Numerical Solutions of Ordinary and Partial Differential Equations
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Lecture - 11
Convergence and Stability of Multi Step Methods

Good morning. In few classes, last classes, we have learnt on multi step methods, both explicit and implicit. So, now it is very important to learn about the convergence and stability aspects of multi step methods. We have learnt a similar aspect for single step methods. However, the concepts for the multi step methods are more involved. So, let us have a quick recall on the multi step methods, before we discuss on the convergence and stability aspects.

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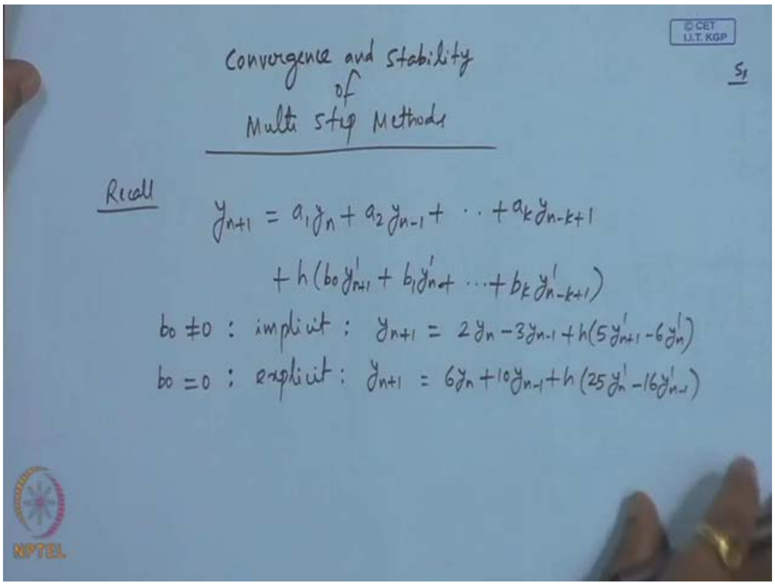

So, convergence and stability of multi step methods, so recall we have general linear k step method. This is just b n. So, this is a general and in this, if b 0 is non zero implicit, b 0 is zero explicit example. So, this is an explicit, so this is implicit and explicit. So, this is explicit method because to compute y n plus 1, we are asking past points y n and y n minus 1, whereas here to compute y n plus 1, we are asking y n n minus 1 n plus 1. So, this is implicit and this is explicit.

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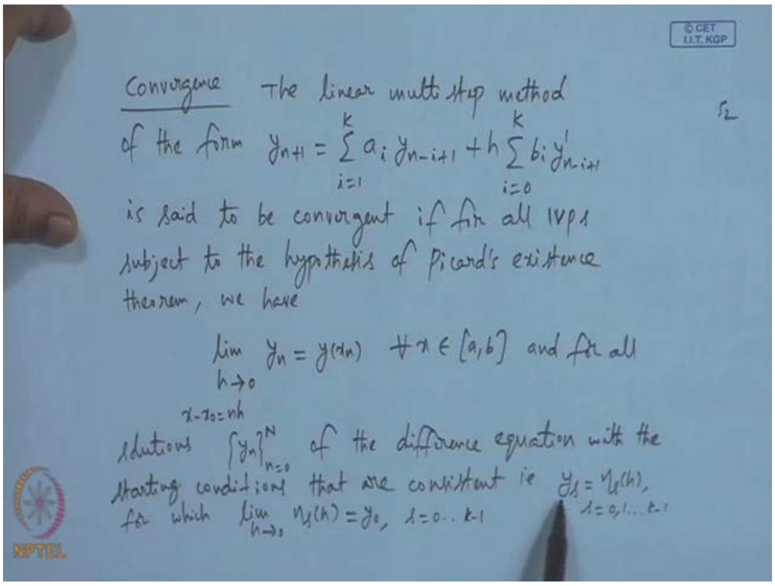

Now, what is the definition of convergence? So, the linear multi step method of the form, so of the form. So, let us say this is star of the form, we can even write it because star means of the form y n plus 1 is of the form is said to be convergent if for all IVPS subject to the hypothesis of Picard's existence theorem. So, that means if you recall when we defined IVP, we assumed that the solution exists a subject to the Picard's existence theorem. So, whatever the hypothesis under those for example, Lipchitz condition etcetera, so we have h goes to 0, y n equals to y of x n where x minus x 0 is n h for all x belongs to a b and for all solutions of the difference, see, for this is we have approximated the given IVP with the difference equation. So, we are now talking about convergence of this particular solution of the difference equation.

