutions of Ordinary and Partial Differential Equations
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Lecture - 14
Predictor – Corrector Methods

Good morning, I hope you have done some exercise on understanding the multistep methods. So, we have discussed in previous lectures both explicit and implicit multistep methods. So, in case of explicit, if you are given past points then one compute the value at a particular grid point. And in case of implicit, of course we should know value at that point. However, that can be computed initially using some explicit method.

Now, what is the main purpose of these explicit implicit methods? So, when you have explicit, anyway you get the solution up to some accuracy depending on the order of the method. Now, in case of implicit definitely the main motto is to really improve upon, otherwise there is no fun in having an implicit method, where your right hand side also requires the value at that grid point.

So, what is a real motivation? The real motivation is to improve upon the solution. So, let us see what this combination is called, what is the combination? Yes, explicit implicit, so if this combination is used. So, what is this called? This is called predictor corrector methods. So, let us discuss what is the motive behind predictor corrector methods?

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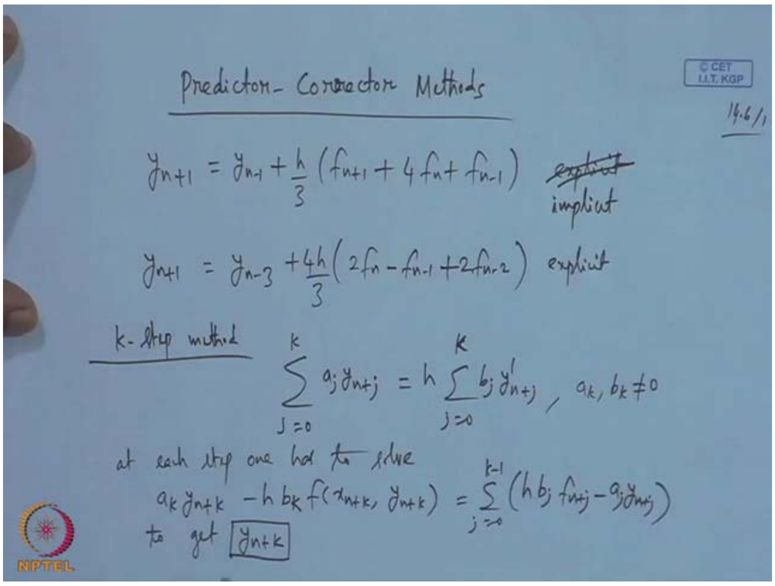

So, for example we have explicit method, so this is explicit, sorry this is implicit because we have fn plus 1. So, this is explicit method. So, a general k step method as we have seen where a k and b k non zero. So, when this is given what is a task at each step? One has to solve. So, one has to solve this what for to get y n. So, this is the general k step method and at each step at each grid point one has to solve. So, I plugged out the y n plus k coefficient and here of course, so this is obviously an implicit method, okay?

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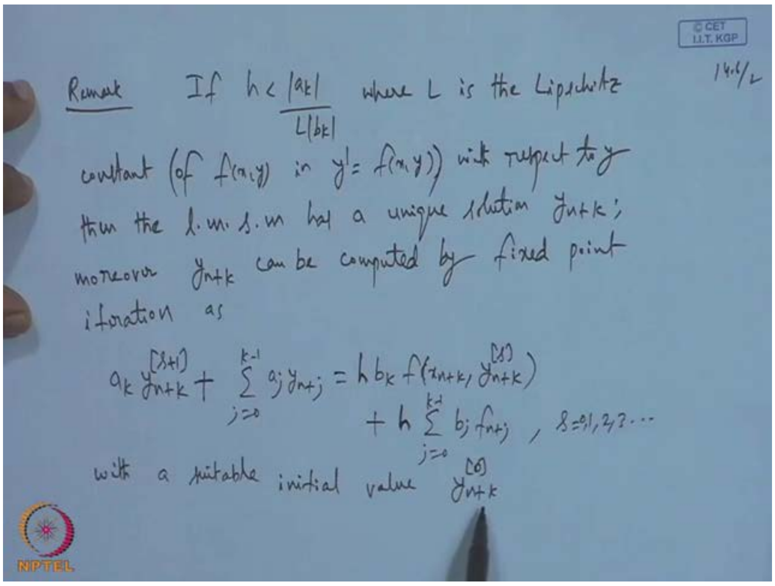

So, when we have to do this the remark if h is less than by L b k, where L is the Lipschitz constant of f of x y in this one with respect to y, Lipschitz constant with respect to y. Then the linear multistep method has a unique solution, y n plus k. Moreover, y n plus k can be computed by, see if you look at it y n plus k and you have a processor, right? So, this resembles some kind of fixed point iteration so computed by fixed point iteration as follows a k.

