Functional Analysis

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Module No. # 01
Lecture No. # 09
Normed Spaces with Examples

Last lecture, we have given many examples of the vector spaces and seen that, most of them, they are the metric spaces.

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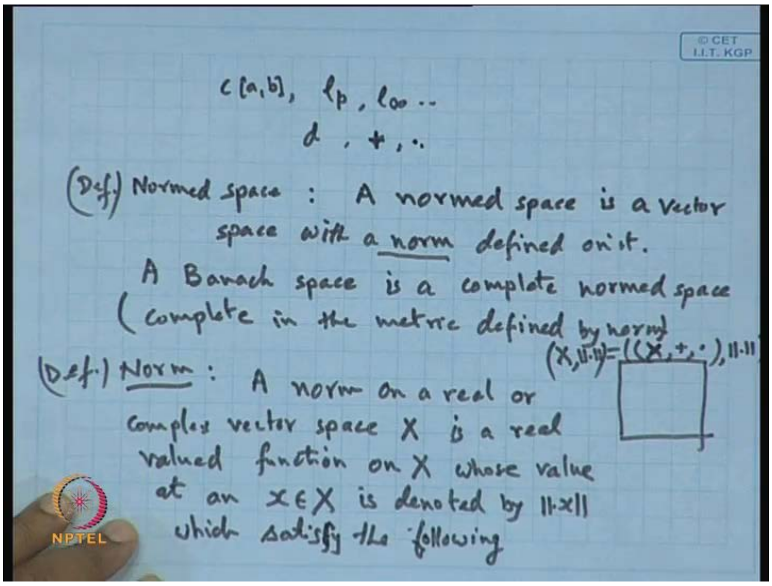

For example C a b, l p, l infinity, these are all vector spaces and the metric, which we have defined on it, in a usual way, the distance between the two points and they form the metric space. But, so far, the notion of the distance d and the addition and scalar multiplication, which you are using and formation of the vector space for these classes, does not have any relation. So, the definition of the metric space, or the way in which we are getting the metric space, will not be a very fruitful thing. We cannot develop a fruitful theory, when we start with the concept of the distance notion, in this fashion.

So, we are interested to define the metric in some other way, which has a relation between the addition and a scalar multiplication, as well as the notion of the distance. So, for this, we introduce a concept of the norm, which has a relation with plus and dot and then, with the help of the norm, we will introduce the notion of the distance on it, so that, we can get a useful theory for further developments. So, before it, what is the normed space? We define the normed space, we define the normed space, normed space. A normed space is a vector space, vector space, with a norm, with a norm defined on it.

A Banach space, a Banach space is a complete normed space, where the completeness, complete in the metric, in the metric introduced or defined by norm. So, what is the normed space? It is a vector space X with an operation dot, plus and dot vector addition and scalar multiplication and at norm, is introduce on it. So, this complete structure, we call it as a normed space and in short, we denote as this. We understand, X with an operation plus and dot, and norm is a norm defined on it. Banach space is a complete normed space; complete means, every Cauchy sequence in it, must be a convergent one. So, when you say a Cauchy sequence convergent, it converges in the norm of X, or in the metric introduced with the help of norm on it, clear.

So, this involves the concept of norm, first. Vector space, we have already introduced. Now, what is this norm? We define the norm as, we define the norm as, a norm on a real or or a complex vector space, capital X, a real or complex vector space capital X, is a real valued function on X, whose value at an x, belonging to capital X, is denoted by norm of x, which satisfies the following properties, which satisfy the following…Yes, it satisfies following properties. So, a norm on a real or complex vector space X is a real valued function on X, whose value at an x is denoted by norm of x, which satisfies the following properties.

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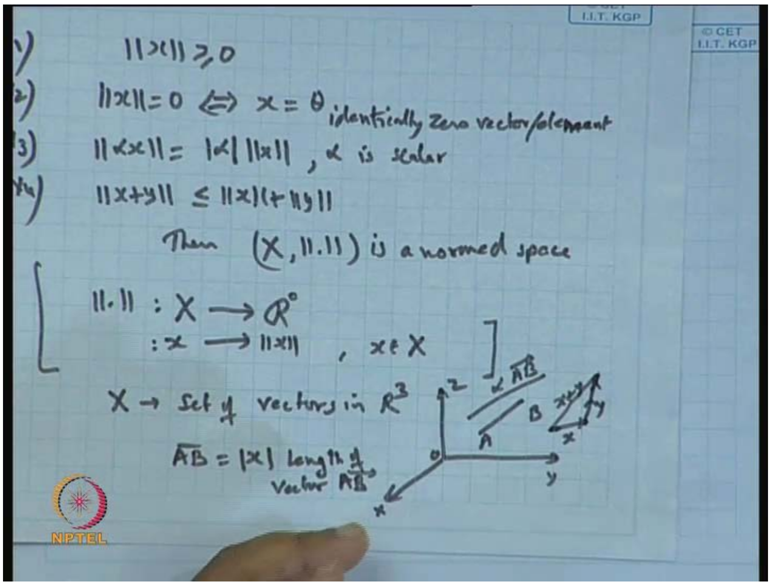

