**Functional Analysis**

**Prof. P. D. Srivastava**

**Department of Mathematics**

**Indian Institute of Technology, Kharagpur**

**Module No. # 01**

**Lecture No. # 08**

# Vector Spaces with Examples

So, so far, we have discussed, the concept of the metric space, various examples and also the relation with others functions and all structures, various definitions, concepts, and so on. So, in the metric space, we have started with a set X, an arbitrary set and then, picking up the two points, we have introduced the concept of, notion of the distance on it. But, so, we have not yet considered the relation between the distance, notion of the distance and the addition or scalar multiplication between the points of this X; say, if x and y are the two points, then, we, one can add x plus y and one can also multiply alpha dot x, if the corresponding set is a structure, like a vector space.

So, once we replace the set X by a vector space, and then, we introduce the notion of the distance, as we have take, introduced earlier, then, it does not give any relation between the algebraic structures, that is, the addition and a scalar multiplication and the metric distance notion. So, we do not get a useful theory further. So, in order to have a useful metric space or useful theory, we wanted to define a metric, concept of the metric or notion of the distance, in a different way, by introducing a new concept of norm on a class, which we call it as a vector space. And then, with the help of that norm, we introduce the notion of the distance and that way, we are able to connect or give a relation between the addition or a scalar multiplication and the distance function and that will give you your usual structure.

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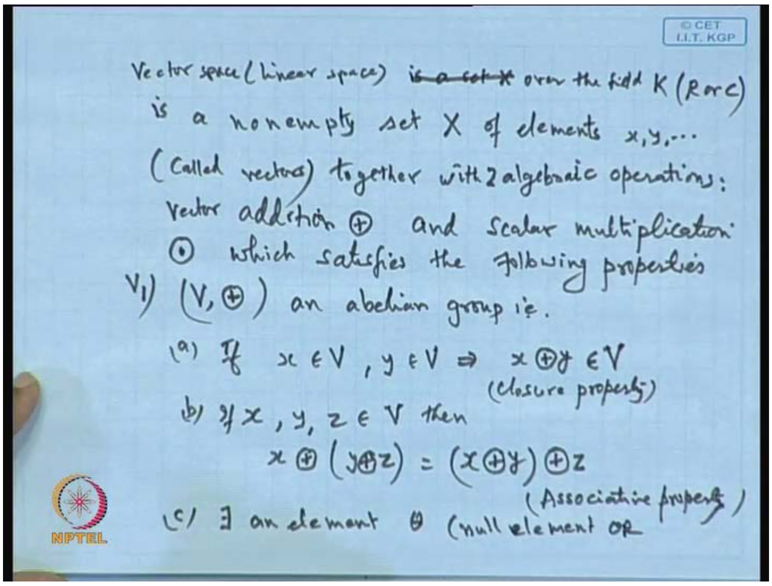


So, today, we will discuss, what is the vector space first, and then, how to introduce the concept of the distance with the help of norm on this vector space. So, let us see, what is the vector space. Vector space, sometimes, we, some authors, we also call it as a linear space. Vector space or linear space is, is a set X, vector space is a set X, over the field, over the field of vector space, over the field, vector space over the field K; here, K, we will take R or C; R means set of real number or C is a complex. So, a vector space over the field K is a nonempty, is a nonempty set capital X of elements, of elements x, y and so on, which we called a vectors, called as vectors, together with the two operations, with algebraic operations, operations, the two algebraic operations, with two algebraic operations, addition, which we call it as a vector addition, denoted by, say this, and scalar multiplication denoted by this.

So, with a two algebraic operations, one is vector addition, other one is a scalar multiplication, where the scalar is taken from the field K of this, which satisfies the following property, which satisfies the following property, properties. The first property is, say we take V 1, the set V, with addition, must be an abelian group abelian group; abelian group, we mean, that is, the following conditions are satisfied. If x is a Vector elements belonging to V, y is an element belonging to V, then, if we combine these two element by the operation addition, then, a new vector is obtained, must be the point of V; that is, we call it that, addition a closure property. This property is called the closure property or we say, the addition is closed with, a vector is closed with respect to addition.

Then, if x, y, z are the three points, then, x, y and z are the three, any arbitrary three points of V, then, the x plus y plus z, this will be equal to x plus y plus z, which is called, known as the associative property, associative property.

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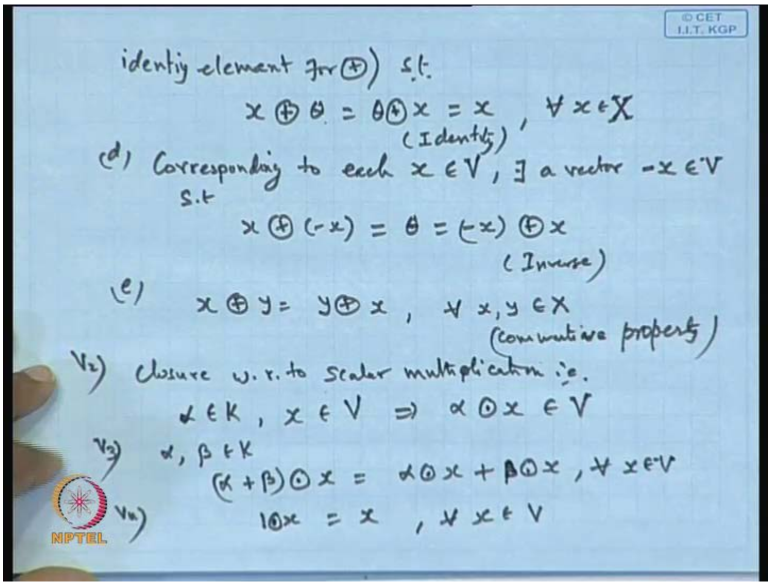


As third, there exists, ((this is time for they are exist)), of an element theta, which we called as the null vector, null element or identity element for addition or identity element for addition, such that, x plus this null element is the same as this plus null element, theta plus x, must be x and this is true for every x belonging to capital X. So, theta, which we denote as a null vectors or the identity element with respect to the operation addition, is satisfy this condition. Then, d, corresponding to each, to each vector x belonging to V, there must exist a vector, minus of x, which is also in V, such that, when we add these two vectors, the total should come out to be the identity elements for addition, whether it is added towards the left or towards the right.