So, this is a processor which is demanding y n plus k, in fact s starts from 0. So, this is however with a suitable initial value look. So, this is our multistep method implicit, so to essentially we need to compute y n plus k, but the processor demands y n plus k. Therefore, it has to be solved like a fixed point iteration method, but then in order to solve we need some initial guess. So, that is suitable initial value. So, this is a implicit method, okay?

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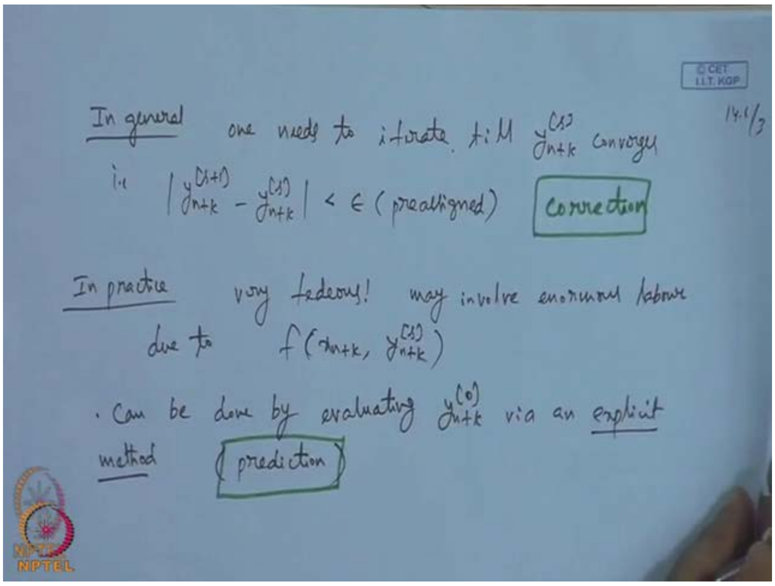

So, now what happens in general and in practice? In general one needs to iterate till y n plus k s converges, that is obviously the difference between two consecutive is less than epsilon, which is a pre assigned. So, this is in general see once you have such method, you plug in some initial value and compute. See for example, s is 0 we know initial y n plus k 0, you put it here. Compute, then right hand side will give y n plus k 1, put it there for s equals to 1, then we get 2. So, we compute, but the question is how long we do it? How long we do it?

We do it until the difference between two consecutive is less than epsilon, that means it agrees up to decide across e pre assigned. This is in general, but in practice very tedious may involve enormous labor due to f of x n plus k, right? So, if this is a case then what one should do it? See, your method demands that you have to time and again compute the function f with some initial guess, then put it back and then you get some improvement and then again you compute f and then you compute y. So, we are doing these processes, but then if your initial choice is not really good enough then you will be forced to spend lot of labor on it. So, definitely it depends on initial choice, that means one should not go blindly some initial guess, okay?

So, let us see what should be done. So, this means so this can be done by evaluating this via an explicit method. So, it make sense since we need reasonable enough approximation, you do it by explicit method. So, this is so called prediction and then this particular thing, this is correction. Why correction? We compute until the difference. So, you keep on computing until the difference between two consecutive agrees to some accuracy. So, this is the correction, okay?

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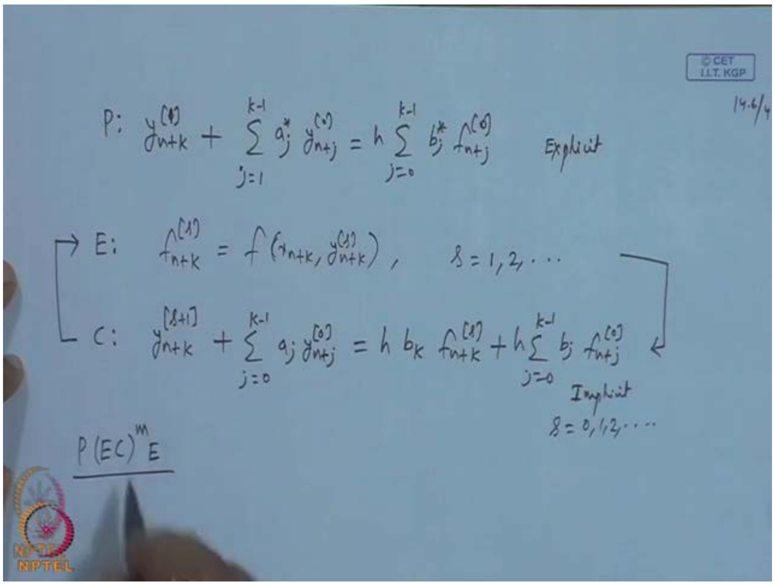

So, let us see the how the general procedure evolves. So, prediction, so I put stars because the coefficient have normalized. So, this is an explicit, so we need so 0 fn. So, the past values up to k minus 1. So, this is an explicit which is used for prediction. So, the moment we use something for prediction, we need to evaluate f. So, we evaluate, so let us say initially we compute by explicit method, then we need a some initial values. So, this should be 1. So, then we need some initial values. So, once we compute we evaluate this, then we need a implicit look.

