Numerical Solutions of Ordinary and Partial Differential Equations
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Lecture - 12
General Methods for Absolute Stability

Hi. In the last class, we have discussed about root condition, which has direct implication on zero stability of the method. So, before we proceed further on some more things related to stability, so let us state root condition as a necessary and sufficient condition.

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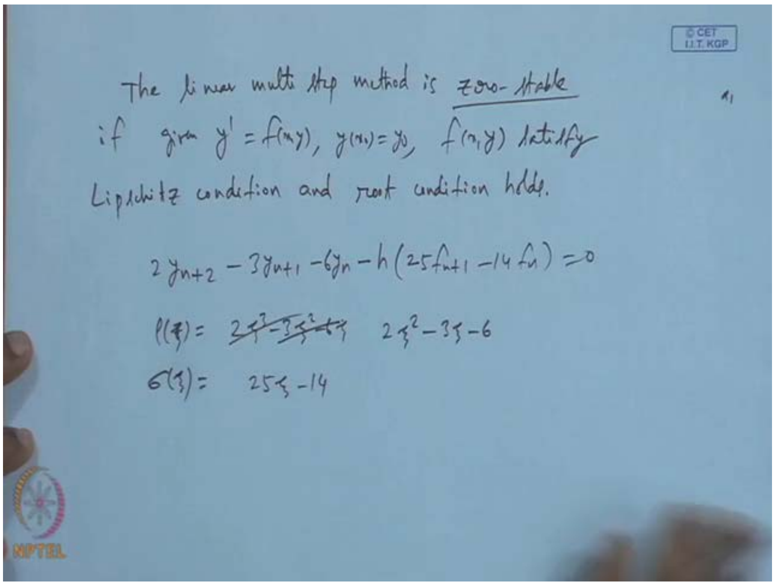

So, we have not stated that. So, the linear multi step method is zero stable, if given y dash equals to f of x y f of x y satisfy Lipchitz condition and root condition holds. So, that means the linear multi step method is zero stable if given this f satisfy Lipchitz condition and root condition holds. So, as I mentioned for a given method say 2 y n plus 2 minus 3 y n plus 1 minus 6 y n minus h 25 f n plus 1 minus 40 f n equals to 0, we can write, so 2. So, this is n plus 2. So, 2 zeta cube zeta, this is n plus 2, so this is not, this will be 2 square minus 6 and this will be this will be…

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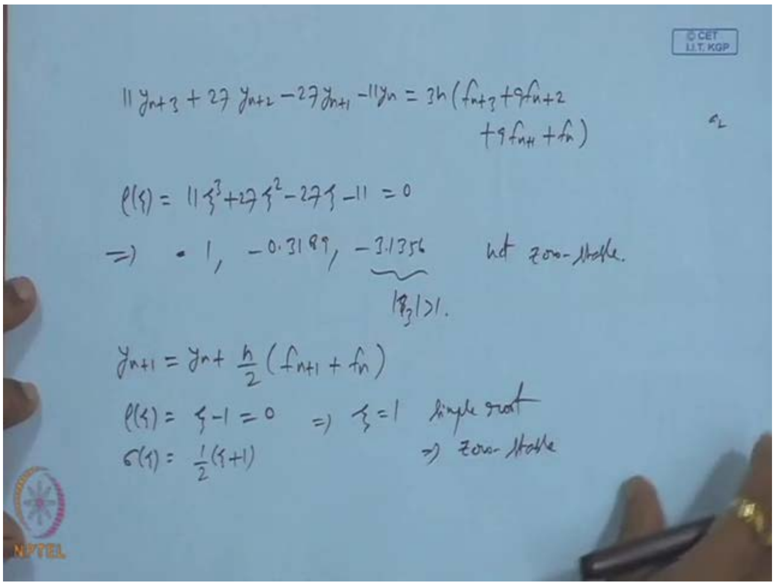

So, that is how we identify. So, now consider another example let us say so the first characteristic polynomial 11 and the roots if we compute. Now, what is the root condition says root condition the roots must be within the unit disk, but unfortunately for this greater than 1, therefore this is not zero stable. So, we can compute accordingly. So, for example, consider so for this is so this implies simple root hence zero stable. So, that is how we compute.