So, as h goes to 0, this approximated is h is equal to the exact for all x within the central and for all solutions of the difference equation with the, with the, with the starting conditions that are consistent. So, what is this that is some y s equals to say eta s of h where s equals to 0, 1, k minus 1 for which limit h goes to 0 eta s of h, this equals to y 0 s equals to. So, that means these are all y s are as h goes to 0, they go to initial condition. So, this is general condition and one must understand that it is within the framework of h goes to 0. So, this is convergence. So, there are some necessary conditions.

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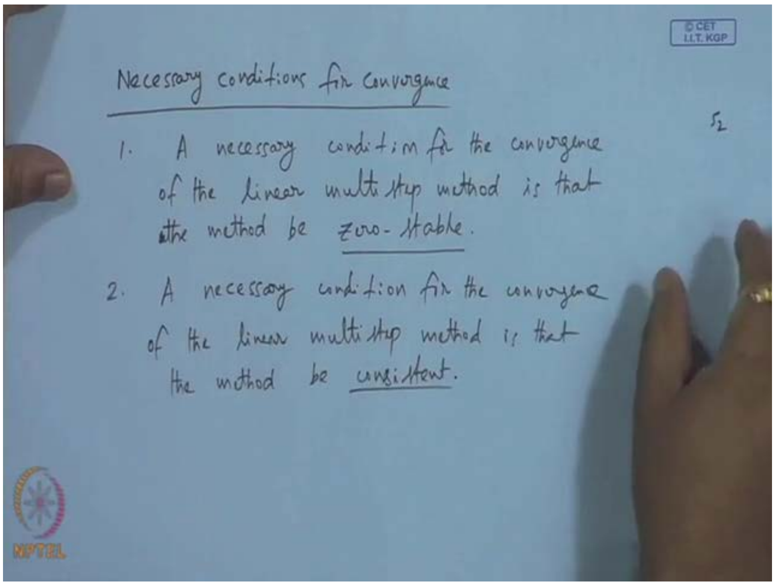

For a linear multi step method to be convergent necessary conditions for convergence, one necessary condition for the convergence of the linear multi step method is that that the method be zero stable. I am underlining two necessary conditions for the convergence of the linear multi step method is that the method be consistent, I am underlining. So, we have learnt a method. Now, the approximation, how this approximation gives converging solution? So, the convergence we have defined. Now, there are some necessary and sufficient conditions.

So, what are they? In some sense, these are conditions are telling one is zero stability and the other is consistent. So, in some sense, one can state necessary and sufficient conditions for convergence or stability and consistency are the necessary and sufficient conditions for convergence, but we do not know. We did not discuss what is zero stable and what is consistent. So, let us try to understand what is zero stable and what is consistence of a linear multi step method. So, what we are discussing is zero stability, zero stability.

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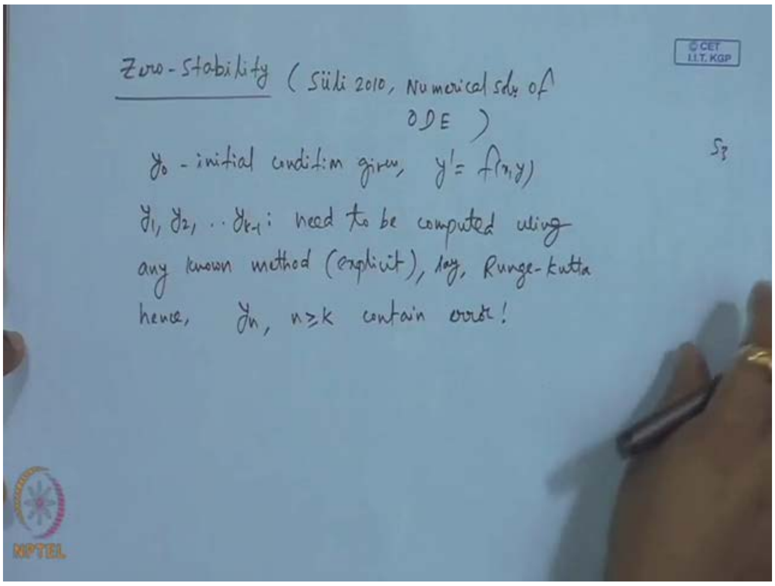

Some of the definitions I am taking from book Suli, this is numerical solutions of ordinary differential equations. So, let us define what is zero stability y zero initial condition given and our IVP is f of x y, y 1, y 2, y k minus 1 need to be computed using any known method of course explicit because for a multi step method, this is IVP, but we need past points that at past points. So, this we have to compute using any known method explicit say Runge-Kutta. So, what is the general strategy, one can compute using Euler's method, Taylor series, Runge-Kutta.