So, this is the identity elements. Here, we say identity element; here, we say inverse exist. Then, another property is the commutative property; x plus y is the same as y plus x, for every x and y belonging to capital X. This is the commutative property with respect to, commutative property with respect to addition. Now, if this five properties are satisfied, then, we say the class V, with respect to the operation addition, is an abelian group. If only the first four properties are satisfied, then, we say, it is a non-abelian group; means commutative property does not hold good. So, that way, we can say.

In case, if there is only one side, property is true, for example, c, in case of c, if only one side is true, x plus theta equal to x or in the second, d also, if there is only one side, then, we say the left identity, right identity; like this, we define the left inverse or right inverse; the concept is correspondingly, can be defined, ok. So, first part is that, the set V, with respect to the addition, must form an abelian group. Now, second one is, V 2, with respect to a scalar multiplication, it must be closed. So, closed, closure property with respect to scalar multiplication; that is, if alpha is an element of k, x is an element of V, then, alpha dot x, this should be an element of V. So, we say, the V is closed with respect to scalar multiplication. Then, another property in this is, say V 3, alpha, beta are the two scalars in K, then, alpha plus beta x is the same as alpha dot x plus beta dot x. And, here also, we can say, V 4, 1dot x is x, for every x belonging to V; this is for every x belonging to V. V 4 and then, distributive properties. This is distributive.

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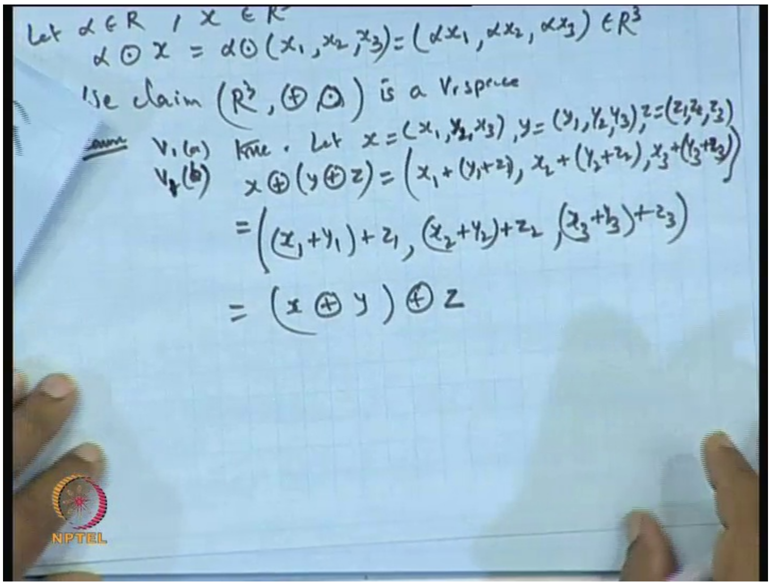


Also, V 5, we can say, alpha dot x plus y is the same as alpha dot x plus alpha dot y. Now, if, your x and y are the elements of V, for any arbitrary element x and y in V. Now, if we look these two properties, V 3 and V 5, the V 3, here the scalars are added and then, it is operated on x; but in case of this, the vectors are added and then, operated, alpha is operated on x. So, basically, you can say, the scalar multiplication is distributed over vector addition. So, we can write like this. Then, alpha beta dot x, that also properties, 6, alpha dot beta dot x, ok.

Sir, (( )) this is cross sign…

Yes, because, this is alpha plus beta; this is a ordinary plus, with respect to the scalar.

So, in circle is for…

In circle is for the vector addition, but this is not a way of, ordinary way. Because, what happens, when vectors are given, say, suppose, I take a vector in three dimensional plane, then, vectors will be, each vector will have a component, say, a 1, a 2, a 3. This is the vector, ok. Now, when we add the two vectors x and y, b 1, b 2, b 3, then, basically, we are taking x plus y. But when we add, then, these a 1, a 2, a 3, these are scalar; either it may be real or may be complex number. So, addition is done as per the real or complex rule. Say, addition of the two real number is real; addition of two complex is complex, and component y is, we added, first coordinate is added with the first coordinate; second coordinate added second coordinate, and like this.

So, in case of the vector, that is why, we require the vector field; because each vector, when we say vector, it has a component. And, once it has a component, what are those elements, whether they are real number, they are complex number or anything else. So, those must belongs to a certain class, and that class, called a field. Here, in case of functional analysis, we restrict our class, either on the real line or may be a complex plane; that is all, ok. So, there are different type operations. To distinguish these two, we have use that plus under the circle as the vector addition and dot within the circle as a scalar multiplication; scalar multiplication behind, a scalar is multiplied with a vector, then, the operation dot within circle will be used.

But if the two scalars are multiplied, then, there is no need; there is no need. And, we can, just like here, when you are taking alpha dot beta dot x, I will simply write, alpha beta dot x. In V 6, you see the difference. Here, alpha and beta are simply a real number. So, just a product is defined, ordinary multiplication; but here, dot is used; because, x is a vector quantity. So, that is why, it will be…And, in the left hand side, beta dot x will becomes a vector quantity. So, same way, you will use the same dot product. So, that is why, the difference must come out like this. Similarly, here, in this case…So, these properties are satisfied, then, we say, if a vector V satisfying with these property... So, how many total properties are there, if you look, there are five properties here, in V 1 and another five here, in there. So, total is ten properties.