The properties are, first property, we say N 1; that is, property of the norm. The norm of x is greater than equal to 0; it cannot be a negative quantity. Then, N 2, the norm of x equal to 0, if and only if, x is an identically 0 vector, theta, is identically 0 vector or 0 element, identically 0 element. N 3, the third property is, the norm of alpha x equal to mod alpha into norm of x, where alpha is a scalar quantity, and this alpha belongs to the field of the vector space X; if x is real, the alpha will be real; if x is complex, the alpha will be a complex. And, last property is, the norm of x plus y is less than equal to norm x plus norm y. x plus y, norm. So, if this four properties are satisfied, then, we say, this X with the norm, is a normed space. Then…

So, then, X, with this, is a normed space. So, basically, what we are getting is that, norm is nothing, but a function, from a vector space X to R; you can say, the R 0, nonnegative, real quantity, a real valued function, such that, the image of x goes to norm of x. Each point x and image x, for any x belongs to capital X; and that, this norm satisfy these four properties.

So, when this satisfies, we say it is a norm and the corresponding space is called the normed space. If we look this vector space X, a set of vectors in the three dimensional plane, suppose R 3, then, we have these properties, which are enjoyed by vectors in a three dimensional plane. First thing is, the length of any vector A B, the length of this vector A B, can never be 0, negative, length of the vector can never be negative. So, if I denote the length of the vector A B as x, mod of x, this is the length of the vector A B; then, this length can never be a negative. It means, it is always be greater than 0 or at the most, 0, when b coincide with a; that is, the x becomes 0. So, the first property says that, it is the generalization of the length of the vectors.

Second property is, first and second, third property says, if suppose, I multiply a vector A B by a alpha, where alpha is a scalar quantity, so, mod of…If alpha is less than 1, then, it will reduce; alpha is greater than 1, over mod 5, greater than 1, less than 1. So, accordingly, it has a, length is changing. So, what that effect, it will affect the mod of alpha into the length of the vector. Then, third property says that, the sum of any two sides of a vector in a triangle can, can never be less than the third side. So, third side, this is our vector x; this is vector y; this will be the x plus y vector. So, sum of these two sides of the vector will always be greater than equal to the length of the third side.

So, that is why, this property of the triangle property also. So, this concept of the norm, which we have introduced, is nothing, but the concept of the length of the vectors in a general metric, general vector space. When we go for that R 3, we get vectors and the length of the vector has been generalized, this concept, to an arbitrary class capital X, which is a class of all the elements, may be a sequence, may be function, which forms the vector space. So, the norm generalizes the concept of the length of the elements of a vectors, in a arbitrary vector space. So, that is the…clear?

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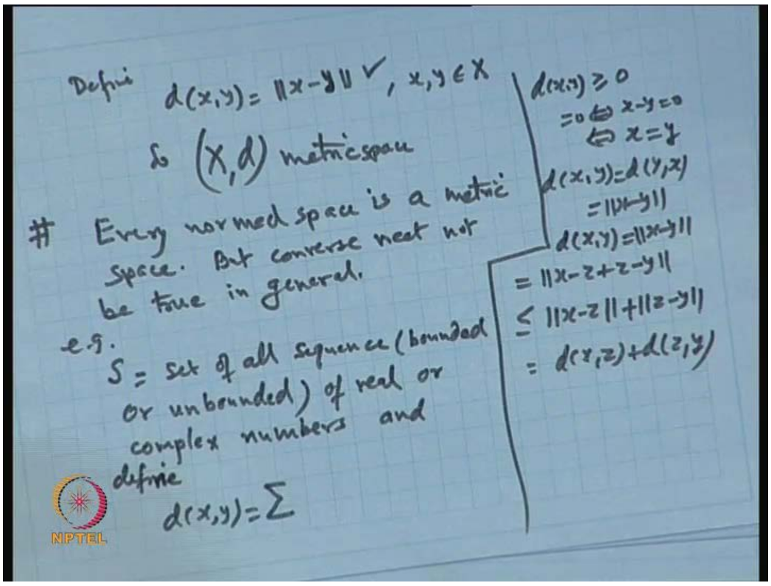

Another thing, which we can note, we can note here is that, if we introduced d of x y as the norm of x minus y, where x and y are any arbitrary elements of X, then, you will see all the properties of the metrics are satisfied. First is, d of x y will always be greater than equal to 0, because the norm cannot be negative. Then, if it is 0, it means, x minus y must be 0. So, x must be equal to y. So, x must be equal to y. The third property, if, suppose, I interchange the x and y; instead of this, I write y x; both will give the same value, because the interchanging x and y will not affect, because, it is a nonnegative quantity. So, negative sign will not affect. So, y minus x, it will remain that.

Then, other property of d of x y is less than equal to d of x z plus z; that also is satisfied. So, all this properties, this property, which is equal to norm of x minus y, which can be written as norm of x minus z, z minus y and that will be again, less than equal to norm of x minus z plus norm of z minus y and that is equal to d of x z plus d of z y. So, the properties of the metrics are satisfied, as soon as we introduce the concept of the metric,in terms of the norm and this gives a metric. So, X d becomes a metric space; X, d becomes a metric space, clear. It means, every normed space is a metric space. So, as a result, you can say, every normed space is a metric space.