What is our ultimate aim? Our ultimate aim is to get as better a solution as possible as accurate as possible. So, for this to happen, the general prediction see Euler method is just it is a, it is a first order kind of method. So, it is first order method. The Taylor series well if we use up to second order, it is fine. Beyond that, the calculations are little tedious and all that.

So, if the sensible method, which is in hand, is RK method and that too fourth order. So, in general, given a linear multi step method, if one has to compute past points, generally one is tempted to use fourth order RK method. So, let us look at zero stability. So, these are initial conditions given and these need to be compared using any known method. Hence, what happens? Hence, all y ns, n greater than equals to k contain error because these are computed using some known method. So, there is an error. So, that gets in to compute the higher one. So, let us define what is a zero stability in this context.

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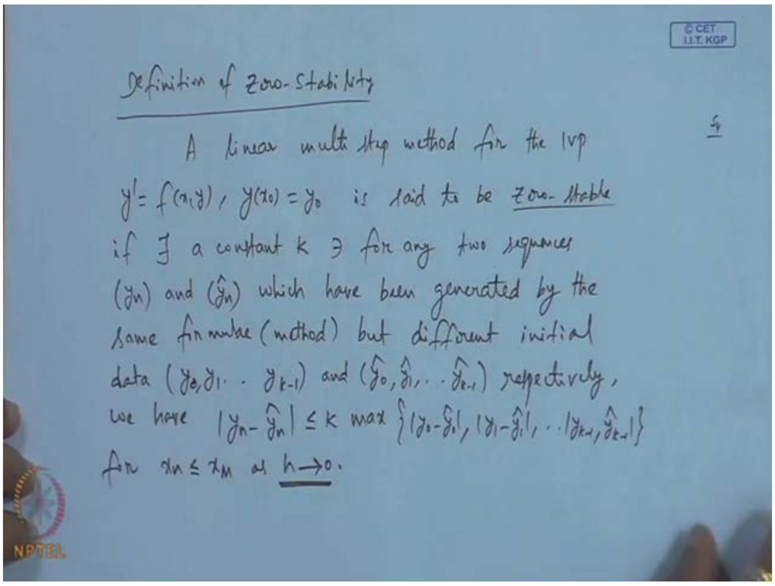

So, the definition and here a linear multi step method for the IVP, y dashed equals to f of x y, y of x naught is said to be zero stable, if there exists a constant k such that for any two sequences y n and y n hat, which have been generated by the same formula that means by the same method, but different initial data mod or what we are going to conclude.

So, we have given an IVP, we have given a linear multi step method. Then we say this method is zero stable, if there exists a constant k such that for any two sequences, two different sequences, which have been generated, see we are talking about the same method for a given IVP. We are talking about the same method and we would like to conclude when do you say this method is zero stable. So, the strategy is very intuitive. What it says? It says when do you say this is sensible? This method is zero stable means it is giving solution, which are sensible that means stability in the sense, if there are errors in the past data, then they are not really magnified to a large extent. They are bounded within some extent.

So, when do you say it is this method is zero stable? We are trying to compare to different sequences. What kind of sequences you start with, some set of initial data and then you compute the past values, start with another a set of initial data and then start with, but the difference equation is same. So, we are talking about these two. These two, they should be comparable in terms of the initial data. So, how they are comparable, let us see. So, this difference is less than or equals to k times max of for x n less than equal to say some x m as x goes to 0. This is very important.

So, what did we do? You start with two sequences, for any two sequences which have been generated by same formula, but different initial data. This is one set of initial data. This is another set of initial data. Of course, the solutions will be different. You start with one set of initial data. You arrive at some solution and you start with another set of initial data, you arrive with the corresponding solution because after all, the solutions are curves.

So, we have this difference should be bounded by constant times maximum of the difference between the solution at a particular grid point. So, at each grid point, you take the maximum and that will be k times of that. So, that will be the nth stage. So, this is the zero stability. So, that means it is not growing beyond the maximum value at a particular grid point. Now, the question why is zero stability?