So, if a class V, under the addition and a scalar multiplication, satisfies these ten property, then, we say V is a vector space over the field K, over the field K and we write it this, V over the field K, with addition and a scalar multiplication, is a vector space, ok. Sometimes, we omit the field K. Once we identify or it is understanding that, all the scalars are taken real, then, we know, the elements alpha, beta, these are the reals. So, we do not write the V R; we simply say V; or K is a complex, we also say V. So, some author, without writing the V K, or some use the V within the bracket K, it means, it is a over the field K. Now, K may be real or K may be complex. If K is real, R, then, this is called the real vector space, is a real vector space. And, if K is complex number, then, this is called a complex vector space. So, real and complex vector space, totally depends on the scalar, on the field, what field you are using and that is…Clear?

So, this gives you concept. Now, one more thing, which I, normally confusion is, when we take 0 dot x, it means, what is the meaning? 0 is the, simply a real point, real number 0; when it multiply by a vector quantity x, then, what you get? You get a vector 0 vector quantity or null vector or the identity elements with respect to the addition. So, these two should be…That also one should keep in mind. Now, there are many examples of the vector spaces. The most simple and useful example is our, say X, which is, say R 3; R 3 means, set of all elements, vectors; elements are say x 1, x 2, x 3; coordinates are x 1, x 2, x 3, where the xis, these are real number. So, set of all tuples, triples in the three dimensional plane. So, these are the vectors, ok.

Now, on this, we are defining the addition and multiplication as follows. Dot as follows. If we take two vectors x plus y, then, this will be equal to, x means, x 1 comma x 2 comma x 3, plus y means, y 1 comma y 2 comma y 3 and the addition is done coordinate wise, and we are getting a vector x 1 plus y 1 comma x 2 plus y 2, x 3 plus y 3, ok. Now, again, x 1, x 2, x 3, y 1, y 2, y 3, these are all real numbers, where yis is also real. So, x 1 plus y 1, x 2 plus y 2, x 3plus y 3 will also be real number. Let it be, this numbers, any number, all say z 1, z 2, z 3. So, it is again an element of V; V means R 3, because it is a triple, a vector in a three dimensional plane. So, if x is a vector in three dimensional plane, y is also vector, if we define the addition in this fashion, you are getting the new vector, which is again a point, point of R 3.

And, if we introduce the concept of dot, then, alpha dot x is a real number. Let alpha be the real number; x be an elements of R 3; then, what we get, alpha dot x 1 comma x 2 comma x 3 and that will be equal to, alpha x 1, alpha x 2, alpha x 3. So, this way, we are defining, ok. This is again a point of R 3. So, addition is…With respect to addition, R 3 is closed. With respect to scalar multiplication, R 3 is closed. So, basically, the first and the, this property is satisfied. The first property, a, closure property, with respect to addition is satisfied. Then, the property, closure property, with respect to scalar multiplication is satisfied. So, whenever you introduce the addition and a scalar multiplication, you must see that, these two properties must hold good; otherwise, it will not form a vector space. So, these two things, you should introduce in such a way, so that, these two must be…

Now, you can verify all these properties are satisfied. With these two operation, if we look, then, what are the property, let us see, start with this. The first property is already true. The second property, associative property, if I look this one, x and y, this is our definition; x plus y is this 1. So, suppose, I take one more point z, then, what happen this? x plus y plus z, this will be…So, we claim, R 3, with respect to this, is a vector space. Why, because, the first property is true, V 1 a is true; V 2 a, V 1 b, V 1 b is, if we take x plus y plus z, then, what happens is, this will be equal to… I am just writing x 1 plus y 1 plus z 1, ok. Then, simply x 2 plus y 2 plus z 2 and then, x 3 plus y 3 plus z 3, y 3 plus z 3, this is, and this will be same as, where x is, let x is, x 1, y 1, x 1, x 2, x 3; y is y 1, y 2, y 3 and z is z 1, z 2, z 3, ok. So, if we introduce this, then, what we get? Over the first bracket, x 1, y 1, z 1, these are the real numbers. So, real number satisfy the associative property. So, we can write x 1 plus y 1 plus z 1. Similarly, you have the second. Again, these are real number and real numbers are satisfying the associative property. So, we can interchange the position of the bracket and we get x 3 plus y 3 plus z 3 and that is nothing, but x plus y plus z. So, associative is satisfied, ok.

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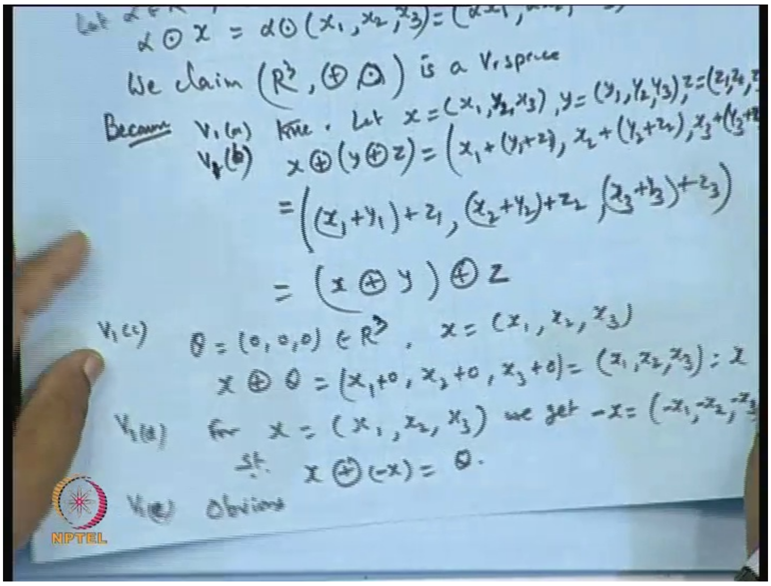