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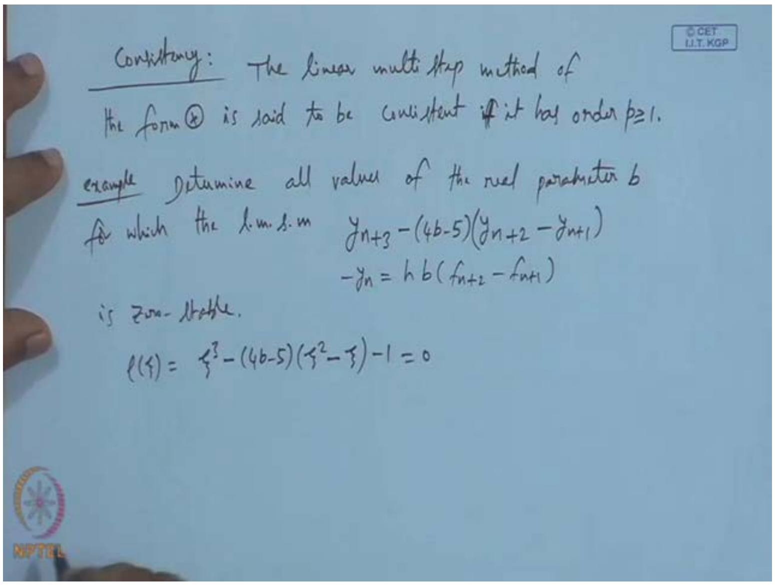

So, let us proceed further. If you recall for convergence, we have defined two things, one is stability and another is consistency. So, the linear multi step method of the form star is said to be consistent. If it has order 3, greater than 1, so these two are together necessary for convergence. So, before we go for further concept of stability, let us do some example. Determine all values of the real parameter b for which linear multi step method. Determine all values of the real parameter b for which this method is zero stable. So, let us identify rho of zeta. So, rho of zeta for this method, look at the distance plus 1 and plus 2 and plus 3, so this is our first characteristic polynomial. So, this can be written as this.

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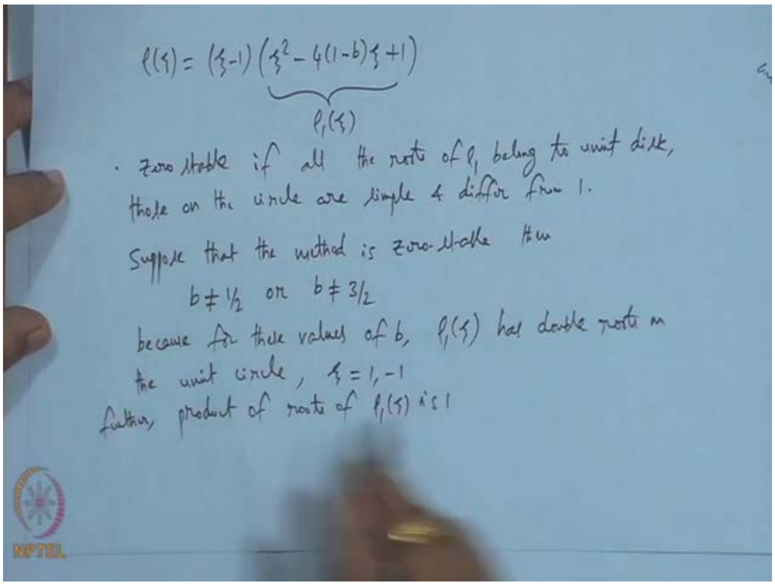

This can be written as, so call this, now the method is zero stable if all the roots of rho one belong to unit disk, those on the circle are simple and differ from 1. Why? This is because already we have zeta equals to 1 as one root. If you have multiple roots, then one of the necessities you have seen of the modulus is greater than 1. Then it blows up and multiplicity, therefore for rho 1, these are the conditions. The method will be zero stable. If this happens, hence suppose that the method is zero stable, then b is not equal to 1 by 2 or b is not equal to 3 by 2 because for these values of b, rho 1 zeta has double roots on the unit circle, that is zeta equals to 1, minus 1 for these values of this further product of roots of rho 1 zeta is 1, further product of is 1. Then but since these values are not equal, roots are none of them are 1 or minus 1, neither of them are plus or minus 1.