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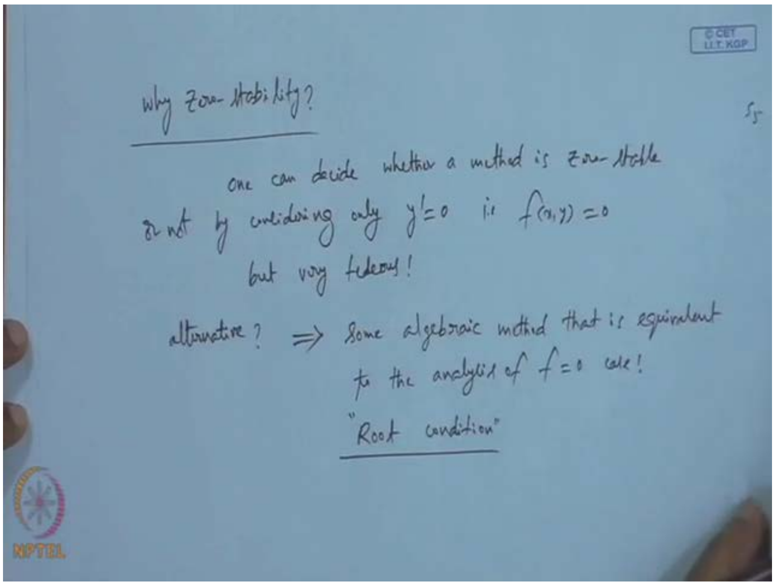

So, it is in some sense, in some sense, one can decide whether a method is zero stable or not by considering only y dash equals to 0. So, that is f of x y equals to 0. So, this is possible, but very tedious the corresponding analysis. So, that is how the word zero has come that means whatever the definition we have given for zero stability; that can be verified by considering just y dash equals to 0, but the analysis is involved.

Hence, f is 0, therefore the zero stability. So, then if that is difficult, what is the alternative? So, the alternative is some algebraic, some algebraic method, which is that is equivalent to the analysis of f equal to 0 case, some algebraic in the sense you verify a certain condition and then conclude that the method is zero stable or the method is not zero stable and things like that. So, what kind of corresponding algebraic condition? So, that is called root condition. So, that means instead of doing this tedious analysis, we pick up this alternative and try to analyze and conclude that a method is zero stable or not. But, before that, so what is this root condition we must know. Unless we know what root condition is, we cannot proceed.

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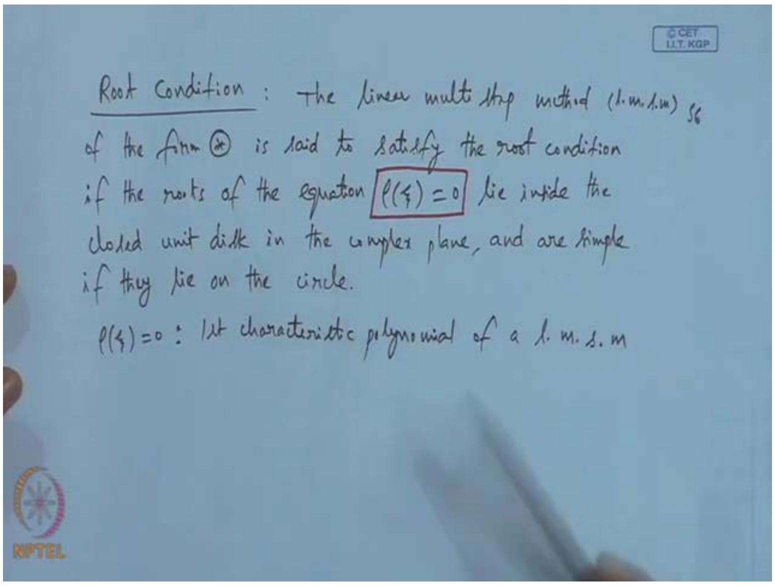

So, let us try to see what root condition is. So, root condition, so linear multi step method of the form start I am forced to write here star because in same method, we have to write. So, let us call star is said to satisfy the root condition if the roots of the equation rho of zeta equal to 0 lie inside the closed unit disk in the complex plane and the roots are simple, if they lie on the circle of course. I wrote something, which does not make any sense. This quantity we have not yet defined, but it some condition what is it? Linear multi step method of the form star is said to satisfy the root condition.

If the roots of the equation rho of zeta equals to 0, so that means this is a polynomial lying inside the closed unit disc in the complex plane and are said to be simple, if they lie on the inner circle. Now, what is so this is first characteristic polynomial of a linear multi step method? So, linear multi step method, linear multi step method, first characteristic polynomial what is it? So, we have not discussed. So, we have to define. So, it is interlinked, so we have defined zero stability.