Then, V 1 c. V 1 c is, there exist on a null element or an identity element, which satisfy this condition, x plus theta is theta plus x plus x. So, we claim, if we take theta, null element as 0, 0, 0, which is in R 3 and it satisfy the condition…If we take x as x 1, x 2, x 3, then, x plus theta is the same as x; because it is nothing, but x 1 plus 0, x 2 plus 0, x 3 plus 0; that will be the same as, x 1, x 2, x 3, that is equal to x. Similarly, other way round also, ok. And, V 1 d. For corresponding to each x, for x, which is x 1, x 2, x 3, we get minus x as minus x 1, minus x 2, minus x 3, such that, x plus minus x, this is equal to null. Because, x 1 minus x 1 is 0, 0, 0. So, a vector is obtained, clear. So, we are getting this. Then, e property, V 1 e also satisfy obviously; you can verify it. x plus y is the same as y plus x. So, the property V 1, all the five property under V 1 are satisfied. Therefore, it forms a abelian group. And then, V 2 is already defined; it is belongs to V. So, it is satisfied. Now, V 3, if we look the V 3, you can see, alpha plus beta dot x, is alpha dot x plus beta dot x; I think this can be done. So, V 3, V 4, you can check, it satisfied.

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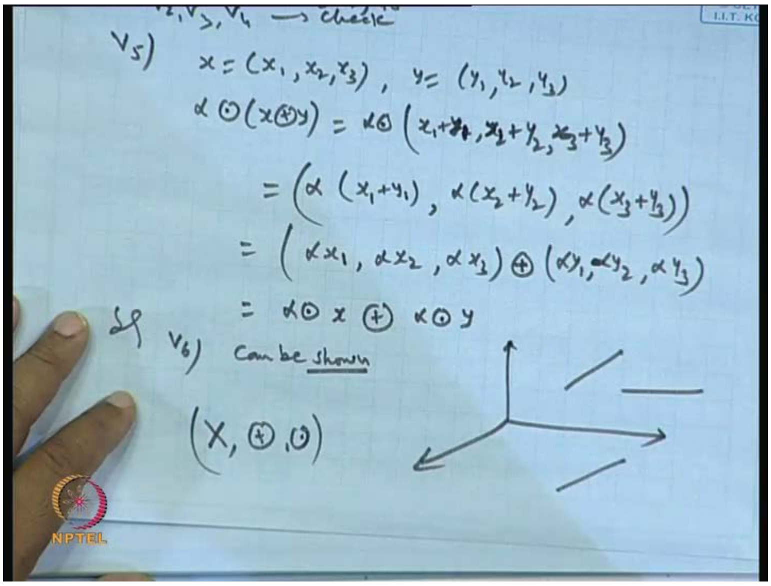

And then, V 5, V 5, and V 6. So, V 5 is alpha dot… So, V 2, V 3, V 4, check. That is easy to check, easy to check. Then, we go for the V 5. V 5 says, if alpha dot x plus y equal to alpha dot x plus alpha dot y. So, let us take x as x 1, x 2, x 3, y, y 1, y 2, y 3, then, we take alpha dot x plus y, x plus y. So, that is equal to same as, alpha dot, what is x plus y? x 1 plus x 2, y 1, x 1 plus y 1, sorry, x 1 plus y 1, x 2 plus y 2, comma x 3 plus y 3, x 3 plus y 3; that is alpha dot. Now, this, multiply alpha multiply by this. So, x 1 plus y 1 comma alpha times x 2 plus y 2 comma alpha times x 3 plus y 3; and then, it can be written as alpha x 1 comma alpha x 2 comma alpha x 3 plus alpha y 1 comma alpha y 2 comma alpha y 3. I just showed, cut short the two steps, ok.

Then, take alpha outside. So, it is alpha times of x plus alpha times of y. So, we can say the V 5 is satisfied. Similarly, V 6 can be proved, clear. So, this forms here. It means, in a three dimensional plane, if we consider the vectors, these are the vectors, collection of all such vectors and then, addition and scalar multiplication is defined earlier. So, it forms a vector space. So, since the elements R, R 3, elements of R 3 are the vectors and they satisfy all the ten property of the vector space, therefore, any structure X, any arbitrary set X, with addition and scalar multiplication, if it satisfy those ten property, we call it as a vector space. Why the vector word is used, because, as a particular case, when X is replaced by R 3, it satisfies those ten properties, clear. And since, those are very standard notations, to call it as a vector, so, that is why, the correspondingly, we say it is a vector space. But it does not mean that, always the elements of X must be a vectors; it does not mean; it may be anything. It may be sequences. It may be functions; maybe anything; only thing, we have to define suitably, the two structure, addition and a scalar multiplication, so that, all the ten properties are satisfied, ok.

