Numerical Solutions of Ordinary and Partial Differential Equations
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Lecture - 10
Multi- Step Methods (Implicit)

Hello, so in the last class we have learnt explicit multi step method, so when do we say it is explicit. Suppose in order to compute value at n plus 1 stage if the process is asking k passed points; that means n n minus 1 n minus k plus 1. So, then we say it is explicit multi step method, so know in this lecture we are going to discuss implicit. So, when do we say implicit as I mentioned before to compute value at n plus 1 stage, if the processor on the right hand side demanding the value at n plus 1 stage. So, then that is called implicit, so let us see how are we can derive implicit expressions.

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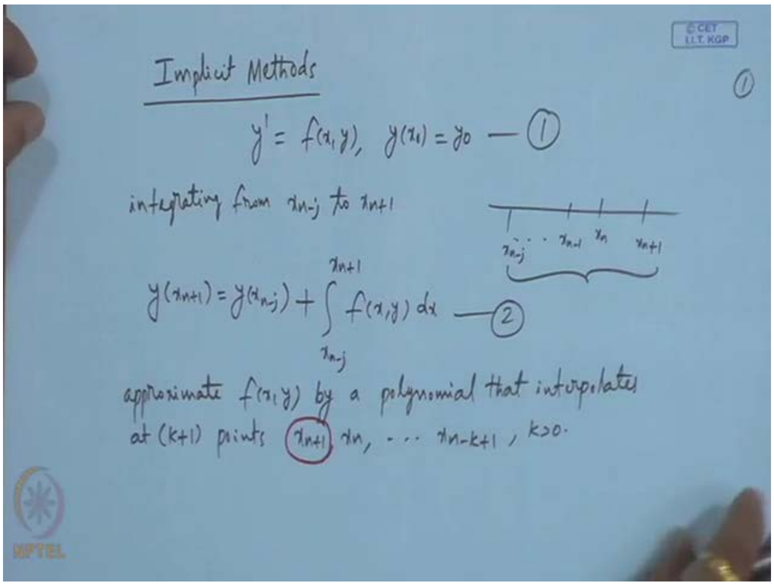

So, implicit methods y dashed is f of x y, so this is our initial value problem, now for the explicit method we have integrator from only within one interval, for the implicit we are stretching to the left. So, we integrate x n minus j to that means more than one interval, so more than this is the interval of integration then we get right. Further, how are going to approximate this f if you recall in the explicit method, we have interpolate this using k passed points. Now, that we are calling implicit that means in the interpolation we need to consider n plus point as well approximate f of x y by a polynomial that interpolates at k plus 1 points x n plus one x n. So, please make a note x n plus 1 included, x n plus 1 is included, now similar to the explicit method we have to use Newton's backward.

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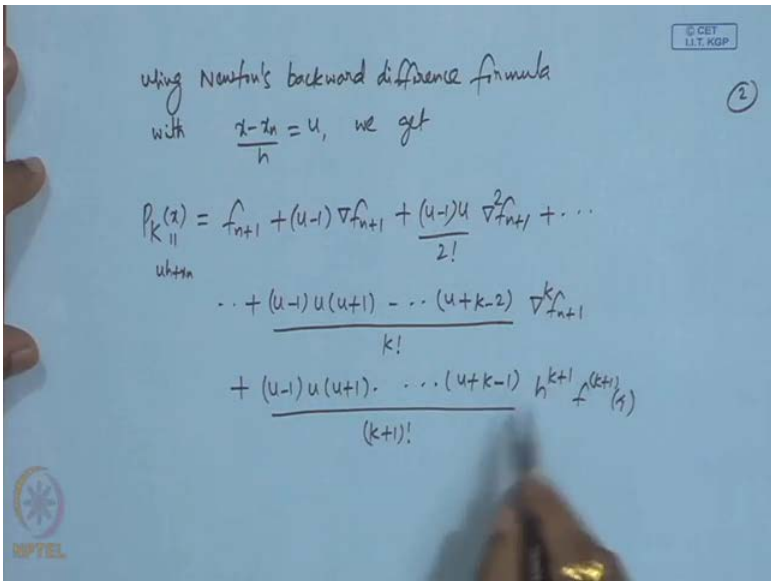

So, let us use, using Newton's backward difference formula, with this variable we are going to get of cause k-th degree polynomial, we have k plus 1 point. So, I am not writing in the terms of the x variable, so directly I am writing in terms of u, so this indeed. So, this is the reminded term again I am not giving the polynomial in terms of variable x because we have done explicitly when we have discussed explicative methods, so similarly one can do it, so this can be left to the exercise.