Every normed space is a metric space, clear. Because, whenever we take norm, it can be converted into metric. But converse need not be true; but converse need not be true, in general; that is, all the metric spaces may not be a normed space. For example, if we consider the class S, the set of all sequences, bounded or unbounded sequences, bounded or unbounded sequences of real or complex number, numbers and introduce that notion and define d x y as sigma 1 by 2i, i is 1 to infinity, mod of x i minus y i over 1 plus mod x i minus y i, where x is, x is x i, i is 1 to infinity, belongs to capital X, y which is y i, i is 1to infinity, belongs to capital X, ok.

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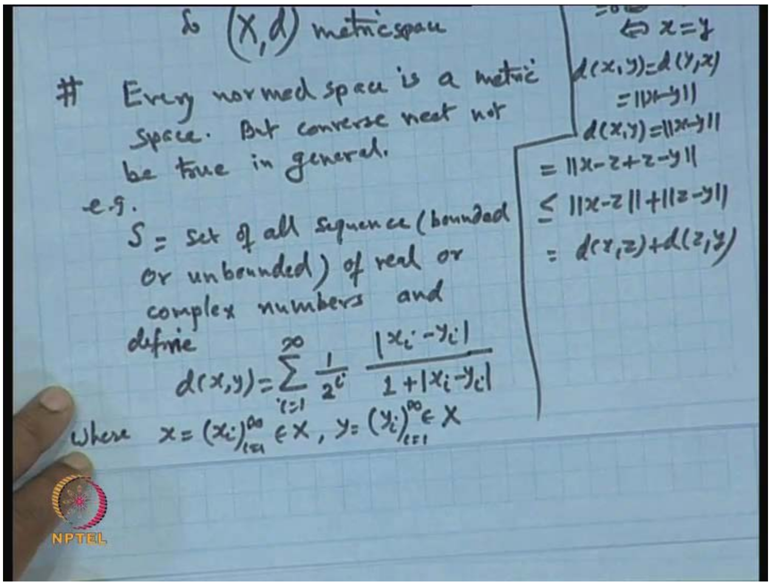

So, if we look this metric d, then, we have already seen that, this forms a metric space; that S d will be a metric space, is a metric space. You have already seen. Is it a normed space? The question is, is it a normed space? The answer is no.

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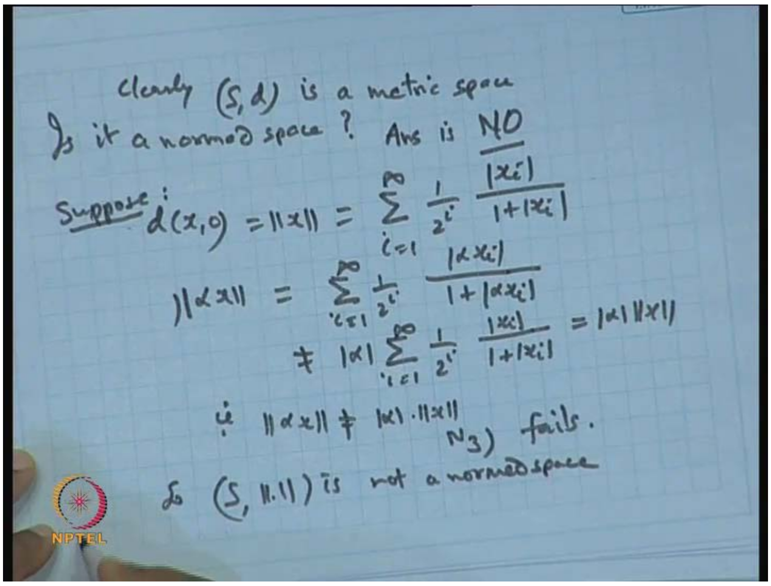

Why? It means, those properties of the metric space, which we have introduced, those property of the normed space, all the properties are not satisfied; that is, this property which we have introduced, the norm of x is greater than equal to norm x equal to 0, alpha x equal to this and so on is not… all property may not be satisfying. So, let us suppose, it is a normed space. It can be defined with the help of the norm. So, if it is so, then, let us see that, d of x 0, suppose, it is the norm of x, which is equal to sigma 1 by 2 i mod of x i over 1 plus mod x i, i is 1 to infinity, is it not? Because, what is this norm x? It is the length of the vector. So, from origin, you are choosing the distance; so…

Now, let us suppose, suppose d x y, x 0 can be written in terms of the norm as follows. Then, what should be the norm alpha x? This is equal to what, mod alpha x i alpha x i. Now, can you take alpha outside from here? If I take alpha outside, it is not possible; it cannot be, this cannot be equal to mod alpha into sigma of this. It cannot be, that is, it is not equal to mod alpha into norm of x, is it not? Because, this is not…So, mod, norm of alpha x is not equal to mod of alpha into norm of x; that is, norm alpha x is not equal to mod alpha into norm of x. So, the condition N 2, N 3 fails. Therefore, S cannot be a normed spaces. So, S, under this, is not a normed space.