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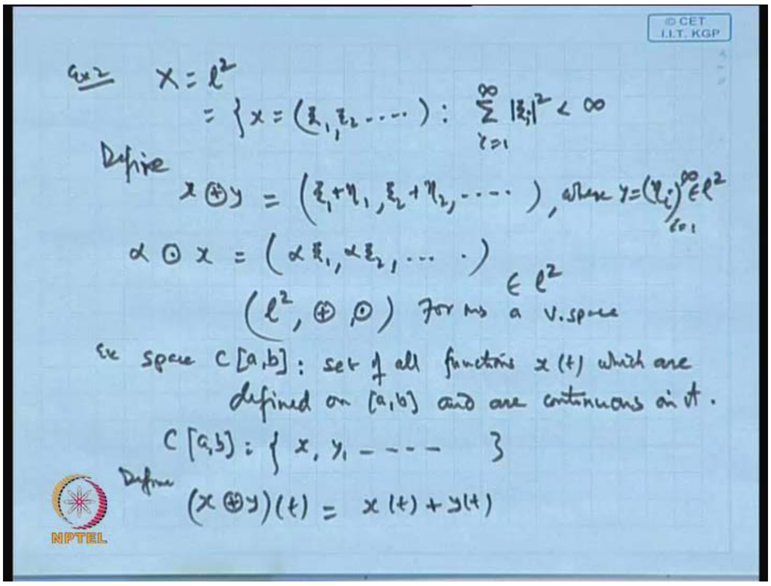


For example, if I take another problem, which is a, say L 2 space, L 2 space. Now, what is L 2 space? L 2 is the set of those sequences x i 1, x i 2 and so on, infinite sequences, such that, sigma of mod x i i square is finite, is it not. L 2 space, this is finite. So, when you are choosing these sequences, L 2, then, find out the, define the structure, x plus y as x i 1 plus eta 1, x i 2 plus eta 2 and so on, where y means, eta i, belonging to L 2, i is 1 to infinity. So, this one and if we define the dot product as alpha x i 1, alpha x i 2 and so on, then, we say, they are all the elements of L 2, closure property. And, if we go, just like a previous case, we will see that, this L 2, under these two operations, forms a vector space, forms a vector space, clear. Now, if this xis are complex, then, it is a complex vector space; if it is real, then it is a real vector. So, there no need for modulus sign, if it is a real vector, point. Then, we have another one, which is also interesting, the space C a b. C a b is the set of all functions x t, which are defined, which are defined on the closed interval a b and are continuous on it, on it.

So, set of all continuous function, defined over the closed interval a b and are defined, well defined, is continuous. So, C a b, let it be the set of all x, y and so on. Let us define the operation two, addition and scalar multiplication as, x plus y, x plus y, as, since these are the function at the point t, so, we define like this: x t plus y t, x t plus y t, ok. These are the functional values and this is the function. So, we are using that, if this plus sign, for the functional, we are like this. And then, alpha dot x t, we are defining as alpha into x t, alpha into x t, ok.

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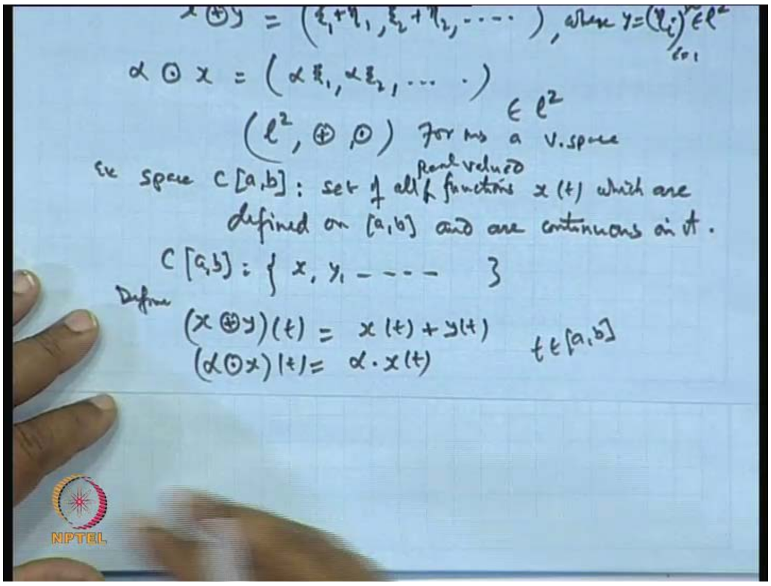


So, we are getting this alpha dot x t is alpha dot, where, what is t? t is an interval belonging to a b, point in the interval a b. Now, if we look this definition, here, these are the functions, x and y are functions, just like f and g; and then, combination of this defined over a t will give the values at the point t, individually and then, addition with x. So, the…Now, with these two operation, we will see, it forms a vector space. Closure property is already satisfied. Why, if x is a continuous function, y is a continuous function, then, x plus y must be a continuous function. Then, any continuous function, constant times of the continuous function remains continuous. So, this is continuous. So, both the closure properties, with respect to addition and scalar multiplication are satisfied, ok. Then, associative property holds because, the two, three functions are there; with respect to the plus, they have associative property, because, these are all real numbers; this is the real valued function.

You should write, set of all real valued function, real valued, sorry, set of all real valued functions defined, which are defined on a b and the continuous function. So, this form. Now, what is the identity element, for addition? What is the identity element for addition? Is there any continuous function which satisfy the x plus 0 equal to 0 plus x? So, in fact, a function which is identically 0, that is also continuous function; constant functions are continuous.