So, this is implicit, so we need some initial then we evaluate. Once we know initial, we put it here then we compute one, then we compute f for by putting this we get f, then that we will put it in here. So, this process 0, 1, 2, etcetera. Now, once we do it so then again we evaluate, so we are doing this process, right? So, first we use explicit method predict, then you evaluate this then whatever f that has been evaluated you refine it. So, then again you use the implicit method to compute y n plus k at refined, then again put it and compute f and then put it back, refine it. So, we do this processes, right? So, this suggests a prediction evaluation correction, a finite number of times then we are ready with the evaluation, so this is the method, okay?

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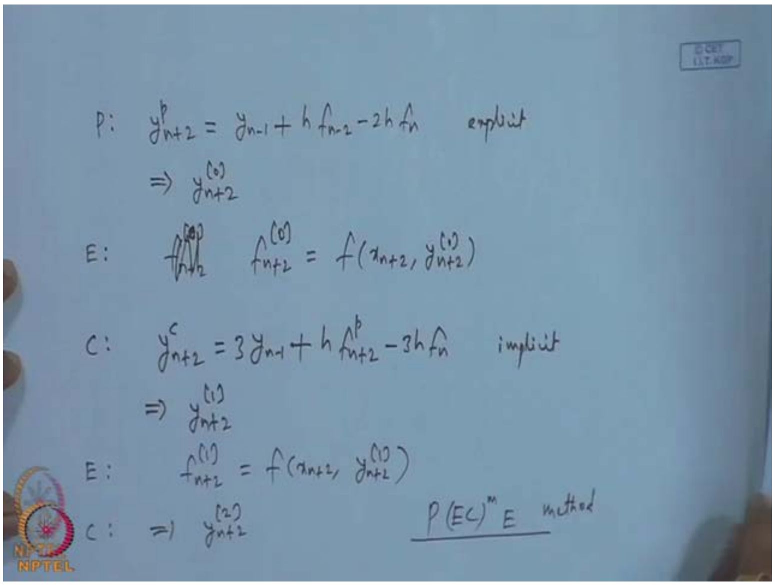

So, for example, for example, suppose we have so using this we get predicted value then we evaluate. Of course, to start with, f n is computed. So, we need some initial value f n, f n minus 2 and we compute this, then f n plus 2, this will be f of we compute this, then we go for the correction. So, let us introduce this notation. So, it is like this.

So, let us see this carefully, we have an explicit method, we have an explicit method using which we compute an initial approximation that we are denoting by y n plus 2 0. Then f n plus 2 0 is computed, then put it here in the correction formula because why this is implicit. So, whatever we compute, put it here to get a refined one. So, using this what we get? Once we obtain, then you evaluate, then we correct to get. So, we continue like this. So, this is PECME method.

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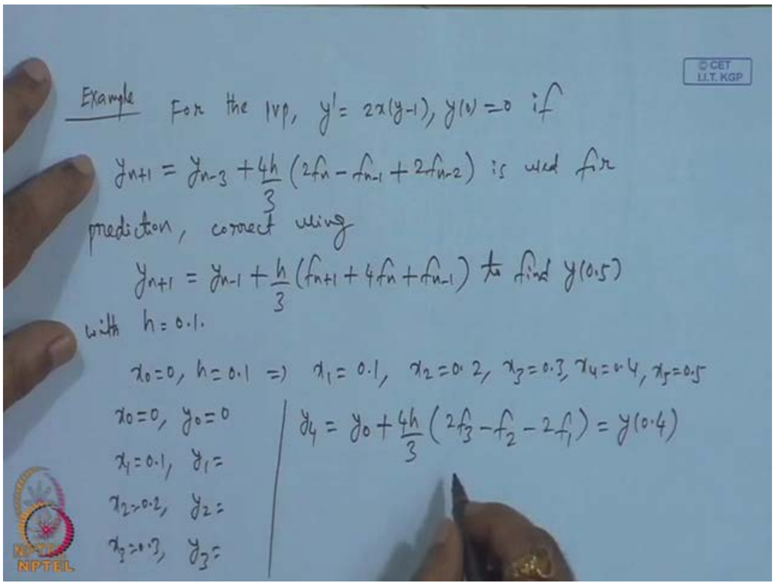

So, let us see how this works out with an example, for the IVP if so this is an explicit method which is used for prediction, correct? Using this implicit method to find y of 0.5 with h equals to 0.1, but the story does not stop there, the reason is look at it. So, this requires more data or at least one statement is missing. What is that? Let us understand what was our x 0 and with h, h is 0.1. This implies x 1 is 0.1 0.2 and it was asked to find y at 0.5.