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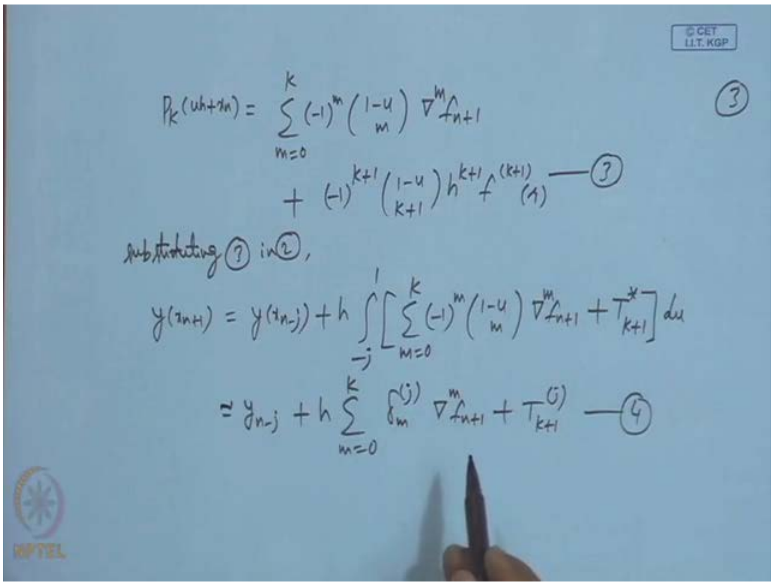

Now, this can be simplified as follows this is the polynomial, so this is the reminder term, now as before substituting 3 into we get y of x n minus j look the transformation. So, here the x 1 minus j accordingly when we use the variable change of variable x minus by h n is u we get a different limit that is minus j to 1 plus T k plus 1. So, say T k plus 1 star until the interval sign, so further of this can be simplified as no other approximation n minus j plus h submission comes out the codes in this case.

So, star within the integration and if you remove the integration it is this that is a notation I follow, now look at in the explicit method if you could recall we have defined gamma m 0 and in this case delta n j. So, what is the difference, so it is only one interval x n to x n plus 1, whereas here x n minus j to x n plus 1, so this is the indication the length of the interval. So, if the length of the interval is just 1 we are using 0 and if it is x n minus j-th x n we are using j.

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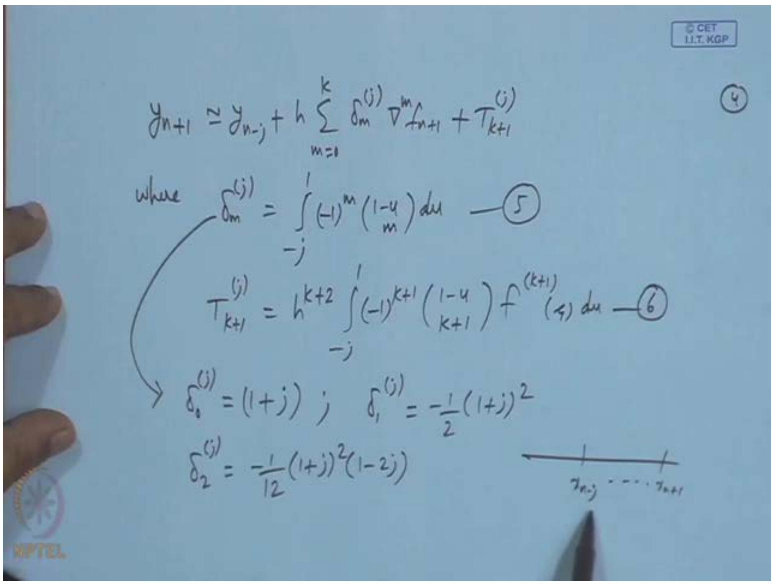

Now, the polynomial become the approximation becomes where the codes this is the error, so this is the error. Now, one can compute the codes 1 plus j, 1 plus j square etcetera, so when we are using more than one interval to the left if we fix a particular j, so that will determine the codes. So, these are functions of j for example, x minus 1 minus 2, minus 3, so lengths of the interval codes vary.