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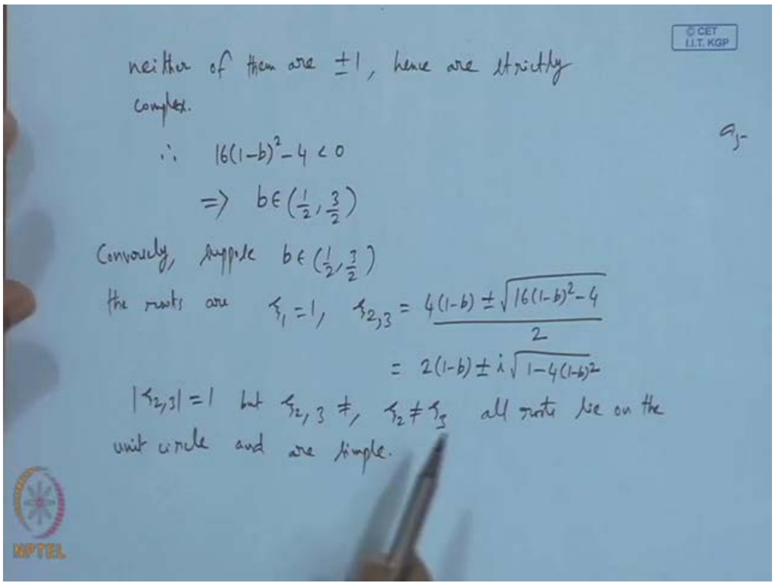

Hence, they are strictly complex. Therefore, the discriminant negative, this implies, this implies b belongs to half, 3 by 2 to b belongs to, so this is this interval for zero stability. Conversely, suppose b belongs to this. Then the roots are 1, zeta 2, 3. These are, so this is 1 and mod 2, 3, this is 1, but zeta 2 or 3 not equals to 1, zeta 2 not equals to zeta 3 and all roots lie on the unit circle and are simple. So, this is the analysis to identify a particular method is zero stable or not. In this particular case, we were trying to obtain the two bounds on the b, the range for which this method is absolutely zero stable.

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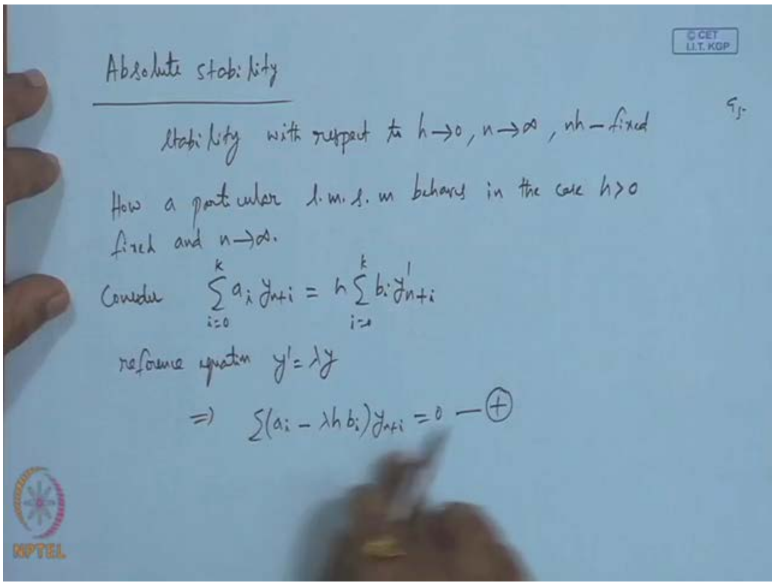

Now, let us proceed to something called absolute stability. If you recall for single step methods, we have discussed this concept. So, the stability that we discussed with respect to h goes to 0, n goes to infinity, nh fixed, so this was the stability. The zero stability h goes to 0, n goes to infinity, so that nh is fixed. Now, we need to analyze something different, how a particular linear multi step method behaves in the case h greater than 0 fixed and n goes to infinity.