So, that means we need some data initial and let us look at the prediction. So, this is to compute n plus 1, we need n, n minus 1, n minus 2, n minus 3. So, one would expect a statement even to use this explicit method, you need this values n, n minus 1, n minus 2, n minus 3. So, we have y 0. So, what is missing? So, x 0 is 0, so then y 0 is 0. We need x 1 0.1, y 1, x 2 0.2, y 2. So, once we know this then let us go for the value.

What is the value? Latest we can compute, see n 3. If n is 3, we get y 4 equals y 0. So, we need this data to compute this, see that means using explicit method first we need to compute y of 0.4 then correct it. Correct it until the desired accuracy, then we stop, then use that to compute 0.5 at y of 0.5. So, that means the missing statements, missing statements like this at least several alternatives, okay?

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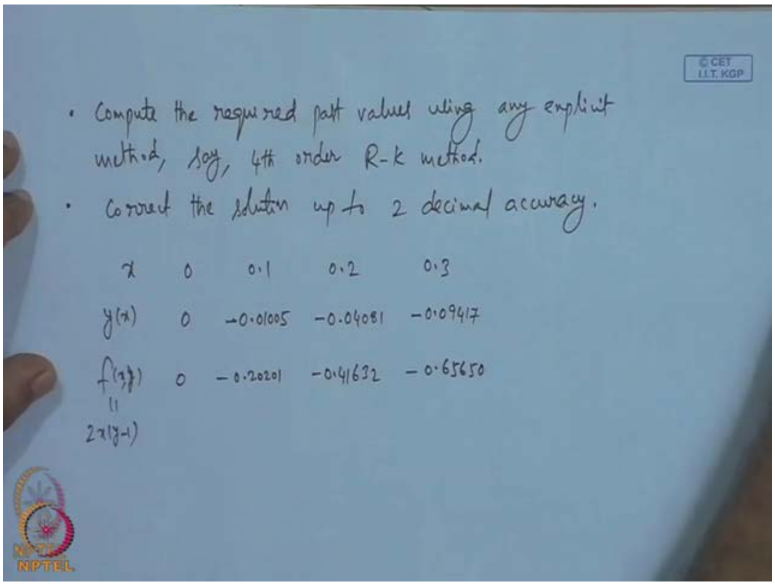

Compute the required past values using any explicit method, say fourth order R K method and correct the solution up to two decimal accuracy. So, these two statements are required. Now, the problem is set what given an IVP, this is the explicit method which is used for prediction, then this is used for correction which is an implicit of this nature.

Now, the past data is required. So, you can compute just to save time, I am just applying the past data. So, this is y, but if you look at our formula it will expect f. So, that means accordingly for a given f we need to compute this. So, for the given f, I am talking about f x y equals to 2 x y minus 1. So, for this value this is, so remember these are to be computed using any explicit method using fourth order R K, but just to save time I have given the data, okay?

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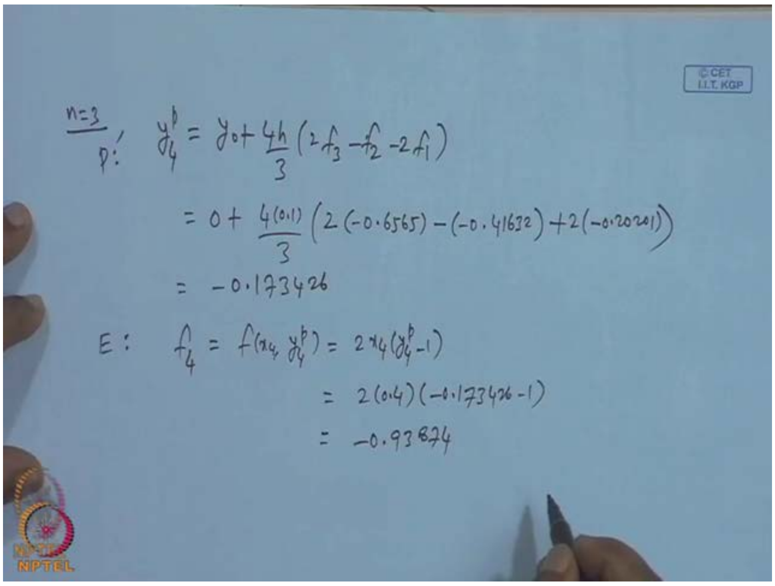

So, now let us go for predicting y 4 using this. So, let us go for n equals to 3, case y 4 p equals y 0 4 h by 3 2 f 3. So, this is y 0 0 2 f 3 minus f 2 2 f 1. So, this you may compute. Now, once we have predicted so this is p part. Now, we need e part, what is e? We have to compute f 4. So, this is f 4.