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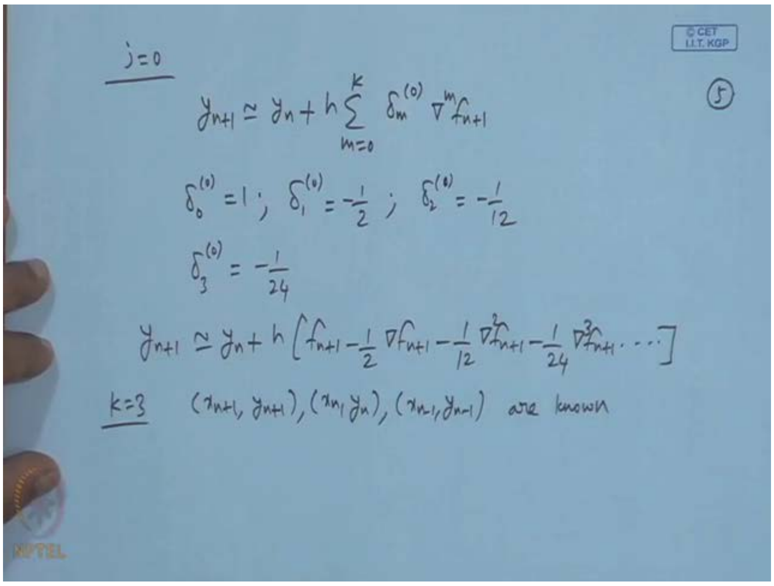

Now, let us take a simple case j is 0 that is again with a one interval, so j equals to 0 case if this is a case where 1 etcetera accordingly the approximation becomes so accordingly this is the approximation. Now, what is the option in our hand k, so how many points we are using to interpolate the polynomial, suppose we use 3 points that includes x n plus 1, so we get quadratic equation suppose these are known.

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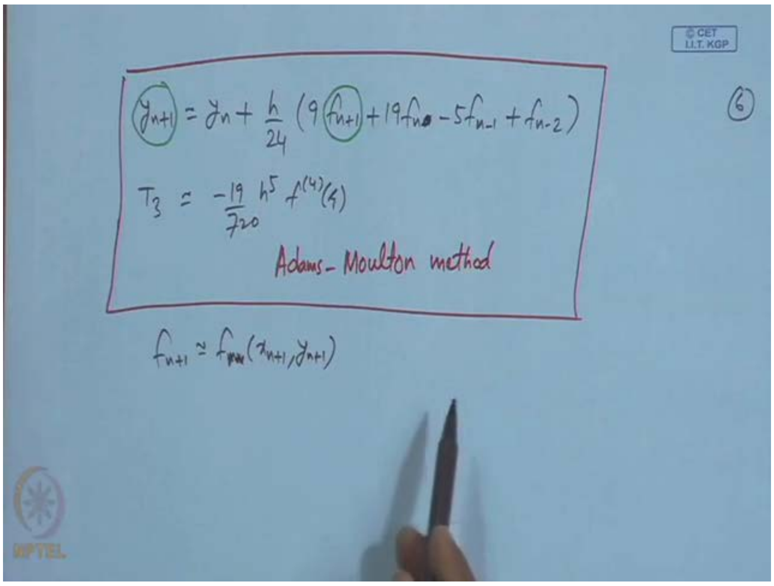

Then we get the polynomial as follows this is f n and the error T 3, so this method is called Adams Moulton method. So, earlier one is Adams backward that is explicit method and this is Adams Moulton method, implicit method why it is implicit, the implicit nature come from because in order to calculate f of. So, this is implicit in this signs, so we have two different concepts that is if the interpolating is including the n plus n-th point then we end of with a implicit for the interpolation. If, we are not using a correct point only past points then we end up with explicit in general these are the popular methods.

Now, for a given general multi step method how do we compute the error, so well this is the method and once we derive we know this is the error because we have derived it. But on the other hand suppose there is a method given we do not know that what is the error, so we have to compute that means the coefficients everything somebody gives you, this is a multi step method. So, can you verify what could be the error for this approximation, this is an important task one has to learn, so let us try to do that.