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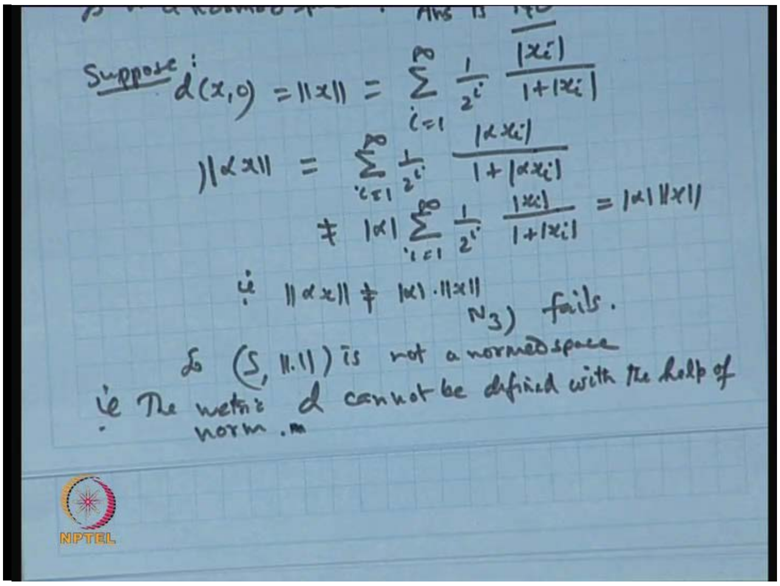

So, though it is a metric space, but it is not a norm; or you can say, the norm, metric cannot be defined with the help of the norm; that is, that is, the metric d, that is, the metric d cannot be defined or introduced with the help of norm, of norm, as norm in the form norm, that is all. Right, d x y equal to norm of x minus y, we cannot introduce, because it does not form. So, what we conclude that, every normed space is a metric space, but all the metric space need not be a normed space, clear.

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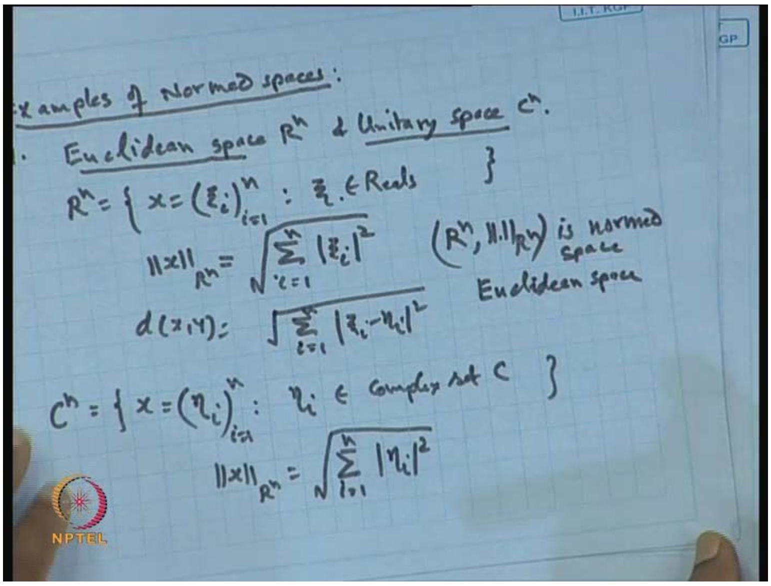

Now, let us see few examples, where it forms for both normed space as a metric space, as well as the metric space. So, examples of normed spaces. So, first example, Euclidean space, Euclidean space R n and unitary space C n, C n. R n means set of all n-tuples, 1 to n, where x i's are real and they are reals only; n-tuples with reals and the metric concept is defined as under root of this thing. So, here we are introducing the norm of x on R n, we are introducing as sigma i is 1 to n mod x i i is square under root. R n is the set of all n-tuples, x i 1, x i 2, x i n, where x i's are real and for each x belongs to R n and we introduce the norm as follows.

Now, it satisfy all the condition of norms; that, as because, we are verifying for the metric and this metric d on R n can be defined as under root sigma i is 1 to n mod x i i minus eta i whole square, that is all, ok. And that is the metric. So, R n with this norm is a normed space, is a normed space and we call it, this as Euclidean space, Euclidean space, it will be normed space. Then, C n is the set of all x, say eta i, eta i, such that, i is 1 to n, where eta i 's are complex number. They belongs to the complex set, complex set C; they are complex numbers, ok. And, introduce on this the norm of x R n as sigma i is 1 to n, mod of eta i square under root, where eta i's are complex numbers; complex, eta i are complex numbers, eta i's are complex numbers C.