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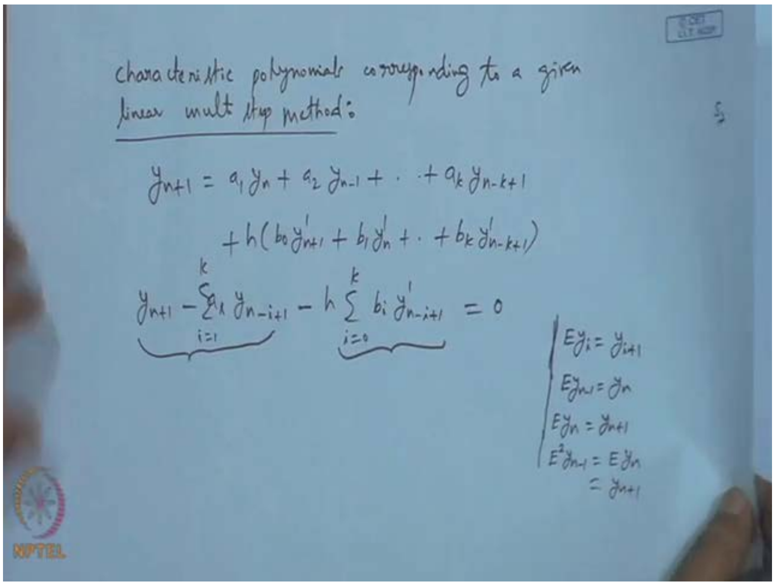

Then, method is zero stable if it is satisfies root condition etcetera we will say. So, let us see characteristic polynomials corresponding to a given linear multi step method. So, what is a method h times? So, this is our method? Now, this can be put it in the form minus sigma a i, all the terms then minus h equals 0. So, this is the linear multi step method. Now, so let us identify these terms. Let us identify these terms.

We have y forward operator defined as follows. Now, we make use of let us look at y n plus 1 is E on y n and y n minus 1 is E square on y, I am sorry, y n plus 1 is E square on y n minus 1 E on y n. So, that means E on y n minus 1 E on y n is y n plus 1. Therefore, E 2 on y n minus 1 is E 1, E is y n and this is n plus 1. So, like this, these can be converted and we have coefficients. So, let us see how this can be simplified with respect to notation point of view.

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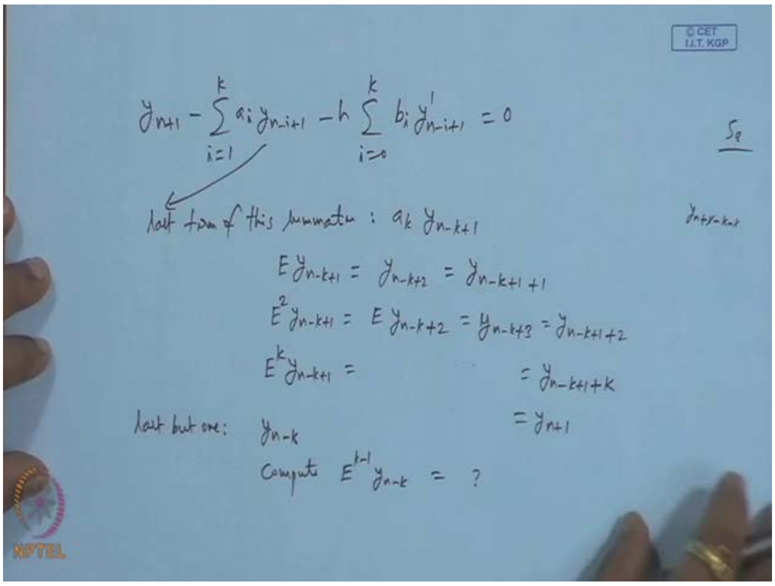

So, we have y n plus 1 minus. Now, in terms of the forward operator define, so what will be the last term last term of, this is last term of this is this summation. So, this is the last term of the summation. Now, E y n minus k plus 1, this will be plus 2. So, this will be k plus 2, so on so forth. Now, 1 E is giving me 1. So, this is like this. So, this is so E k. So, if you do the intermediate steps, did you get this? So, you can work out. See, this is 1 forward plus 2.

So, I am retaining this and this is a plus 1 plus 2. So, E k will be plus k. So, that gets cancelled we get, so that means this relation, this E k on y n minus k plus 1 is y n plus 1. So, this is the last then last, but one. What is the last but one term? Last but one term contains y n minus k y n minus k minus 1. So, what will be last term y n minus k minus 1? So, it will be this. Now, how do we express this in terms of y n plus 1? So, compute E k minus 1 and take it.

