**Functional Analysis**

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**Lecture No. # 07**

**Completion of Metric Spaces and Tutorial**

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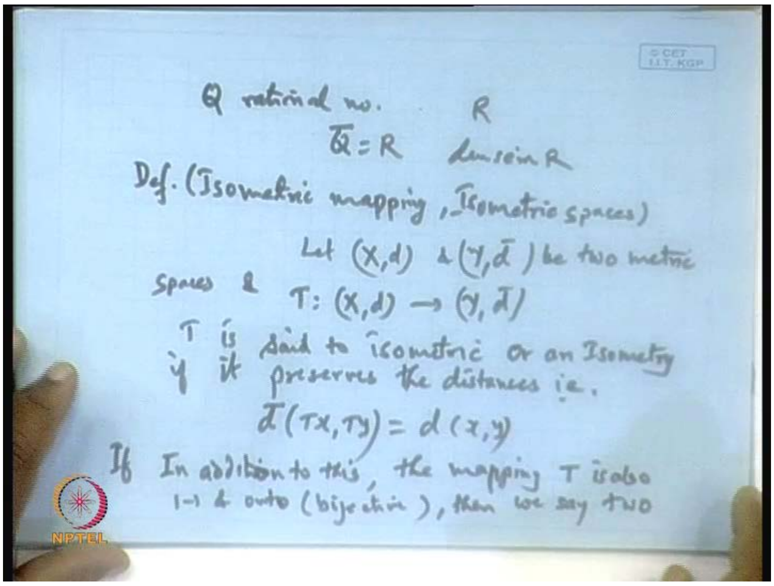


So, last time we have seen few examples of complete and incomplete metric space and one of that example was, a rational number Q, is an incomplete metric space in R with usual to approach, but this rational number has a characteristic, that the closure of this is R, that is, it is a dense in R. So, this suggests a way to convert an incomplete metric space to a complete metric space and this requires the definition or the concepts of the isometric metric. So, before going for the result for completion, we will see, the, how to define the isometric metric, isometric mapping, isometric spaces.

Let (X, d) and (Y, d), (X, d) and Y, d bar we say, 2 metric spaces and T is a mapping from metric space (X, d) to metric space Y, d bar. Now, this mapping T is said to be isometric, is said to be isometric, or an isometry if it preserves, if it preserves the distances, that is, meaning of this is, that the distance of (Tx, Ty) under the metric d bar is the same as the distance between x and y. Then, we say, this mapping T is an isometric or isometry from one metric to another metric space.

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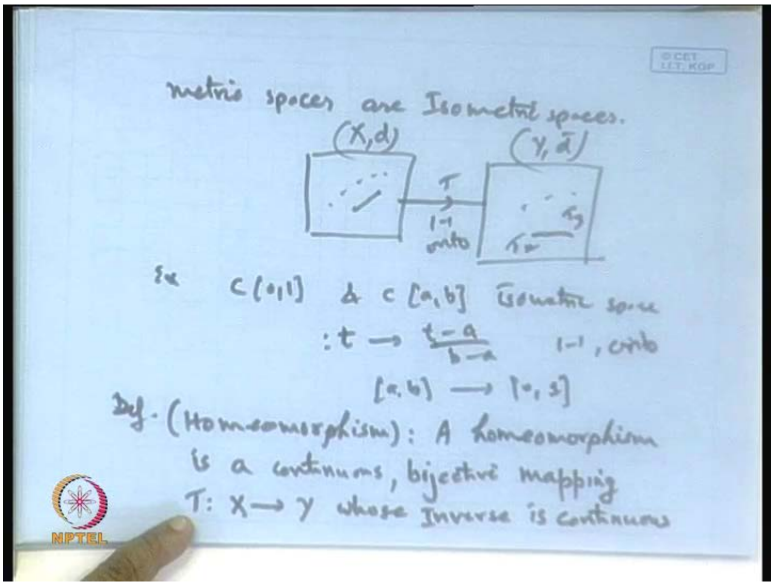


Now, in addition to this, if the mapping T, if in addition to this, in addition to this, the mapping T, the mapping T is also 1, 1 and onto, that is a bijective mapping, 1, 1 map onto mapping from x to y, then we say 2 metric spaces, 2 metric spaces are iso, are isometric spaces, are isometric. That is, a mapping from one metrics to another metric, which is 1, 1 onto and preserve the metric also, then this 2 metrics are said to be isometric metric.

So, basically, when we have an isometric mapping or 2 spaces are isometric, then the nature of the elements may different. Here, that elements points are different here, the points are different. But what is important part is, there is a 1 to 1 correspondence, 1, 1 onto mapping T and if we pickup any 2 points here and find out the distance, the corresponding image is here, will have the same length distance here between Tx and Ty.