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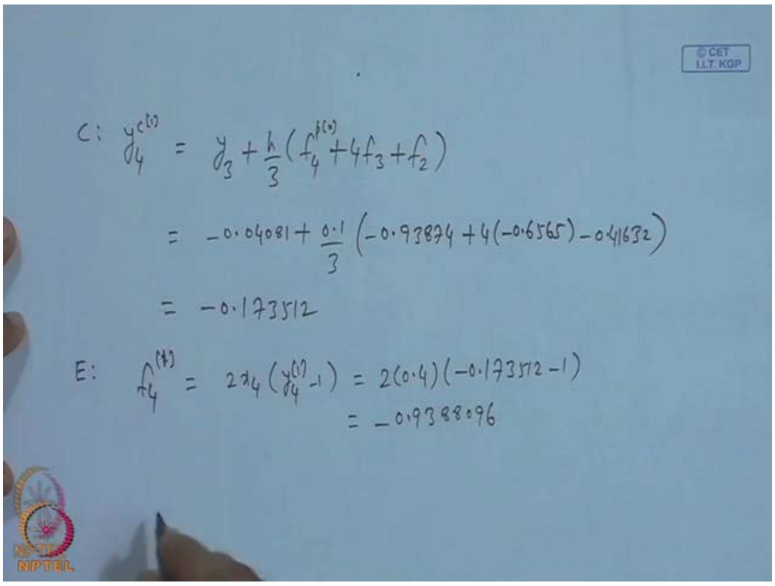

Now, what next yes we have do the correction, so we do the correction using y 4 c. Of course, 1, so this is y 3, so this is and f we have computed, f 4 is in the predicted. So, this is corrected, now this is corrected first iteration, right? Now, what we should do? We should evaluate f 4, f 4 1. So, this is so f 4 now once we have a f 4 1 what we do we should correct it, okay?

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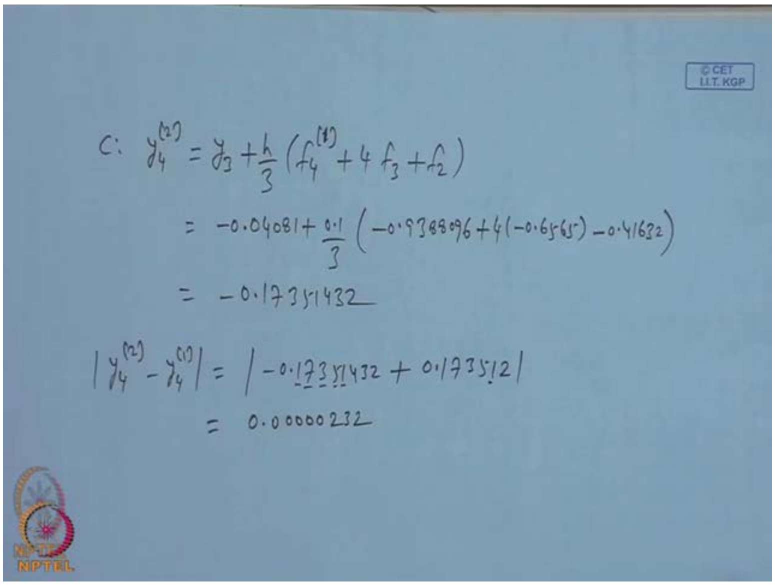

So, how do we correct one, so this is now we have corrected two times. So, let us compute, so this we are getting 0.12345, oh that is pretty. So, 5 1 up to then this is 2, so this is agreeing up to 5 decimals. In second correction of course, we have very fortunate here, but in general this may not be the case.

Suppose, you suppose a best check could be you start any random value, do not use any explicit method, just take any random guess as initial and then you put it in your correction and then try to keep on correcting. So, this is a one solution, let us say you have corrected five times and another method you take say explicit method and then you compute the value and correct it again five times and try to compare. See, you will really find the difference, sometimes one may be forced to put lot of labor. So, that means we need some considerations under which the method really converges fairly accurately, okay?