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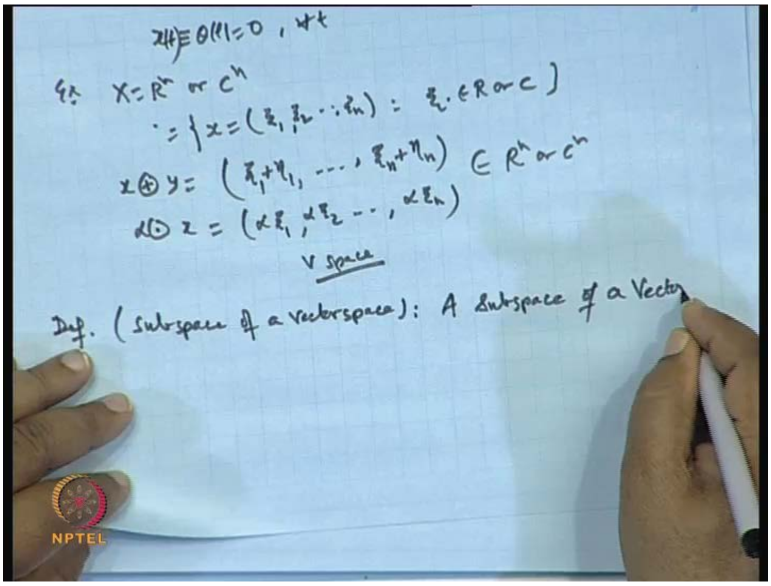

So, the function which is identically 0, will behave as a identity element; x is equal to theta.

For all t?

For all ts; that is, x t equal to 0, 0, clear; that is, x t is equal to 0, for all t, for all t, then, it is a identity function and inverse is, if x is a function, then, minus x t will be a inversion. So, all the properties are satisfied by this, clear. Now, another is, say x is R n or C n, they also form a vector space. What is R n C n? It is the n-tuples, where xis are either real or complex. So, it is the extension of R 3, extension of C 3. And, if we introduce the two operation as x plus y as x i 1 plus eta 1, x i 2 plus eta 2, x i n plus eta n and alpha dot x equal to alpha x i 1, alpha x i 2, alpha x i n, then, these are all belongs to R n or C n and it forms a vector space, ok.

So, there are many many examples of the vector spaces are available, where the elements are, need not be a vectors; just like, in case of the function C a b, the elements are not vector at all; they are functions. So, we get that concept, which is parallel to our, this one. Now, here, we also require few more concept, like a subspace of a vector space, vector space. A subspace of a vector space, of a vector space, a subspace of a vector space X, which is, vector space X is a nonempty, is a nonempty subset, capital Y of X, nonempty subset, capital Y of X, such that, for all y 1, y 2 belonging to capital Y and all scalars alpha, beta belonging to K, we have, we have alpha y 1 plus beta y 2 must be a point of Y. What is the meaning of this? How to define the subspaces?

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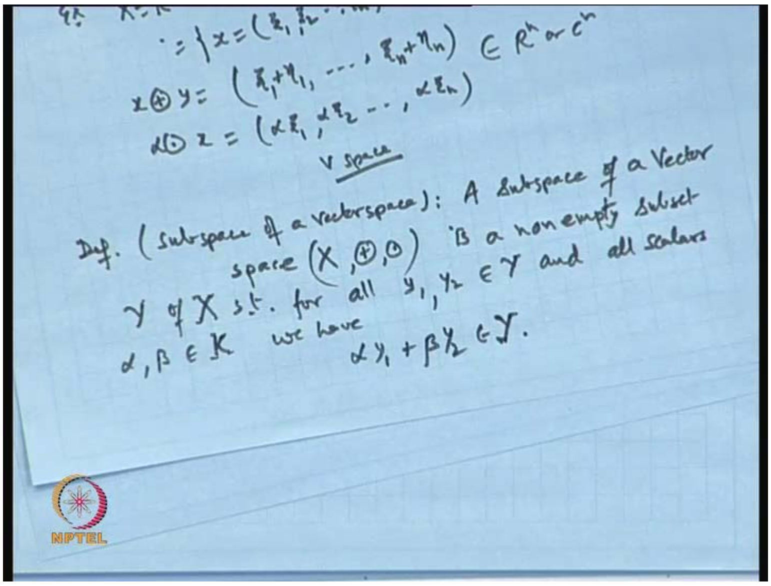

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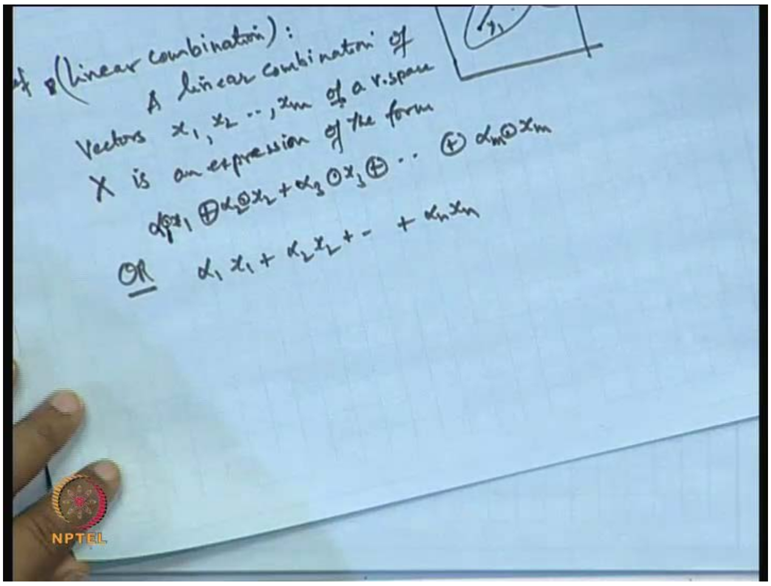

The subspace is, basically, defined like this. Suppose, we have a vector space V, where the operations, addition and scalar multiplications, are defined. So, if we take any two points here, the addition of the two points, are again here, and scalar multiplication is also here. Now, Y is a nonempty subset of V; the elements of Y are also the elements of V, because it is nonempty subset. So, if Y, with respect to addition and scalar multiplication, is also a vector, is also a vector space, then, we say Y is a subspace of V, clear. But, in order to prove the Y, with the addition and scalar multiplication is satisfy, you have to verify the ten properties. Those ten property can be avoided, simply, you have to justify this condition.