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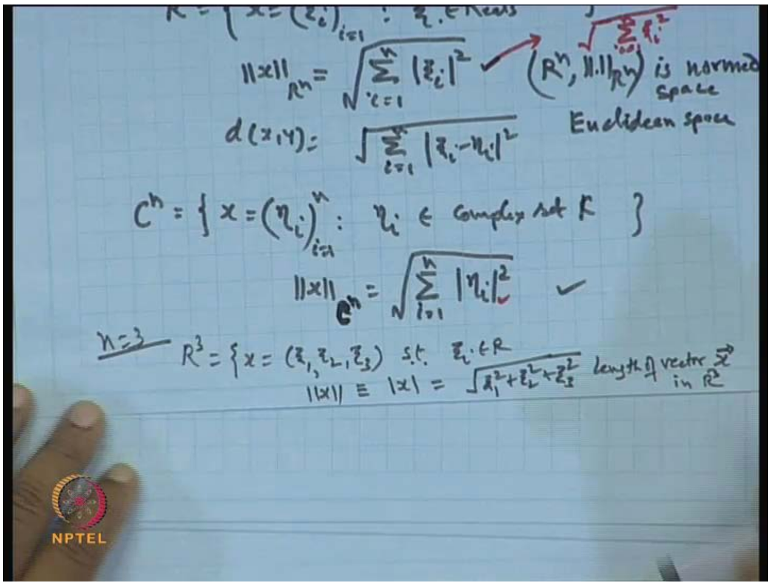

Oh, this is C n. This is C n. Now, if we look these two, actually, the difference between this, this norm of x n R can be written also, there is no point of writing the modulus here; you can write it this, in this form; this can be, in fact, should be written like this; under root sigma i is 1 to n, x i square, that is all. Why, because they are real. So, x i square will be positive, and we get; but here, we have to use this mod sign, because they are complex numbers. So, mod of eta i must be real. So, this basically, we can write this one or mod x i, there will no difference with, but here, you have to strictly use the mod, absolute value of this complex numbers and then, make the square. So, it forms this, clear.

Now, as a particular case when you take R n equal to 3, then, it becomes the vector space. So, when n is equal to 3, then, the R 3 is the set of those point x, x i 1, x i 2, x i 3, such that, x i's are real and the length of this mod x is under root x i 1 square, x i 2 square plus x i 3 square. And, that is the length of the vector, of vector x in the three dimensional plane; and, which is nothing, but norm of x. So, in particular, we are getting the length, clear. Now, they form the normed space, is it ok.

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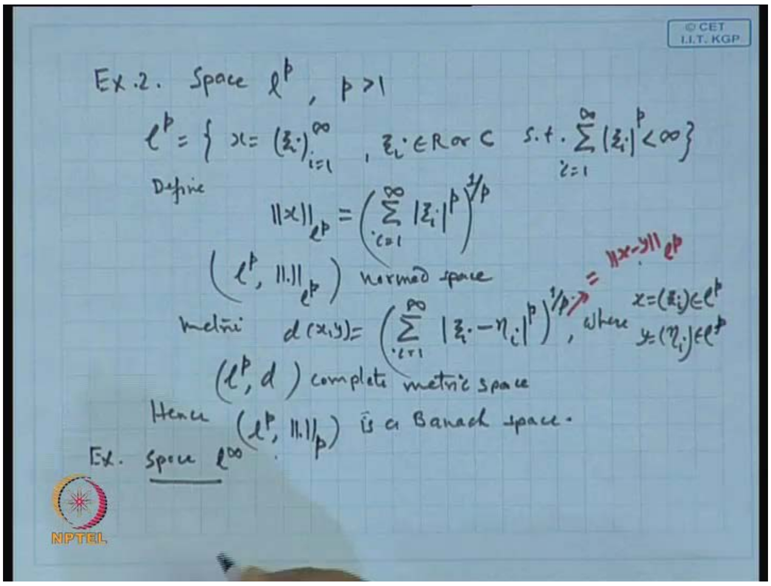

Then, we, another examples, two, the space l p, l p, p is greater than 1. The space l p is the set of those sequences x, x i's are real or complex number, R or C, such that, such that, sigma i is 1 to infinity mod x i i power p is finite, ok. And then, introduce the norm x l p as sigma i is 1 to infinity, mod x i i power p raised to the power 1 by p. So, it will form the normed space l p. It can be easy to verify that, this forms a normed space. And, in fact, if I introduce the metric d x y as sigma i equal to 1 to infinity, mod x i i minus eta i power p power 1 by p, where x and y are the points in the l p space, in the l p space, then, this forms a metric; l p under d is a complete metric space, is a complete metric space.

Hence, l p with this norm is a Banach space, because, what is the definition of Banach space? We have introduced the Banach space as a complete normed space and completeness is tested under the metric, introduced with the help of norm. So, this is the metric, which we have introduced with the help of norm that, d x y is the norm of x minus y, l p and that is equal to this. So, this way, because this is nothing, but, if we take this part, it is nothing, but, this is equal to norm of x minus y, is it not. So, d of x y defined in terms of norm and with this metric, it forms a complete metric space. Therefore, l p with norm p is a Banach space. Similarly, now, here, you see, the elements are not vectors, are not, they are the infinite sequences.