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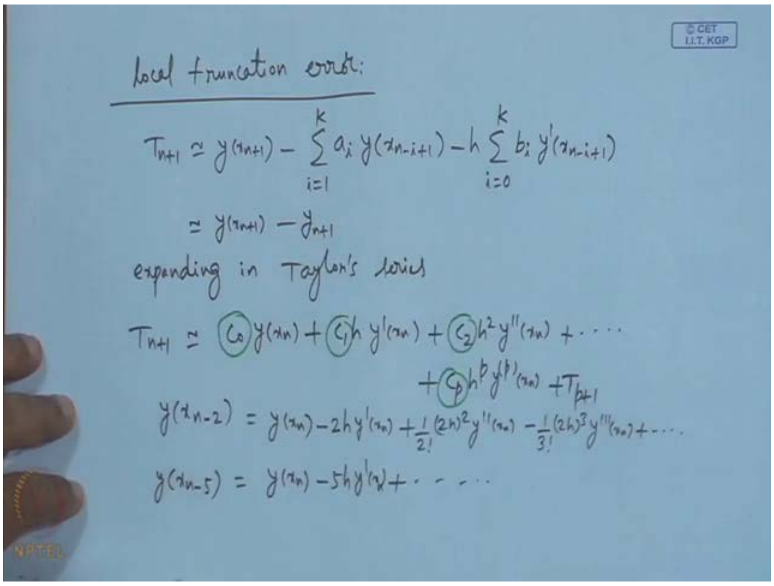

So, that is local truncation error we define T n plus 1 as minus the general multi step method we have given a standard notation. So, this is function values passed function values and these are the derivate values, so this is exact minus. So, this is the approximation right see what is this is exactly this is the method, so if you take minus common this term plus this equals to y n plus 1. Now, expanding in Taylor series let T n plus 1 expressed as c 0 y of x n c 1 h okay that means you expand that in Taylor series than expand this in Taylor series, so for a particular case we will do that.

But, in a general sense say for an example I can take an example and do it, see for an example you need to expand n minus 2. So, this will be minus 2 h plus 2 double square of double h n minus 2 h cube, so in that sense for a general term i, so for example i is 1 then x n, so a n of x n. So, here y of x n and here when i is 1 we get x n, so what will be the coefficient from here 1 and from here a 1, so it is like that right.

Suppose i is 2, then a 2 will be multiplying of x n because we have seen here y of x n is coefficient is 1 and a 2 will be multiplying. Suppose y of x n minus 5, so what will be multiplying y of x n a 5, so that is how you get therefore, if you take the difference and define it as T n plus 1 and expand in Taylor series, put it in this form then we should try to identify this coefficients in terms of what in terms of a i and b i.

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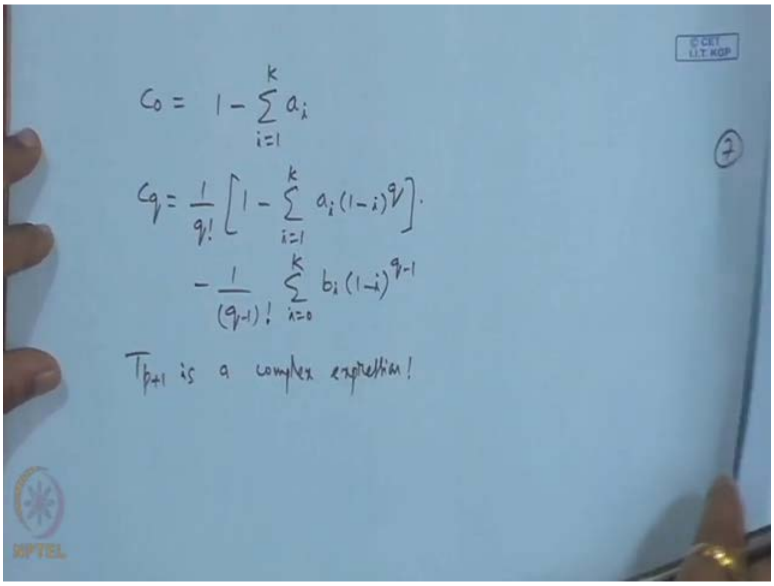

So, this can be done as follows c 0, what is c 0 coefficient of y of x n, in this term y of x n is 1, in this term coefficient of term x n is a i and in this nothing. Therefore, c 0 is 1 minus a i, so if we do it in a general sense we get slightly complex minus, so this is c q and we get quite complex expression for T p plus 1 which is the error, so T p plus 1.