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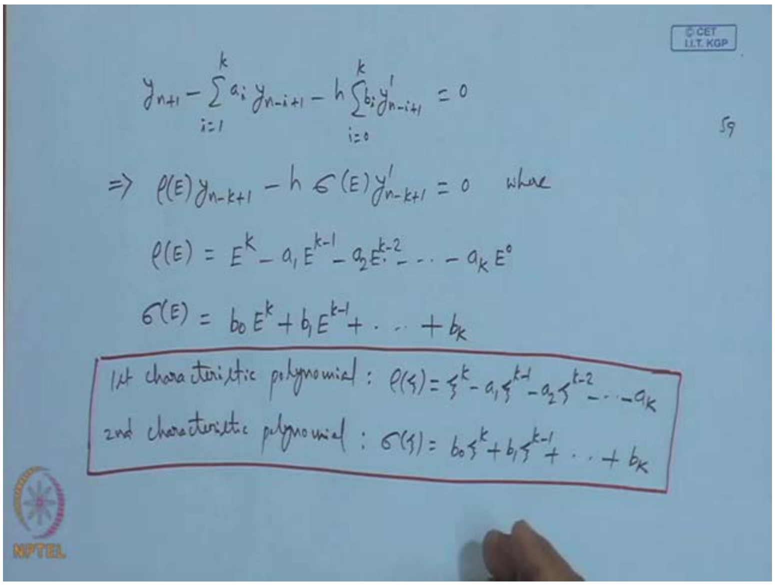

So, then you will get a center. So, accordingly we have, so accordingly I write this as rho of E sigma of E 0 where rho of E n minus 1 k minus a 2 k minus 2, let us try to see. So, if you multiply on this, this is 1, so minus a k y n plus 1, which is last term of summation. For example, first term E k on n minus k plus 1, E k on n minus k plus 1 n plus 1, this is y n plus 1 and k minus 1 with a coefficient a 1. So, that should be y n minus n y n term so on so forth, similarly, sigma E b 0, so these are with plus because I have written minus here.

So, this linear multi step method can be expressed in terms of this. Now, we define first characteristic polynomial of a multi step method. So, this is a polynomial a 1 minus a 2 minus a k. Second characteristic polynomial is sigma of this is b 0, so for a given linear multi step method, so these are important as within the context of stability of a method. So, we will see how these will play a vital role. So, having learnt this, we first and second characteristic polynomial for a given particular method, let us see how do we compute these polynomials just to be more comfortable.

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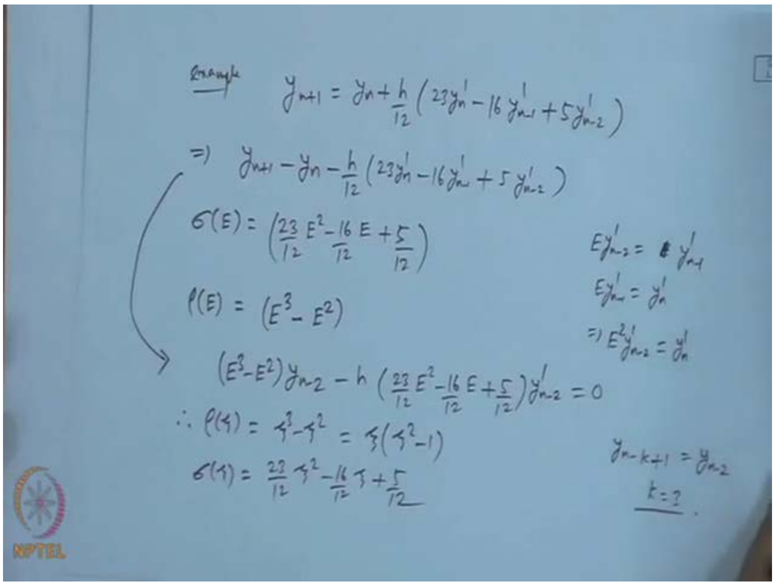

So, for example, so suppose this is our given multi step method. So, this can be written as now look, so this will be E on y n y n plus 1, but we do not have the terms at these grid points. We have the derivatives at these points that is n minus 1, n minus 2, but here we do not have, that means the corresponding coefficients are 0. So, if you recall the number of terms here, if you include this as well and in the summation, then i starts from 0. So, where a 0 is 1, so that means essentially we have k plus 1 terms here, k plus terms here.

That means k plus terms at the grid points and the derivatives k plus terms at the grid points. So, we must always compare with this. So, in that context, before we arrive at this, we may try to see, we have to arrive at y n from here. So, how do we arrive? E on y n minus 2 prime is y n minus 1 prime, then E on y n minus 1 prime is y n prime. This implies E 2 on y n minus 2 prime is y n prime. That means what is the distance from this term to this terms is E 2. Therefore, this suggests your sigma E of the form, so if you compare the general method, h is given, minus sign is given.