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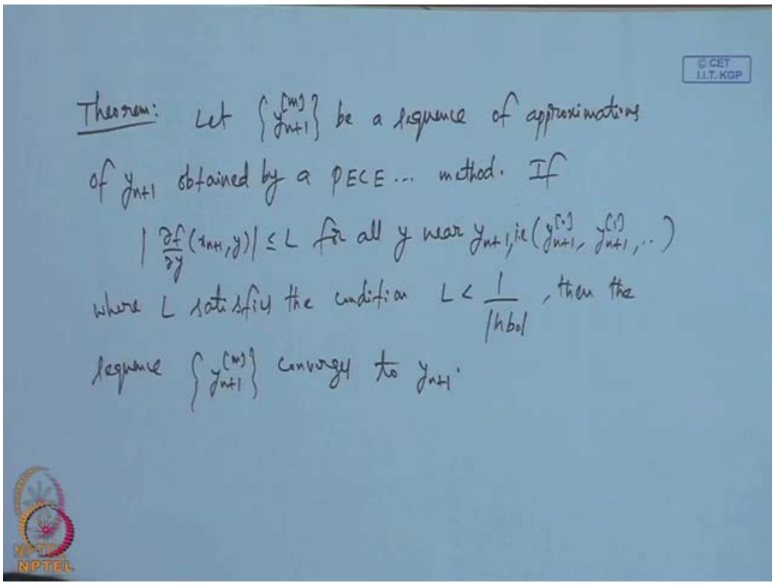

So, let us see what are those considerations? So, essentially we are considering this sequence, so this is lipschitz condition. So, for all y near y n plus 1, that is y n plus 1, 0, 1, etcetera where l satisfies the condition 1. So, this is the coefficient in the multistep method, then the sequence converges to y n plus 1. So, this condition, so this coefficient is important. So, with respect to the general multistep method this coefficient so we should be careful how we have defined. So, this is coefficient of f n, so this is guarantees the convergence, okay?

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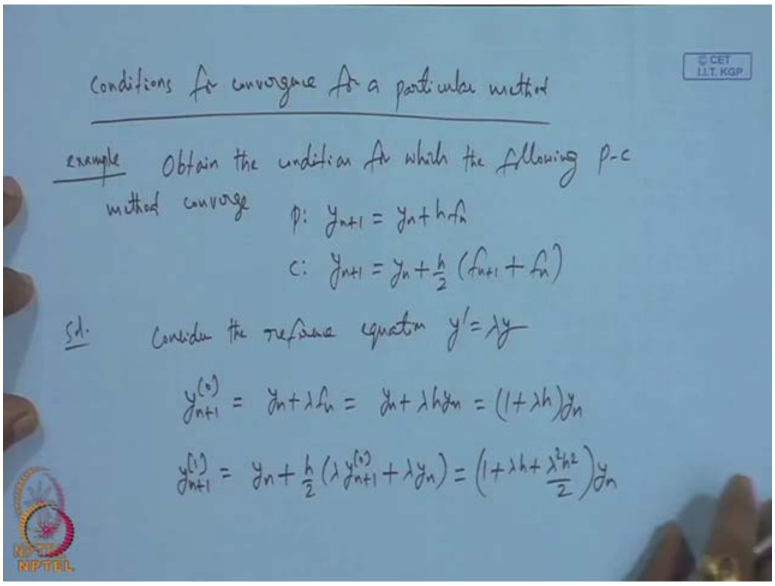

So, let us consider conditions for convergence for a particular method. So, this is to know how to obtain convergence for a particular method. So, for example, obtain the condition for which the following P C method converge, what is the pc method? Prediction is simple correction is modified, right? So, we consider the reference equation, this is reference equation then prediction gives y n plus lambda f n. So, this will be so this is the prediction, now correction. So, naturally this is this corresponds to prediction, but we have predicted. So, we substitute, so then we get suppose we continue the process.

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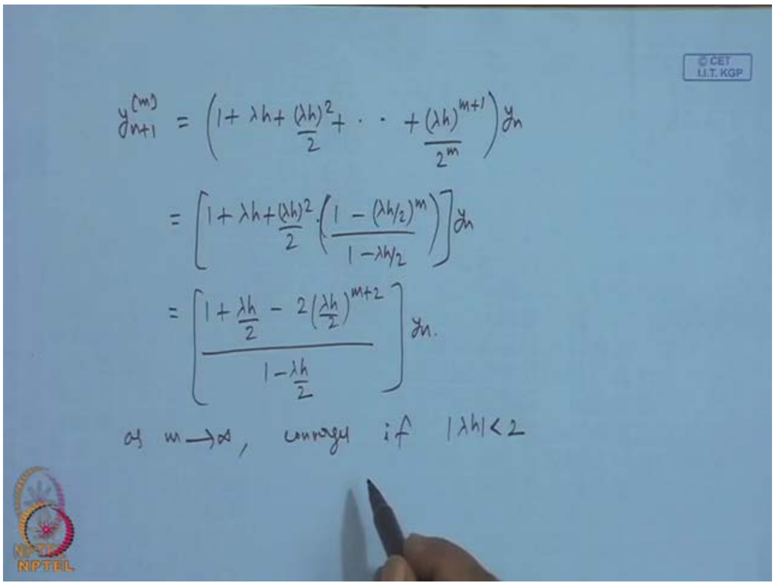