So, how a particular linear multi step method in the case h goes to 0 fixed and n goes to infinity. So, absolute stability with reference to always a particular reference equation, so that is our y dash equals to lambda y. Now, consider our method equals to h multi step method in this form, let us say both i equals to 0 to k; suppose we considered in this, then with reference to the reference equation, we get similar difference equation in this form.

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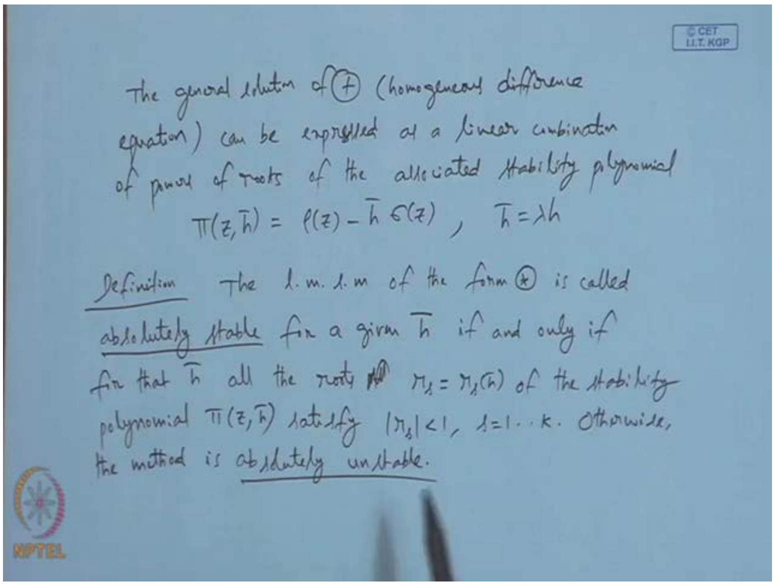

Now, the general solution of this, what is this? This is a homogeneous difference equation or a recurrence relation can be expressed as a linear combination of powers of roots of the associated stability polynomial. So, this is a stability polynomial, first characteristic, second characteristic, and then one can define stability polynomial. So, this is possible. Now, the definition of absolute stability the linear multi step method of the form star is called absolutely stable for a given h bar if and only if for that h bar, all the roots all the roots say r s of the stability polynomial satisfy this condition, otherwise this method is absolutely unstable, absolutely stable unstable. So, what it says this linear multistep method is absolutely stable for a given h bar. See zero stability we defined with h goes to 0 as n goes to infinity. Now, absolute stability we are defining for fixed h.

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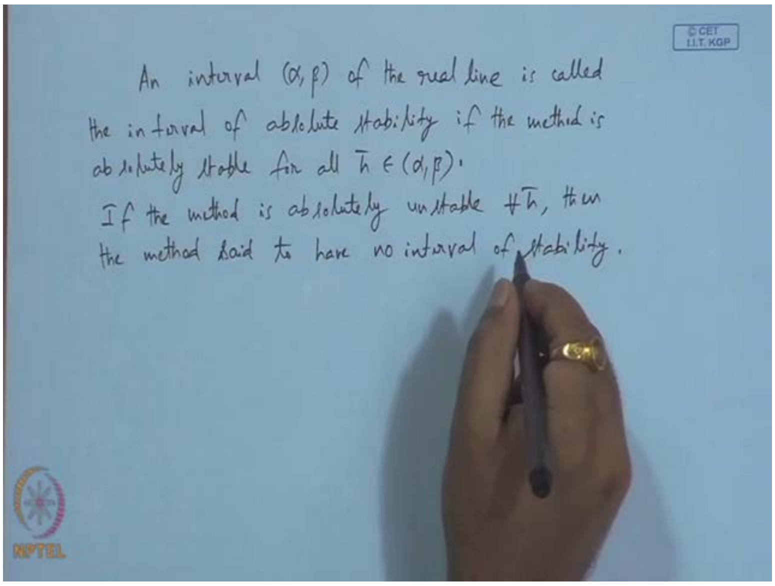

So, similar to single step method, it is our duty to find out an interval, an interval alpha beta of the real line is called the interval of absolute stability. The method is absolutely stable for all h bar belongs to this interval. Further, the method is absolutely unstable for every h bar, then the method is said to have no interval of stability. So, it is a very much essential to find the interval of absolute stability. So, what is the procedure?