So, the elements are real, the infinite sequence of real or complex number and they form the…This is again a complete and completeness, how to test it? Completeness means, if a sequence, Cauchy sequence converges, then, it must be complete. So, when you say the Cauchy sequence converges, it means, that converge to a certain point. So, slowly, the distance between x 1 and x keep, keep on changing and reducing to 0. So, notion of the distance is important, when you test for the convergence of the sequence, whether Cauchy sequence or so. So, that notion of the distance, is introduced with the help of norm. So, a space which is a normed space, if a Cauchy sequence, every Cauchy sequence also converges in the metric introduced with the help of norm, then, such a space, we call it as a Banach space, clear. A complete normed space. So, this l p is one of them.

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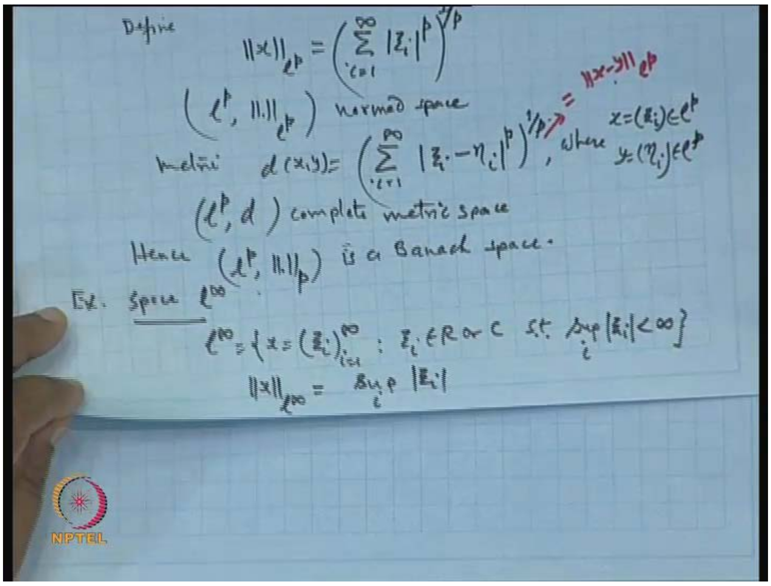

Now, another example is l infinity, l infinity space, the set of all bounded sequences; l infinity is the set of all bounded sequences of real or complex number. Set of all sequences x i i, i is 1 to infinity, where x i's are real or complex and such that, such that, the supremum of this thing, i is finite; that is, set of all bounded sequences of real or complex numbers; and on it, if we introduce the concept of the norm as the supremum of mod x i i over i, then, this l infinity, l infinity under this norm, l infinity under the norm x norm infinity forms this one.

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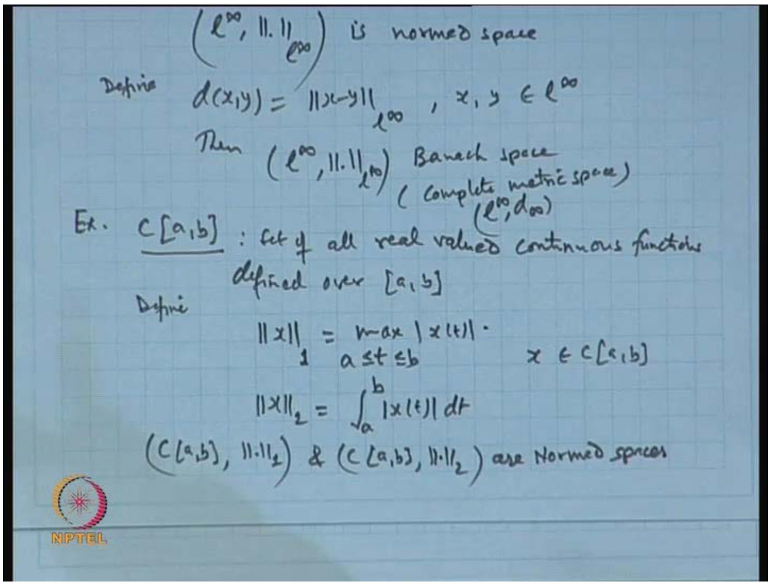

This is a normed space. And, not only this, it will be a complete normed space. It will be a…All the property of norms are satisfied and if we introduce the notion of the distance d x y on l infinity as norm of x minus y l infinity, where x and y, these are the elements of l infinity, then, this l infinity, under norm infinity, l infinity, this forms a Banach space. This forms a Banach space, because under this, it becomes a complete metric space; l infinity d is a complete metric space. So, this will be a Banach space it is a complete metric space, ok. l infinity comma, no.

l infinity comma d, complete, this is a Banach space. So, when you say, complete metric space means, l infinity d, it is complete metric space. Next is a C a b, set of all continuous functions, set of all real valued continuous functions defined over the closed interval a b. And, introduce the concept of the norm, norm of x 1; I am introducing two norms here; one is maximum of mod x t, x t, t lying between a and b, ok. Another way, I am introducing norm of x 2, as integral a to b, mod of x t d t, where the x belongs to C a b, ok. Now, both these forms a norm; C a b, under this and C a b, under this are normed spaces; that, one can verify without any problem. Both forms a normed space, because if we take the norm is greater than equal to 0, because you are taking absolute value, maximum of it, and if it is 0, then, maximum of x t equal to 0 means, the function x is such, which is identically 0, over throughout the interval a b; because once the maximum value of the function is 0, for all t belongs to, the function must be identically 0, clear.