That means one can easily show if you continue we get this. So, we get we can simplify this series using sum of the series formula. So, this is the correction m times as m goes to infinity, this converges. If we can get the conditions from this, obviously the condition is if so from this one can obtain the corresponding convergence region, this condition. So, this is not region, so converges if this condition satisfies. So, having obtained this condition it would be a good idea to compute what kind of a errors involved at each stage, that means you correct a 3 times, what is the error, correct 4 times, correct 7 times, what could be the error, okay?

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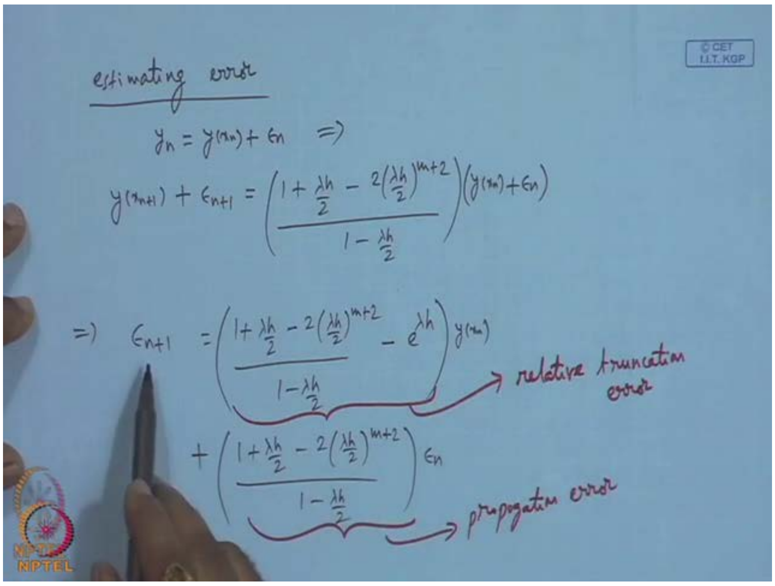

So, let us try to assimilate that so estimating error, so we have y n equals to so this notation we follow. So, from the previous I have substituted for this and for this, the corresponding exact process error. So, this implies minus because exact we know y of x n plus 1 is e of lambda h into y of x n. So, with that notation now what are these two terms. So, this you see this is the due to the approximation from the method and this is the exact.

So, the difference is so this is relative and what about this error at n stage is magnified by this factor for the error at n plus 1 stage, okay? So now we would like to analyze this relative truncation error, you correct it once, correct it twice, correct number of times, what will be the error.

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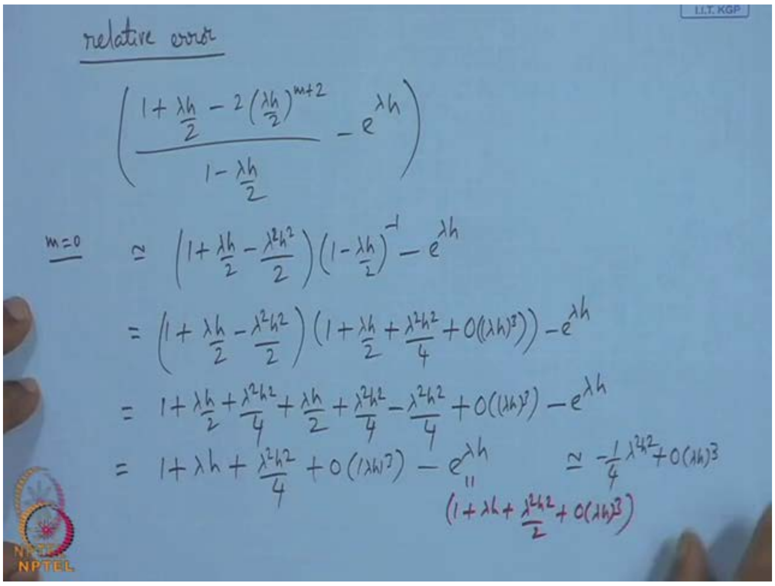

So, let us consider the relative error. So, the relative error is this is given by, so consider the case m 0. So, then this can be written as so this will be m 0. So, lambda square h square by 4. So, there is a 2 there, so this is so I have expanded only two terms. So, this is coefficient of this then so higher order I am not writing plus. So, this is 1 plus these two terms, lambda h then these two terms. We get from these two, one get cancelled and we are getting 4 plus minus.