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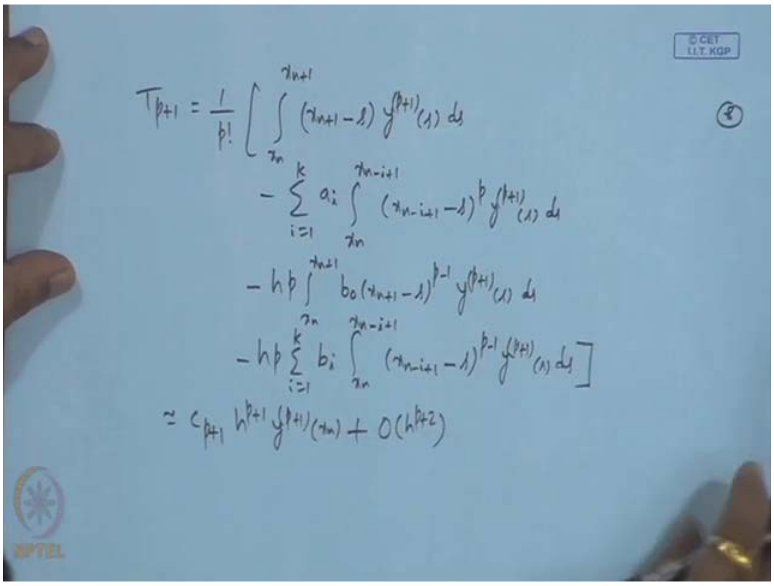

We can write it, so it quite comparison, it is not so much if we understand the logic that would keep us comfortable just that this is an error term and it is of this form. Otherwise it is very complex, so this must be of this form.

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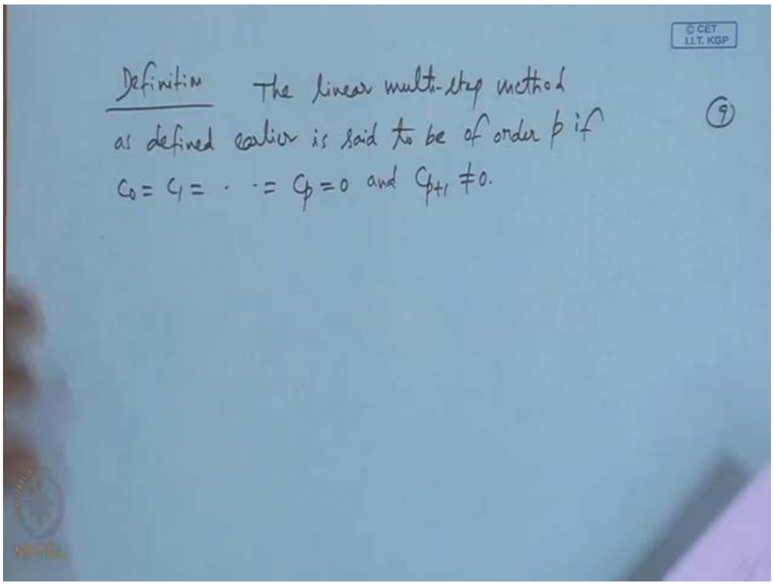

So, when do we say a method is of so much, so the linear multi step method as defined earlier is said to be of order p if c 0, c 1 c p equals to 0 and c p plus 1 is non zero, how come. Look at this, the way we have defined local truncation error exact minus the approximation and that has been expanded in this form.

This is of order p if c 0 that means it agrees and this agrees up to p terms right c 0 rather p plus 1, c 0, c 1, c 2, c p and from, here c p plus 1 is non zero. Hence, the error is coming from this term, so that means in order to determine we have to force each of this equals 0 and calls the system. So, let us look at a specific k s for how do we derive for a given problem how do we determine the coefficients.

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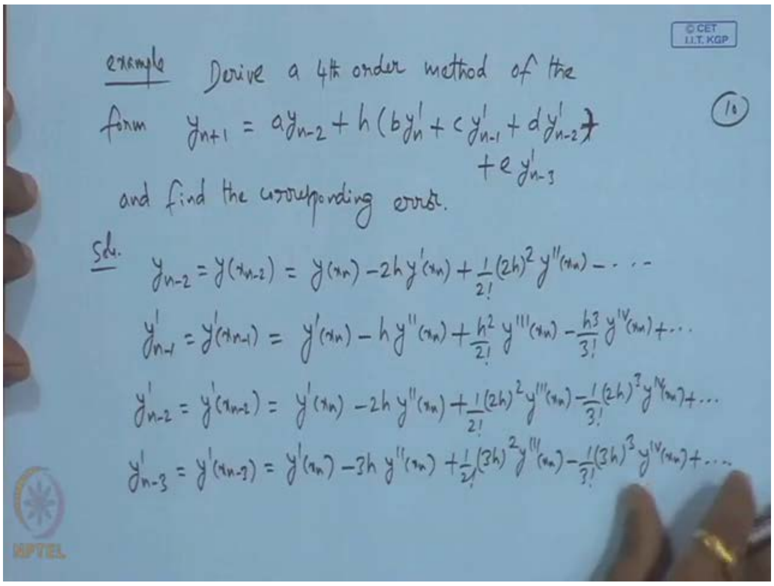