So, I put the polynomial on this. So, all the coefficients must be given to this polynomial. So, how do we give it? We will give 23 by 12. The distance is 2 minus 16 by 12, the distance is 1 plus 5 by 12, 5 by 12 because this will act on this, will multiply y n minus k plus 1 prime. So, what is the k here? So, this should act on the term of the form y n minus 2. So, we have to suitably identify what is our k. Now, what will be this? Look this is up to y n, the distance is 2 and here n plus 1, so naturally the distance must be 3. So, these two terms accordingly this gets transformed as exactly, so we can work it out.

So, n minus 2 prime a n minus 2, therefore this is and sigma of this is so for a given multi step method, this is the first characteristic polynomial and this is second characteristic polynomial. We expect this to be of the form y n minus k plus 1. So, this is y n minus 2. Therefore, that means if this is a linear three step method 1, 2, 3, it is asking past data three grid points, 1, 2, 3. Now, let us go back to our root condition.

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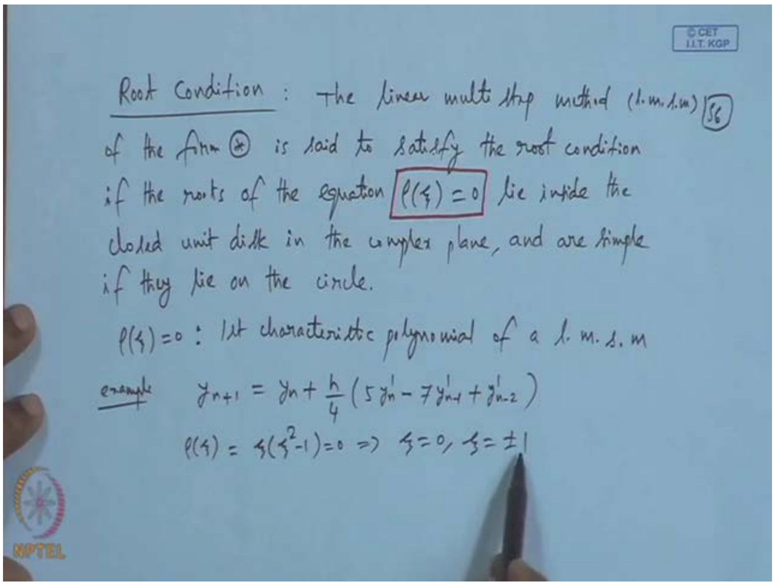

So, what it says, root condition linear multi step method of the form star is said to satisfy the root condition if the roots of the equation that is the first characteristic polynomial of the corresponding multi step method lie inside the closed unit disk. So, these roots must lie inside the closed unit disk. So, that means the magnitude must be within the closed unit disk. If the roots lie on the circle, then we say the roots are simple. So, this root condition has implications on the stability aspect of the linear multi step method. So, let us see some issues related to this. So, we have seen some example.

Again, we will see y n plus 1 equals to y n plus h by 4, 5 y n prime minus 7 y minus 1 prime plus y n minus 2 prime. So, if this is the case, then what is our rho of zeta? So, this is only these two and this is a three step method. So, this will be zeta, zeta square minus 1. This implies, so this is our characteristic polynomial zeta equals to 0, zeta equals to plus or minus 1. So, the linear multistep of the form is said to satisfy the root condition if the roots of the equation lie inside the closed unit disk and roots which are on the circle, they are called simple. So, these within the unit disk and this is these two are simple roots of the corresponding linear multi step method.

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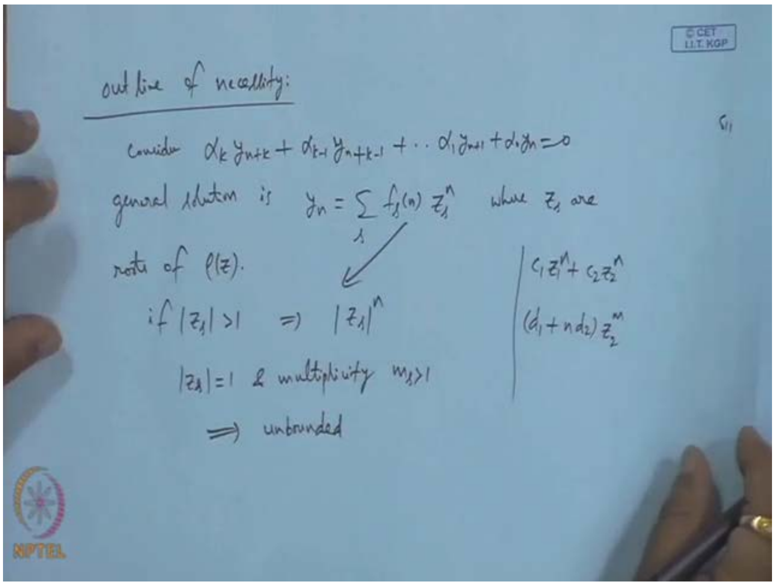