What condition is that, if suppose, y 1 and y 2 is a point in Y, then, if we take the linear combination of this y 1 y 2, that is, alpha y 1 plus beta y 2, then, that vector must be a point in Y. Now, only this condition is enough. Suppose, I take alpha and beta equal to 1,1, then, basically, the closure property is satisfied. If I take alpha to be 0, then, a scalar multiplication is closed. Similarly, if we take beta to be minus 1, alpha to be 0, it is a additive inverse for y 2, and all the properties can be derived. So, for a subspace, it is enough to show that, if we picked up the two point y 1 and y 2, the linear combination alpha y 1 plus beta y 2, must be a point of Y; that is form a vector subspace of this, ok.

Now, there are another concepts, which we have now, the span of M. What is the linear combination? First, let us we linear combination, definition, combination. A linear combination, a linear combination of a vectors, of vectors, x 1, x 2, x m of a vector space X, vector space X, is an expression of the form alpha 1 x 1 plus alpha 2 x 2. In fact, the dot product will be there, ok. And, here, it is this, plus alpha 3 x 3 plus alpha m x m; an expression of this form, alpha 1 x 1 and so on, clear. This expression, we call it as a linear combination; this gives a linear combination of x 1, x 2, x 3. Now, here, if we look the dot and scalar, it is a, gives a very clumsy picture. So, what we do is, we replace this in circle and dot, we simply, the thing, just without writing anything, simply plus and dot. So, we can, in simply way, we can write, alpha 1 x 1 plus alpha 2 x 2 plus alpha n x n, this expression; it is more attractive, rather than to use this circle and plus.

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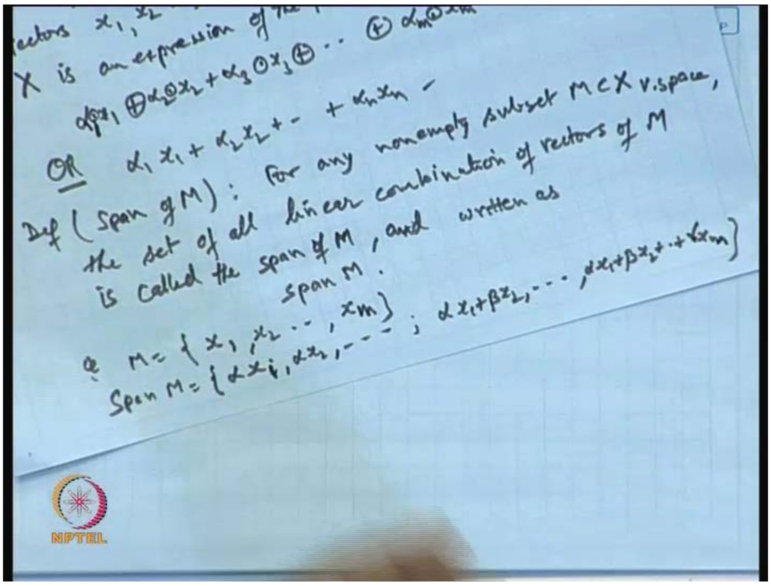

But the understanding is that, between alpha and x 1, that scalar product is defined; between these two way, because this is one vector, this is another vector; the addition is the vector addition. So, this is a understanding for it, ok. So, henceforth, we will use, simply, plus or dot, without any circle. So, that. Then, next is the span, span of M. For any nonempty subset, for any nonempty subset M of X, X is a vector space, X, which is a vector space, nonempty subset M of X, this set, the set of all, the set of all linear combination, set of all linear combination of vectors of M, of vectors of M, is called, is called the span of M, the span of M and denoted by, and written as span of M.

What is the meaning of this is, that is, if M is a set, having the vector, say, x 1, x 2, x n, m, say, then, what will be the span of M? The span of M will be of this type, alpha x i, i is 1 to n, that is, alpha of x 1, alpha of x 2 and so on; or it may be of this type also, alpha x 1 beta x 2 and so on, continue, is it not? Or may be of this type, alpha x 1 alpha beta x 2 and so on and let it be, say, gamma x n, like this. Means, you can pick up any elements of M, find the linear combination; it may be the 1 element, 2 element, 3 elements, all n element, at a time. So, n, m element at a time. So, all combinations, all possible linear combinations, the collection of this, gives you the span; and, this collection span will form a vector space.

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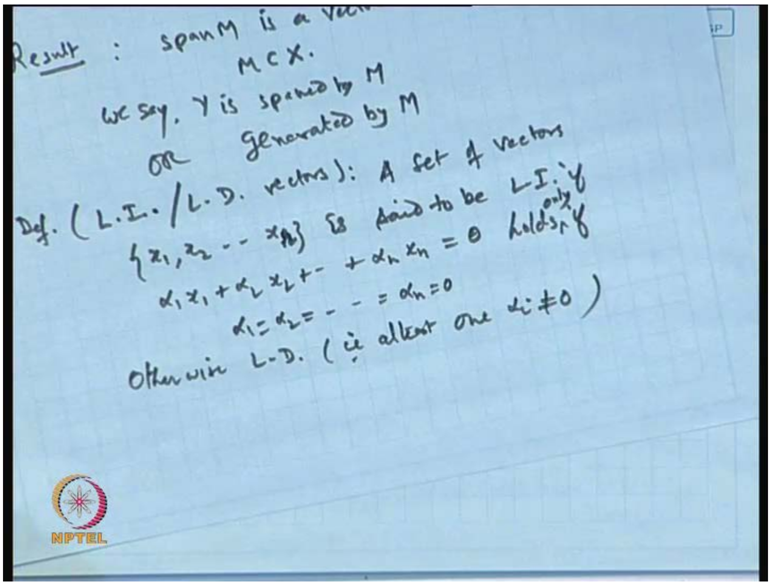