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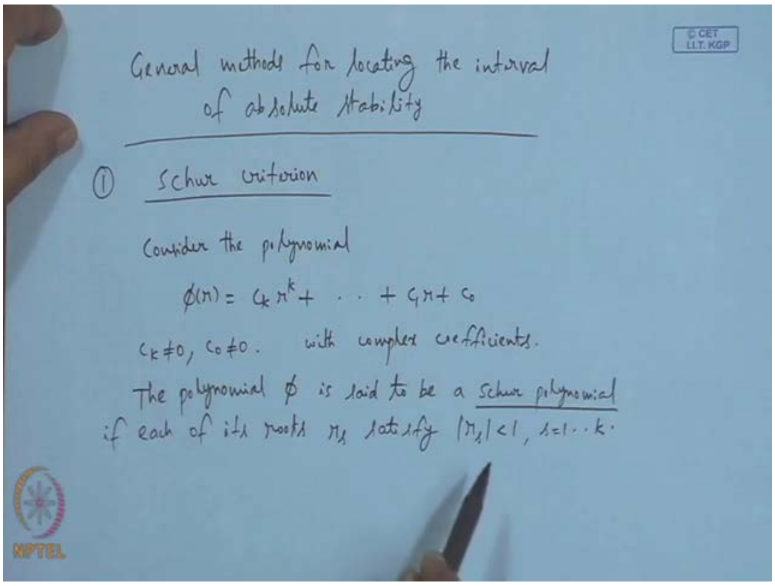

So, general methods for locating the interval of absolute stability, so the first one, first there are several approaches. So, let us discuss couple of them. So, the first one is schur criterion. So, before we state that some preliminaries, consider the polynomial phi r of the form c 0, c k non zero, c 0, non zero with complex coefficients, complex coefficients.

Then, the polynomial phi is said to be a schur polynomial, if each of its roots r s satisfy. So, the polynomial phi is said to, so first of all we consider polynomial where these are complex coefficients and remember c k non zero, c 0 non zero, then the polynomial is said to be a schur polynomial if each of its roots are satisfy. That means the magnitude must be less than 1. So, then we say this polynomial is schur polynomial.

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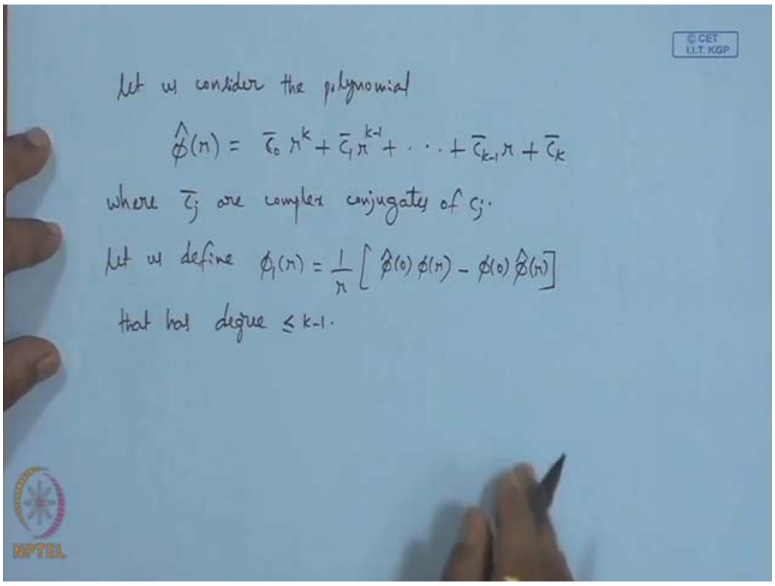

Now, let us consider the following polynomial. What is that? We denote it by phi hat c 0 bar. What are these bars, one can guess easily because the coefficients were complex. Therefore, c j bars are complex conjugates of c j. Please try to recall c k r power k, c k minus 1 r power k minus 1, c 1 r 0. Now, what is the polynomial that we are considering? Look c 0 conjugate has gone to r power k, c 1 conjugate has gone to r power k minus 1, c k conjugate has gone to r power 0, and so it is backwards. Then let us define phi 1 of r 1 over r phi hat of 0 phi of r phi of 0 that has degree less than or equals to k minus 1 that has degree.