So, example derive a fourth order method of the form y n plus 1, a y n minus 2 plus h b y n prime plus e plus prime plus d minus 2 plus prime plus e y n minus 3 for prime and find the number y n, y n minus 1, minus 2, n minus 3. So, we have to expand each of them right, so let us start with see we need say for example y n minus 2, so this is y of x n minus 2, h y dash of x n plus 1 of factorial 2 h square y w dash of x n minus. Next, y n dash keep as it is then y n dash n minus 1, prime y dashed of x n minus h y w of x n plus h square by 2 factorial, y 3 of x n minus h cube by 3 factorial, y 4 of x n etcetera and 2, so this is y dashed of x n minus 2 h then 3, now our method is this.

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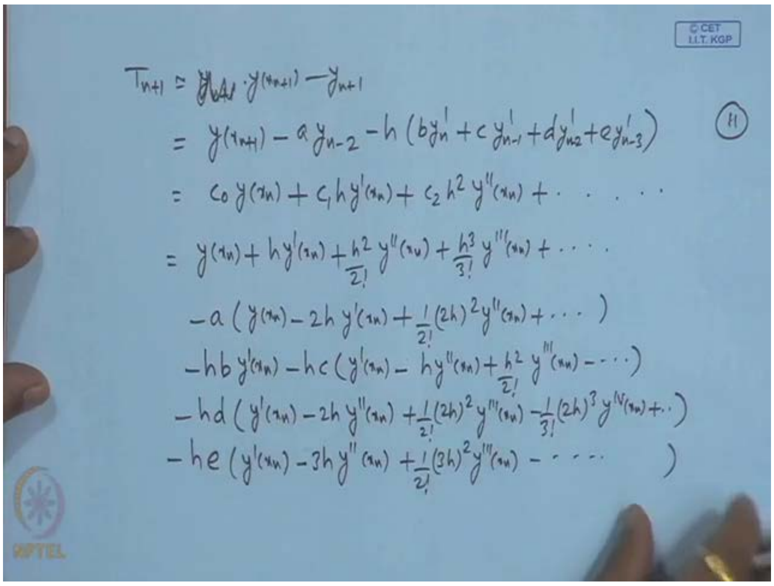

So, what is our T n plus 1 y minus h, so this is was our T n plus 1 and this is written as c 0, this is written as c 0 y of x n plus c 1 h, so we have expanded this. So, the expansion of this then minus a y minus 2 we have expanded, so that is then minus h b y dash of x n minus h c, this one we have expanded then minus h d minus h e. Now, what we have to do we have to collect for c 0, c 1, c 2 as powers of h power 0, h power 1, h power 2 etcetera, so if we look at this collect the coefficients of y of x n this 1 there and minus a there.

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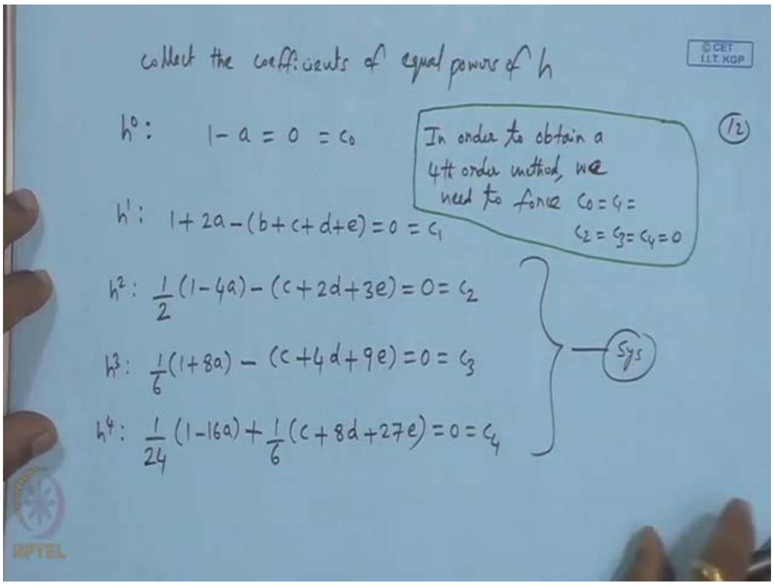

So, if we do that, that means collect the coefficients of equal powers of h, so to start with h power 0, h power 0 is 1 minus a and there is no term, so 1 minus a this is our c 0 in fact and we have to force it to 0 because we are trying to obtain a fourth order. So, remark is in order to obtain a fourth order method, we need to force c 0, c 1, c 2, c 3, c 4, c 0, so h power 0, h power 1 look, so h power 1 this is 1 here then h power 1 in this case minus and minus plus 2 a and h power 1.