Similarly, the other property, if you take alpha x, here the alpha comes. So, mod alpha will come outside. And then, x plus y, again maximum of this sum is less than equal to sum of their maximum. So, again, this is true. Similarly, here, in this case, if it is 0 means, this integral is 0; these are the non, continuous function and no alternative function, because the ((actual)) value is taken consideration; a and b are two finite points. Therefore, this has to be 0, x t must be identically 0 function. So, again, all the properties, we can verify, it forms a norm. So, this two spaces, the same space under the two different notion of the norm, forms the normed space.

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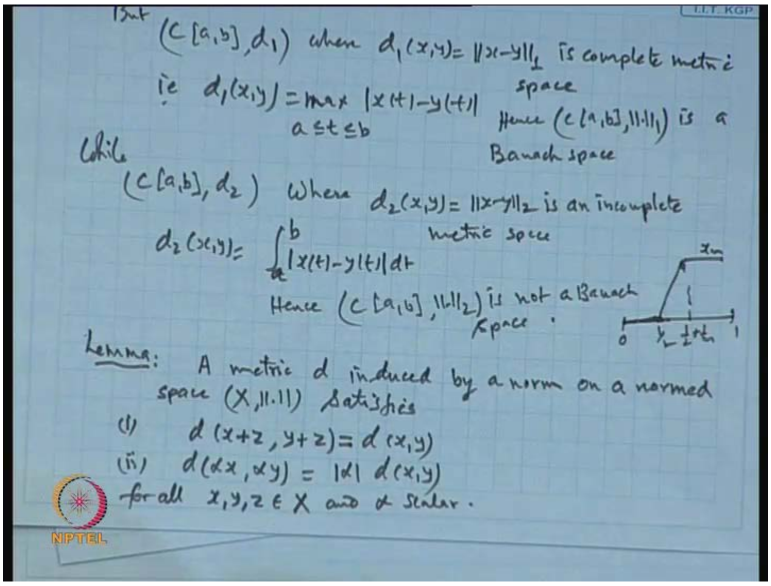

But, if I introduce the concept of the distance notion, here, in t C a b, with the help of first norm, but C a b with the norm d 1, where d 1 x y, which is equal to norm of x minus y 1, is a complete metric space, is a complete metric space, is a complete metric; that is, if I introduce the d 1 as maximum of, maximum of mod x t minus y t, t ranges over a and b, then, under this metric, it forms a complete metric space. Hence, C a b, under the norm1, is a Banach space, is a Banach space. Because, this, we have already verified in the metric space; then, if it is a, metric is defined in this fashion, it forms a complete metric space.

So, corresponding norm is a Banach space. While the C a b, with the metric d 2, where d 2 x y is introduced in terms of integral is an incomplete metric space, incomplete metric space; that is, when we introduced d 2 x y as integral a to b mod of x t minus y t d t, then, it is an incomplete metric space, is it not?. That we have seen also, by taking a Cauchy sequence, Cauchy sequence in this form. Say here, when we start with this and then, go like this and in this fashion, define between 0 to half, then, this is half plus 1 by m and this is 1. So, between 0 to 1, the sequence x n is defined like this. It is a continuous functions, but it forms a Cauchy sequence, but it is not convergent, because the limit point comes out with this continuous function.

So, it is an incomplete metric space. Hence, the C a b, with this norm 2, is not a Banach space. This is simply a normed space, but not Banach and…Clear? Because, it is not complete normed space. So, it is the incomplete normed space of this. Similarly, we can see other properties, other examples also. So, this will be, so many things are there. So, many examples one can say, which forms the metric space as well as the normed space. And, there are examples which forms only the metric space, but not the normed space. One of them is S, that we have seen. Now, this will be, this will be, again, we will have a general results, under what restriction, a metric, if it is introduced by a norm, what should be the restriction, so that, a one can obtain a metric, with the help of the norm; that is, a corresponding normed space, one can get the metric space with the help of…So, that lemma.

A metric d, a metric d induced by, induced by a norm, by a norm on a normed space, on a normed space X norm, on a normed space X norm satisfies, satisfies the following properties, satisfy, number 1, d of x plus z comma y plus z equal to d of x y. Second one is, d of alpha x comma alpha y equal to mod of alpha d of x y, for all x, y, z belongs to capital X and alpha is a scalar quantity. So, what this lemma says, suggest that, if we wanted to test whether the given metric can be introduced with the help of norm, these two conditions must be satisfied. If any one of the condition fails, it means, the metric d cannot be introduced with the help of the norm; cannot be induced by the norm.

In the earlier case, when we have seen, this metric, which we have S of, yes…If we look this metric, which we have defined, d of x y, d of x y equal to 1 upon 2 i mod of x i minus y i over 1 plus x…We have seen that, this is a metric, but not a norm. It means that, condition which we have introduced, both the condition may not be satisfying. Say, example, if I take second condition, introduce here; so, what will be the d of alpha? d of alpha x alpha y. What is this? This condition is, say sigma, i is 1 to infinity, 1 by 2 i and then, mod of alpha x i minus y i divided by 1 plus mod alpha x i minus y i, y i, is it not.