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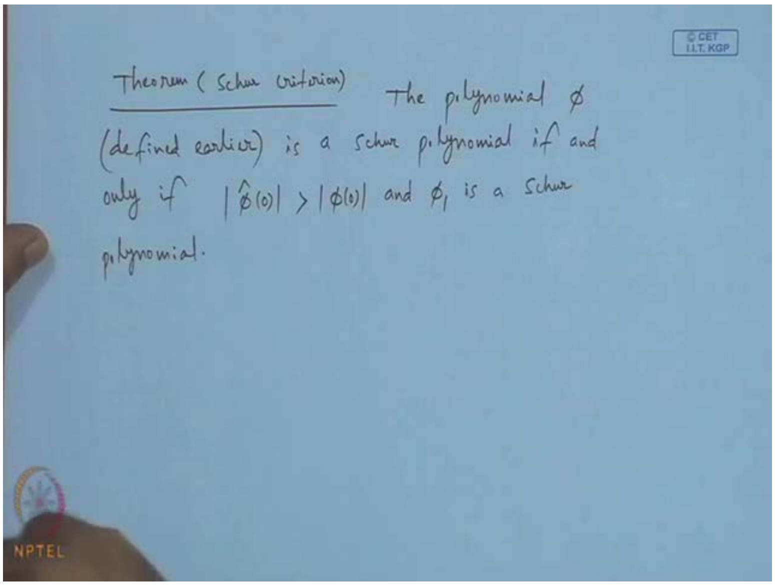

Then, this is the theorem. The polynomial phi defined earlier is a schur polynomial if and only if modulus of phi hat of 0 is greater than modulus of phi of 0 and phi 1. So, what it says that phi which was defined earlier is a schur polynomial, if and only if this condition holds and phi 1 is a schur polynomial. When do you say it is a schur polynomial? That means roots of phi 1 must be less than 1 in magnitude. So, let us check with an example.

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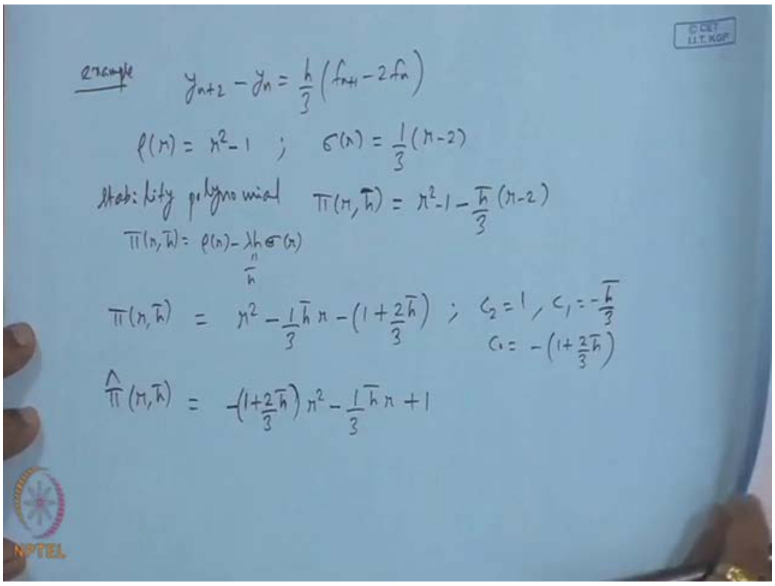

Consider this example characteristic polynomial phi of r square minus 1 because n plus 1 is missing, this is r minus 2. Hence, the stability, the stability polynomial, sorry r h is given by r square minus 1 minus the stability polynomial. See, we have defined like this.

So, this is h bar, so this minus so equal to r square minus, this is our stability polynomial. Now, we would like to use the schur criterion. So, what it says? So, this is our polynomial. Now, consider this as our relative polynomial. Then we have to compute phi hat where the coefficients are conjugated and given backwards. Therefore, pi hat is given by so this is c 2, c 1, and c 0.