So, there is minus b and here there is minus c and there is minus d and minus c, so I repeat again h power this is 1 plus 2 a minus b minus b minus c minus d, so we get 1 plus 2 a minus b plus c plus d plus e. So, this is our c 0, this is our c 1 then h 2, the coefficient of h 2 h square look at, here 1 by 2, now here we have 4 h square, so 2 get cancelled, so 2 a, so if you take 1 by 2 common coefficient is 1 by 2. So, here 1 by 2 if you take common we have 4 minus 4 a minus 4 a, then in this case plus c because minus h square minus c is minus plus.

So, just plus c then in this case h square that is 2, d 2 d and in this case coefficient of x square is 3 e. So, we get the following equation 2 by 2 if I take 2 1 minus 4 a from where this is coming, this is 1 by 2, common 1 by 2 common. So, 1 from there minus 4 a then minus c plus 3 d plus 3 e, so how will we get coefficient of h square, see h square, so minus plus and see this is h square and here plus 2 d and here plus 3 every time then coefficient of h cube.

So, look at h cube 1 over 6 if, you take 1 over 6 common, 1 there and here h cube, so the next term is 1 over 3 factorial 2 h cube y 3. So, we have 8 h cube if you take 1 over 6 common minus 8 a, so this is with minus sign, this will be plus 8 a, so 1 over 6, 1 plus 8 a because it is see it is alternating its minus 2 plus and minus. So, this must be minus 1 over 3 factorial, that is 1 over 6 if you take common and here and there 2 cube is 8 plus 8.

Now, look at the next, here h square and h cube minus half c, because 1 over to c e and here we have, so if you take a half common, here we have c 3 and here we have 4 d minus 4 d and here we have 9 e minus 9 e. So, we get minus c plus 4 d plus 9 e, now h 4, h power 4 look at it here, so 1 by 4 factorial that will be taking common, so coefficient is 1 there then we need to extend this one over 4 factorial 2 h power 4 y 4 x n.

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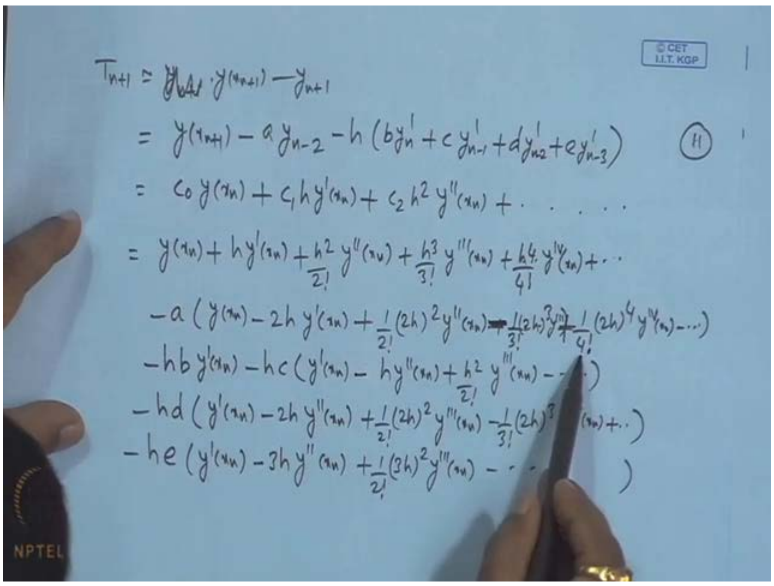

So, here we have 8 there, so this is a plus sign minus 8 a, so if we take common that will be this 4 factorial if we take common, we get mm, so look at carefully. So, one coefficient then we 16, so we have plus 16 right this is a minus 16 a, so we get 1 by let us do it, 1 by 4 factorial is 24, 1 there 1 here, so 4 factorial taken common. So, this will be 16 minus 16 a, so then here h 4 this will be minus h cube by 3 factorial y 4, so h 4 with the plus sign this will be 6, 1 over 6 c, so this is plus 1 over 6 c then. Here, this is 1 over 6 common minus minus plus 8 d.