So, we have to see some outline of necessity means why what you will do with root condition. So, consider our method in a more general form. We convert to a multi step method. Then the corresponding first characteristic polynomial is of this form and the general solution is this our difference equation because of whose general solution is of this form f s of n z s n where z s are roots of this. Now, if the magnitude is greater than 1, this suggests that if the magnitude of this is greater than 1.

So, this suggests that what will happen to this and the other case is so this z s equals to 1 and multiplicity m s greater than 1. So, these are the two troublesome cases. What are they? The magnitude is larger than 1. So, then we had to worry about this and the other case is multiplicity is greater than 1 of a unit root. Then you see this as a specific format from our solutions of difference equations or any recurrence relation. You can see if the root has a multiplicity, so then we have, so the general is a linear combination is say c 1 z 1 power n c 2 z 2 power n. This is general if you have multiple roots, then some d 1 plus n d 2 and say some z 2 has multiplicity m.

So, then we have this kind of, so in these cases, there is a chance that leads to unboundedness. It leads to unboundedness. So, this is just an outline of necessity. Why root condition is really necessary? So, what is to do with root condition? This is the first characteristic polynomial and you compute the roots. Then depending on the behavior of the roots, the solution behaves.

So, how it is possible? See this from here. The general solution is of this form. So, the roots are magnitude is larger than 1. There is a chance that it will get unbounded. If the roots are within this that means on the boundary of the unit circle, but multiplicity 1, then this may happen and then again that may lead to unbounded solution. Therefore, root condition plays a vital role that is how root condition has become into play.

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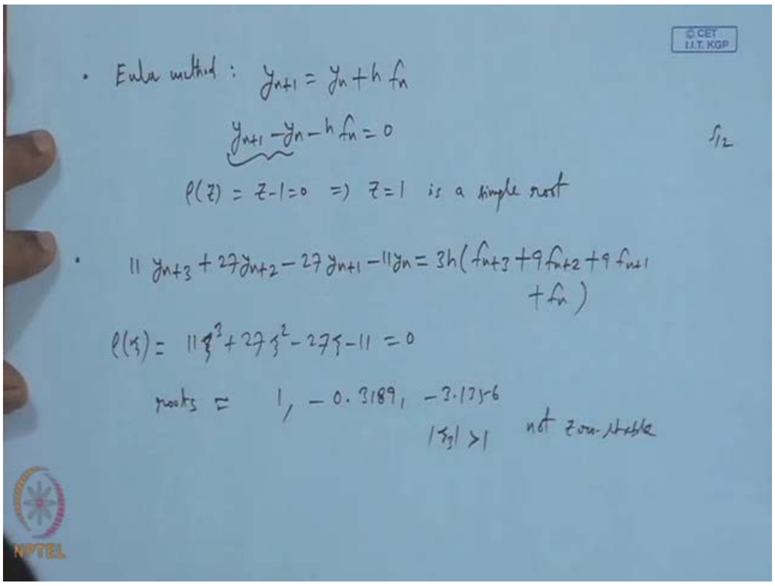

So, for example, consider an Euler method y n plus 1 equals y n plus h f n. So, what will be characteristic polynomial? So, rho of z and implies this is the characteristic polynomial z equals to 1 is simple root that is one is simple root. Then consider another method, so for this method, we zeta, otherwise zeta here. So, this is 11 zeta cube minus 27 zeta 11, this is our characteristic polynomial and the roots are approximately.

So, mod zeta 3 is greater than 1, therefore the method is not zero stable because it fails to satisfy the root condition for this case. So, root condition plays vital role, how the outline I have given. So, if one of the root magnitudes is one, then the solution is unbounded. Therefore, root condition plays a vital role. It becomes a necessary and sufficient condition for a method to be zero stable. So, we learn, further more details about this in the next lecture. So, you may practice for a couple of formulas how to compute first characteristic, second characteristic polynomial.

Thank you.