Result is, the span of M is a Vector subspace of X, where M is a subset of X. When M is nonempty subset of X, the span of M will form a vector subspace, under the same operation of the vector space X. So, it forms the vector. And, we say, by ((ignoring)) the M, span of a vector. And, we say the, this is generated by M, this subspace is generated by M, spanned by M or generated by M. We say, we say, is a subspace Y and we say Y is spanned by M or generated by M, clear. This will generated by M. Then, linearly dependence and independent, linearly independent or depend linearly dependent vectors. A set of vectors x 1, x 2, x n or x r, is said to be linearly independent, if alpha 1 x 1 plus alpha 2 x 2 plus alpha n x n equal to 0 or theta, 0 vector holds, if, if all alpha's are 0; only if all alpha's are 0; holds, only if, only, you can use the word only also; if all alpha's are 0. Then, we say, x 1, x 2, x n is a linearly independent vectors; otherwise, linearly dependent. At least, one of the alpha is is different from 0; that is, at least, one of alpha is is not 0, then, it is a linearly dependent vectors, ok.

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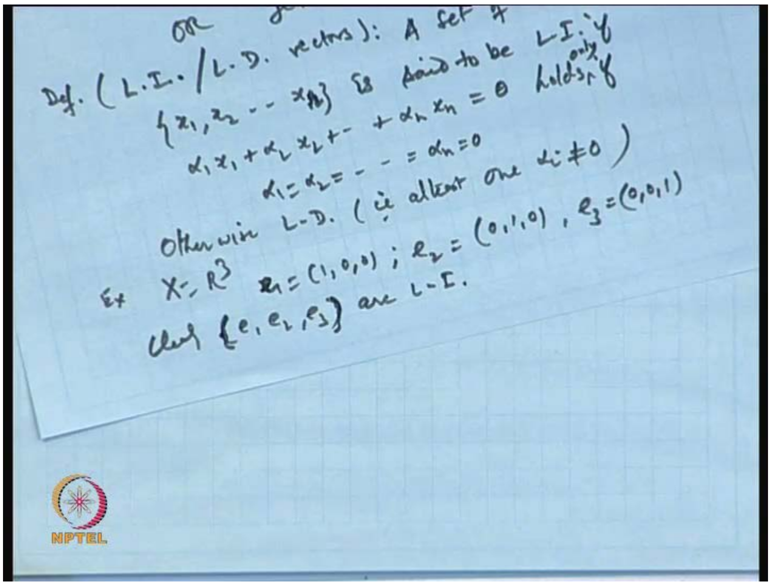


So, it will be…For example, if we take, say R 3, and find out the vector, say, x 1 as 1, 0, 0, e 1, let it be e 1; e 2 as 0, 1, 0 and e 3 as 0, 0, 1. Now, these vectors e 1, e 2, e 3, these vectors are linearly independent; they form a linearly independent set.

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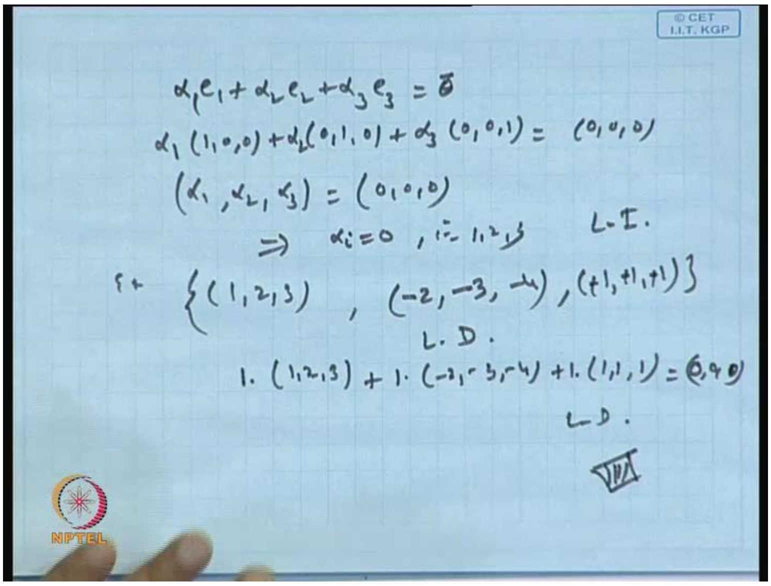


Why, because, because, if we take the alpha of some scalars, alpha of e 1 plus, alpha1, plus alpha 2 e 2 plus alpha 3 e 3 is a vector 0, then, it means, it is of the form alpha 1 into 1, 0, 0, alpha 2 0, 1, 0, alpha 3 0, 0, 1, which must be equal to 0, 0. So, that is the same as, alpha 1 comma alpha 2 comma alpha 3 is 0, 0, 0. This implies, alpha is are 0, when i is 1, 2, 3. So, it is a linearly independent vectors. But, if the vectors, if we take the vector like, 1, 2, 3, say minus 2, minus 3, minus 4 and minus 1, minus 1, minus 1; no, plus 1, plus 1, plus1, yes. So, if we look these vectors, this set of vector is a linearly dependent vector. Why, because, if we multiply this one by 1, 2, 3, then, again one by this, again one by this, this is 0, is it not? But alpha is are not 0. So, it is a linearly dependent vector. So, in fact, any one of the vector can be expressed as linear combination of the other two; if it is linearly dependent vectors, then, one of the vector can be expressed linearly dependent vector.

So, I hope this is, will give a sufficient hint and sufficient material for the vector space concept, which we will use in defining the norm. So, next class, we will discuss, what is the norm and how to introduce the metric. Thank you.