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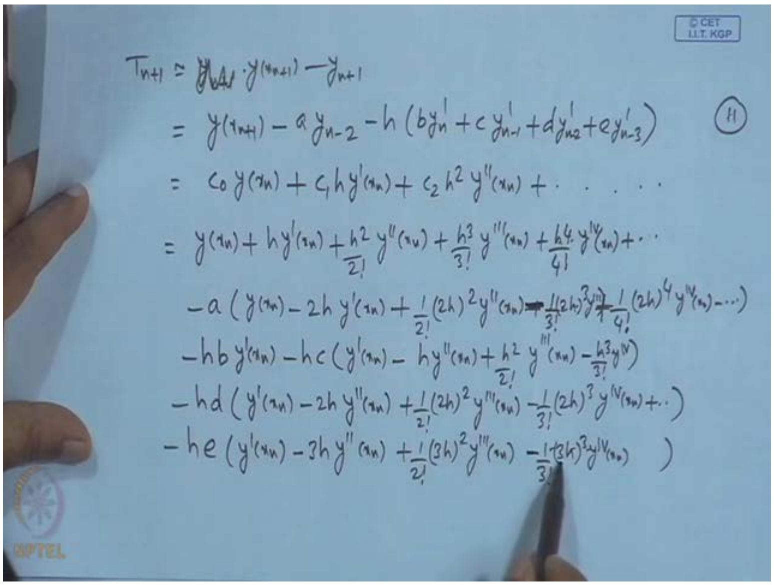

Then, here we have to go for one more y 4 x n, so here one of 3 factorial common minus minus plus 27 e, so this is equals to 0, this is our c 4. So, the remark is since we are asked to determine fourth order method, we need two 4 c two c 3 c 4 0, now we have a, b, c, d, e, so 1, 2, 3, 4, 5 equations let us call this entire system.

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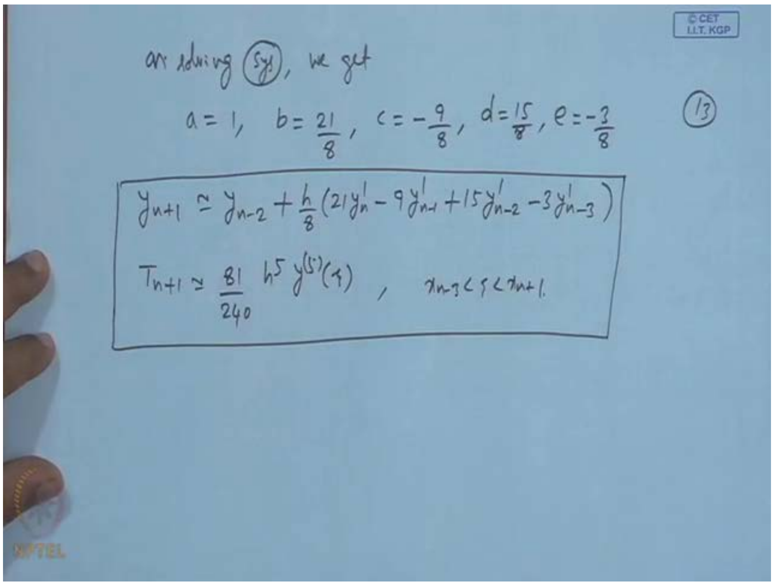

Now, on solving we get a, on solving sys we get a equals to 1 you may check please b is 21 by 8, c is minus 9 by 8, d is 15 by 8, and e is minus 3 by 8 and hence the method we obtain plus h by 8, 21 y n prime and if we take the next terms that means non 0 c 5 81 by 240 h 5. So, this is the fourth order method that we have asked to derive, so that means what did we do, we were given multi step method were the coefficients are not known it was asked derive a fourth order method.

So, then what did we do we have expanded each of the terms and then we have considered the residual that is different between the exact approximation and substituted the expansions. Then we have collected the equal powers of h and then we have determined the coefficients that will desire the method. So, accordingly if suppose somebody asks derive say third order method then we expand and collect the coefficients of equal powers of h and force them to be 0 of c, 0 c 1, c 2, c 3 and then c 4 onwards will contribute the error right.

So, this is very important this will give an idea of how generally the local truncation error is computed. If you do not know just a multistep method is given then what will be the error, so the same method and suppose you do not know the coefficients and your asked to compute a method of order then also similar method we do. But, then from the system of equations we try to make them force them to be 0 up to the desire order and solve the system to determine the coefficients and further determine the error.

So, this gives a fairly a good idea of how in general the implicit or explicit multi step methods are derived. Now, we have to discuss coming lectures may be some problems, apart from problems we have to discuss about some theoretical considerations like stability, convergence some things like that until then good day.

Thank you, bye.