Now, we have to consider the other way. So, c 0 r square, so this is, this is our phi hat. Now, what should be checked? The polynomial phi is a schur polynomial if and only if phi hat of 0 moduli is greater than this and phi 1 is a schur polynomial. So, let us try to do that. So, what was the, so we have pi like this, pi hat like this. Now, we have to check mod, mod pi of 0.

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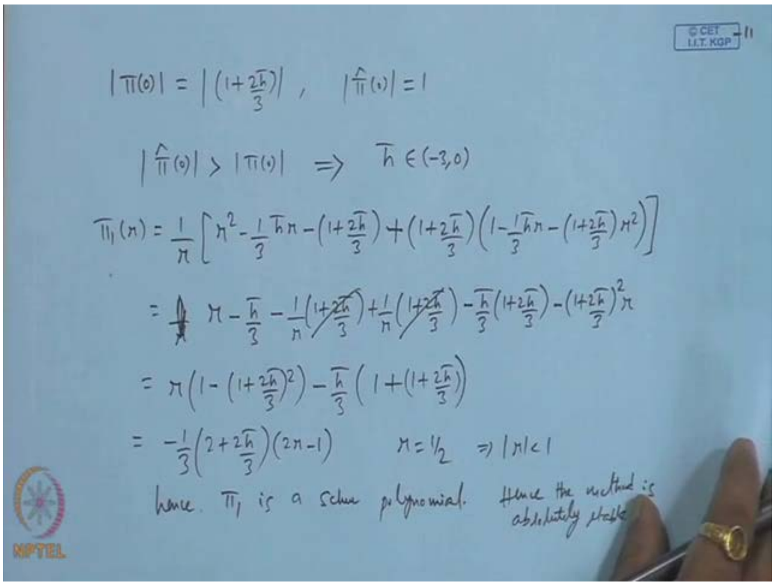

This is mod 1 plus, then so the condition mod is greater than this implies h bar belongs to, so this can be verified very easily. Now, what is the next step? So, this is the range that means the polynomial phi is a schur polynomial if and only if this happens. That means the polynomial is schur polynomial only when h is in this range. Further, we have to check phi 1 is a schur polynomial. So, let us check.

Now, pi 1 r is 1 over r. So, we are computing as defined pi 1 r, we are computing pi 1 r. So, pi hat 0 to pi hat 0 is 1, so times phi of r that is phi of r, this is exactly this minus pi of 0, this minus 1 plus 2 h bar by 3, 1 minus 1 by 3 h bar r. So, this is 1 over r, r gets cancelled. So, we can just put r minus h by 3 minus 1 over r, then plus 1 over r with 1 there, then minus h by 3, 1 plus this.

If it gets cancelled, then this 1 minus 1 plus 2 h bar by 4 square r, so this is so r 1 there and this is the coefficient. Then these two get cancelled. Then minus h by 3, if I take common 1 plus, so this can be written as so this is pi 1 r. Now, the definition is polynomial pi is said to be schur polynomial if each of its roots satisfy modulus less than 1.

Now, further the schur criteria says the polynomial phi is a schur polynomial if and only if this and this is only a, so r equals to half, r equals to half is our root, so which implies, hence pi 1 is a schur polynomial, pi 1 is a schur polynomial. Now, three a multi step method of the form is called absolutely stable for a given h bar if and only if for that h bar, all the roots for a stability polynomial satisfy this. So, we have verified this. So, if it is a schur polynomial if each of roots satisfies, this it is interrelated. We have shown that phi 1 is pi 1 has this; therefore pi 1 is a schur polynomial. Therefore, the roots of their characteristic polynomial are less than 1. Hence, the method is absolutely stable.