Now, if you see this first two terms get cancelled here and what is the third term third term? So, I would like to expand, so this with a minus in front. So, if you see minus get cancelled, this get cancelled and you get minus. So, this is a one fourth and this is a minus sign. So, minus one fourth, so this is of order, what I am trying to say is this is not 0. So, this is not 0, so this is non zero coefficient of lambda square h square. So, if you take m equals to 0 then the error is like this, okay?

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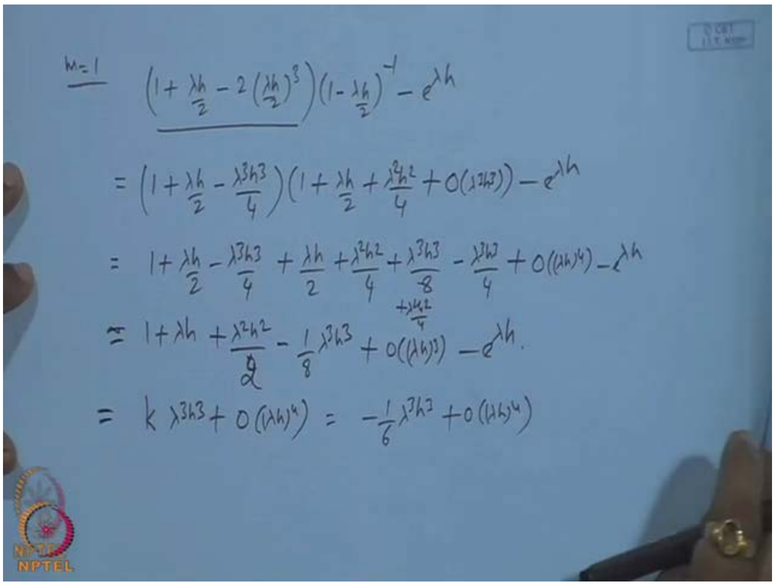

Suppose, you take m equals to 1 then this will be this 1 minus minus. So, this is 1 plus, so this will be lambda cube h cube by 4 1 plus. So, this is a cube term, so I can write it then there is a so one can show this, so this you get lambda h and plus h square by 2. I need this is cube there. So, by 4 there is another, so this let me check. So, once we get from here and then there is another 4.

So, there is another term and this is by 4, so 2 times, so this I have to add one more this, then from these two. We get this is we get minus 1 over 8, you may check, if I am making a mistake with the coefficients, but this tells the procedure. Now, you can see they these three terms get cancelled with this. So, you get some non zero coefficient say some k. So, probably we may get minus 1 over 6 lambda cube h cube. So, this is m equals to 1.

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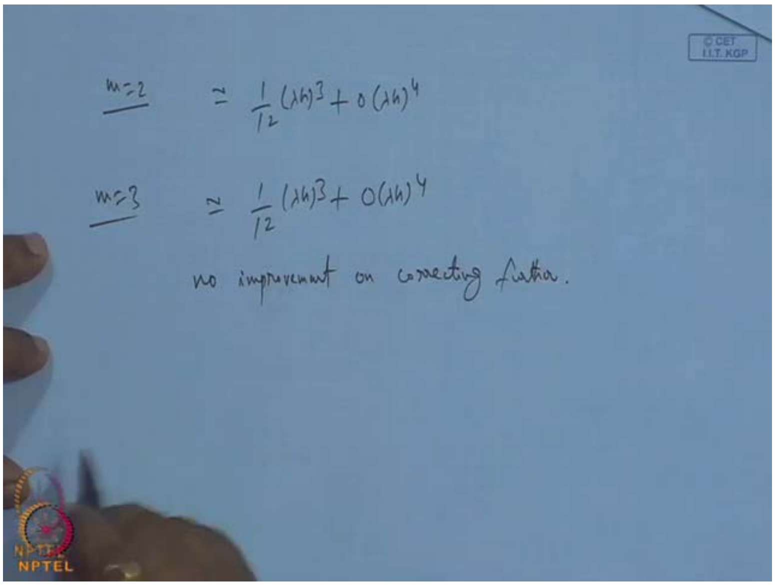

So, if you continue m equals to 2, we should show that some coefficient and in particular for this method if you continue, we can verify subject to this coefficients, but that means there is no improvement on correcting further. So, this gives a very a nice estimate for this particular method, if you keep on correcting, it tells that there is no point in correcting beyond certain stage.

So, with a reference equation if you test it, so then for a general formula even if your f is quite complicated, you need not correct beyond because there is no point in correcting beyond certain stage. And the certain stage here for this particular method is m equals to 2. So, the general predictor corrector methods works in this way, so we need explicit method for predicting, implicit method for correction and we correct up to desired accuracy. So, let us work out a few more details in the next class.

Thank you.