This will be…So, can you take alpha outside? You cannot take alpha outside, because as soon as you take alpha, this 1 plus alpha. So, you cannot take mod alpha outside. It means, the condition second y is not satisfying. So, this shows that, if any one of the condition fails, then, the corresponding metric cannot be induced by the norm.

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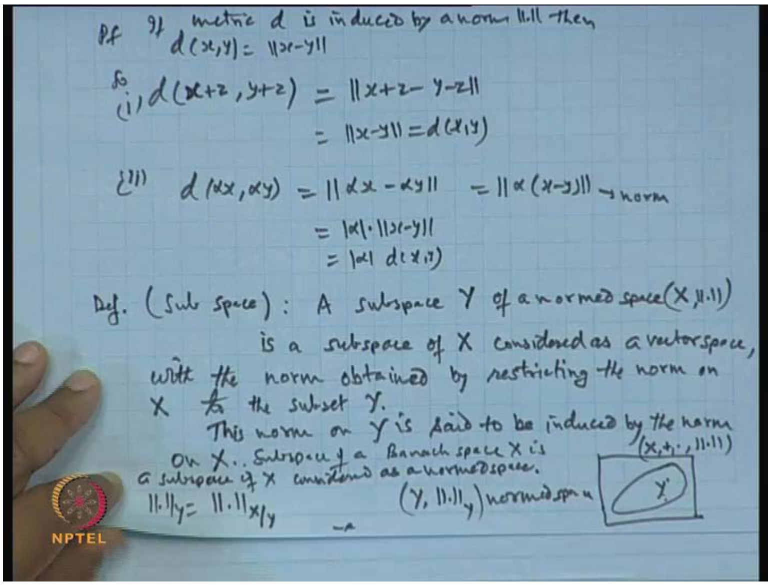

Let us see why it is so. The proof of this lemma is as, is very simple. If a metric d is induced by the norm, then, d of x y will be written as norm of x minus y, is it not. If that metric d is induced, if metric d induced, is induced by a norm, then, one can express d x y as norm of x minus y. So, what will be the d of x plus z comma y plus z? Is it not the same as norm of x plus z minus y minus z? But, z z get cancelled and finally, you are getting x minus y, which is equal to d of x y, clear. Second part. So, first is true. Second, if d of alpha x alpha y, then, this will be equal to norm of alpha x minus alpha y. So, this will be equal to norm of alpha x minus y, which is equal to mod alpha into norm of x minus y, because if this is a norm… So, once it is a norm, alpha can be taken outside with the modulus sign. So, we get this, which is equal to mod alpha d of x y, clear.

So, what we see that, if a metric is induced by a norm, this two condition has to be satisfied. And, that is the criteria to judge, whether the given metric can be induced by norm or not. Lemma is a part, is a very, some identity you wanted to prove it; that is a identity which are used for the further…That, we call it as a lemma or (( )), it is just small results, you can say. But, theorem is that, when, in general, anything, which is completed, is it not? Theorem (( )) and corollary, as a part of the theorem; corollary comes, a means, the result which can obtained with the help of the theorem. Lemma is used to prove, in, which is helpful in proving the theorems. It is not that, very big, important results we call it as a…

Now, the another concept of this metric space, we have the subspace and the corresponding sequence. So, definition of the subspace, or sub-normed space of a normed space. A sub-space Y of a normed space X norm, is a subspace of X, is a subspace of X, considered as, considered as a vector space, vector space, with the norm, with the norm, obtained by restricting, by restricting the norm on X, the norm on X, to the subset, to the subset Y, ok. This norm on Y, this norm on Y, is said to be, said to be induced by the norm on X. The subspace of a mean, if X be a vector space and then, if it is also a norm, a notion of the norm is introduced, so, x with norm is a normed space. A Y, which is a non-empty subset of X, this will form a sub-normed space, a subspace of a normed space, if the Y, with addition and multiplication forms a vector space; that is, Y must be a subspace of X and the restriction of norm on Y, of the norm of X, when it restrict on Y, then, this restriction, under this Y, must be a normed space; that is, this norm must be the restriction of the norm of X on Y, clear.

This is the norm of X on Y. So, we are denoting…Clear? So, if this norm, it is a normed space, it is a subspace, then, we call it (( )). Then, how to define the subspace of a Banach space? Subspace of a Banach means, it should be a simply a subspace of that space; need not be, x be complete; y need not be a complete space, ok. So, subspace of a Banach space, we mean, it is simply a subspace of X, considered as a normed space. So, subspace of a Banach space is, subspace of Banach space X is a subspace only; considered as a subspace; this is a subspace of X, considered as a normed space, not a Banach space. So, this is important, clear.

Means, that concept should not be in mind that, because, we are taking the subspace of a Banach space, so, subspace should also be a Banach space; it is not so. It is simply, must be a normed subspace, that is all. If, in addition, it is also a Banach space, that is a different matter. Suppose, it is a closed, Y is a closed subspace of a Banach space, then, it has to be a Banach also, clear. So, but, that condition in defining the subspace of normed space, the completeness is not required. So, thank you. Thanks