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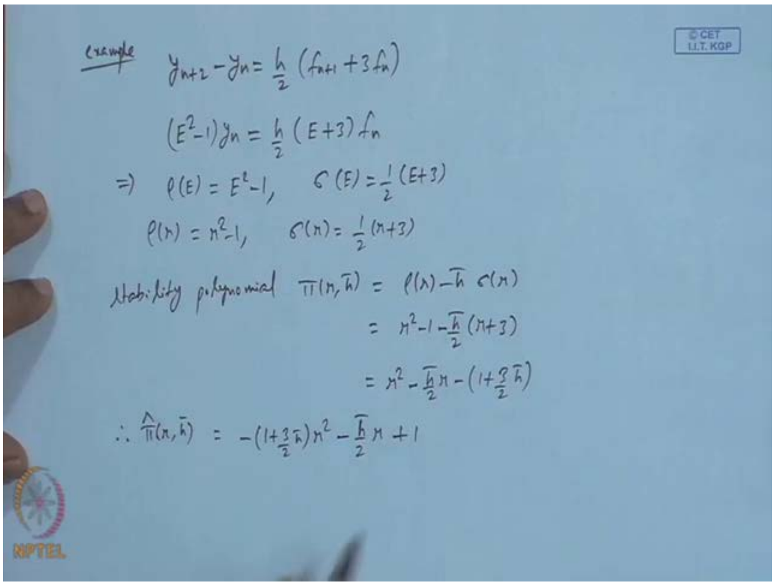

So, let us look at one more example y n plus 2 minus y n equals h by 2. Now, this can be written as so let us do everything in one go, E square minus 1 y n h by 2 E plus 3 f n. This implies rho E equals E square minus 1, so therefore plus 3. Therefore, now the stability polynomial, this is given by, this is given by this equals r square 1 plus 3 by 2. Therefore, pi hat of this r square r plus 1.

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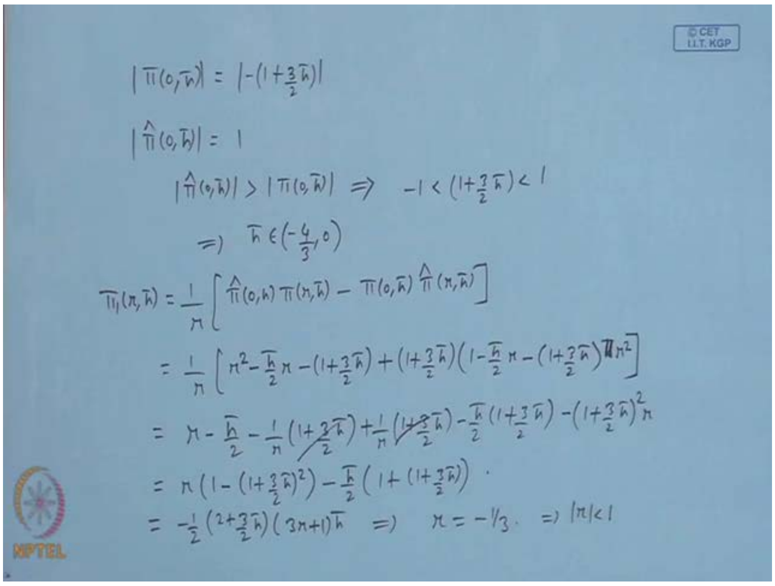

So, we have to compute this equal to 1. Then let us schur criterion says the polynomial pi is a schur polynomial if and only if this is 0. So, let us so implies this implies h bar belongs to this range. Now, on the next step is to construct this pi 1 and show that this is a schur polynomial. So, this is 1 over r. So, what is this polynomial pi 1? Pi 1 has to be constructed. So, we have to construct this, this is 1 and pi of r. So, this will be r minus this, this minus, so this gets cancelled.

So, this implies minus half 3 r, so this implies r equals to minus 1 by 3. So, this implies mod of r is less than 1. So, schur criterion says the polynomial pi is said to be schur polynomial if each of its roots modulus is less than 1. So, pi 1 is a schur polynomial, then so the polynomials pi is a schur polynomial because pi 1 is a schur polynomial. The polynomial phi is a schur polynomial and accordingly what it says for absolute stability, these stability polynomial roots are within modulus unity.

So, this is schur criterion, helps one to conclude the region of absolute stability. For example, when we say h bar that is lambda h is within the range minus 4 by 3 to 0, one must choose a combination of lambda and h because for a fixed h, you have to choose a lambda such that the lambda h is within the range. So, this trade off, suppose lambda, suppose for a given lambda, one cannot choose h, which makes the lambda h beyond this interval. So, the schur criterion is very much useful. So, in the next lecture, we shall discuss some more criterions for absolute stability. So, you may try for some other methods with schur polynomial, until then good bye.