So, whatever the properties regarding the convergence, cauchyness, etcetera are required, that retains or the property, which involves the metric concepts will remain the same. So, they are basically considered as a carbon copy to each other, though the points are different, but so far the properties, metric properties are concerned, they behave more or less the same sets. So, this is the advantage of this.

Now, examples are, say C C 0, 1 and C a, b; these are isometric spaces, you can define the mapping T. Suppose, I define a mapping, such that T goes to t minus a over b minus a, so the transfer a to 0, b to 1. So, interval a, b is transferred to 0, 1 interval and this mapping is a 1, 1 mapping onto mapping and if we define the distance as the maximum distance, then you will see, both will have the same distances. So, these mapping 2 spaces are isometric spaces.

Then, another concept, which we require is homeomorphism, homeo, this is a new concept, homeomorphism, homeomorphism. A homeomorphism is a continuous, continuous bijective, continuous bijective mapping T from x to y whose inverse is continuous, is continuous. It means a mapping from one metric space to another metric space, which is 1, 1 onto continuous and inverse is also continuous, then such a mapping is called the homeomorphism mapping.

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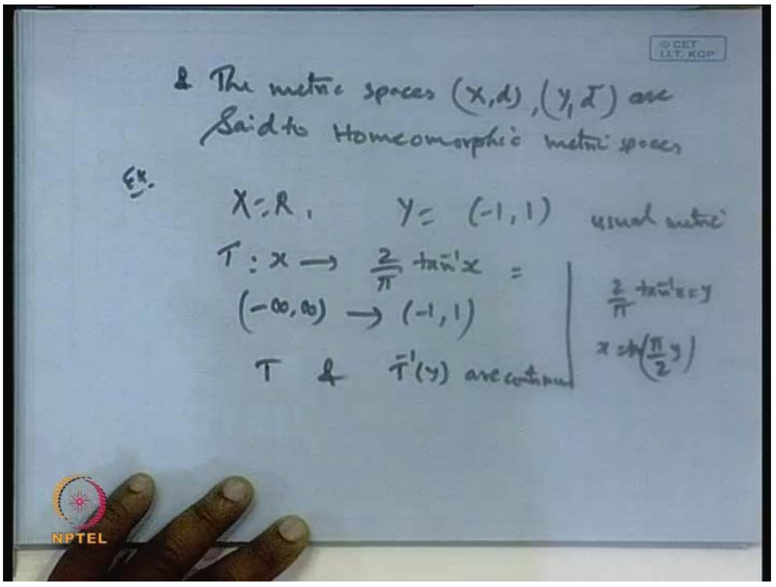

Homeomorphic, homeomorphism mapping from one metrics to another and if this 2 spaces are said to be homeomorphic spaces, the metric spaces x y (x, d) y, d bar, these are said to be, y, d are said to be homeomorphic metric spaces.

Again, here we give one example, say suppose I take R and 0, x is R, y is minus 1 to 1. And define a mapping T from this to this as x to 2 by pi tan inverse x or tan inverse x is a r, is a metric space under the usual metrics, y is an open interval, again is part of R and the same metric d x by, we can define as a usual metric. Then, if we define a mapping T, which stands for x to 2 by pi tan inverse x, then minus infinity to infinity is transferred to minus 1 to... when x is minus infinity tan inverse minus pi by 2. So, this becomes minus 1 and when it is infinity. So, when x tends to y by 2 it goes to infinity, so it tends to 1. So, it is open interval minus 1 to 1.

Now, this mapping T is a continuous mapping when we take the inverse. What is the inverse, inverse will be equal to what? pi by 2 tan x tan of pi by 2 y. So, this is the inverse mapping, T inverse y, suppose this is equal to y, is it not. So, we can write there, inverse of mapping is the inverse y, that will be equal to from here because this will be 2 over pi tan inverse x is y. So, we can write x equal to pi by 2 y and tan of this clear. So, accordingly you can get, again this is continuous function. So, both T and T inverse, T and T inverse are continuous is 1, 1 onto mapping, 1, 1 onto continuous, therefore this is a…

Now, one thing, which we observed here in this, that in case of the homeomorphic mapping, you are getting a mapping T from complete metric to incomplete metric space. It means, in case of the homeomorphic mapping is not required or is not necessary, that only the complete metric will transfer to the complete metric, but in case of the isometric mapping, this is true if the 2 spaces are isometric. Then, similar type of cauchy sequences will behave in the other metrics and we have the corresponding properties of the metric, are same as in one or second; so that is 2 differences.

Now, the question, if suppose we have an incomplete metric space, how to convert it into a complete metric space? And that is known as the completion of the metric space.

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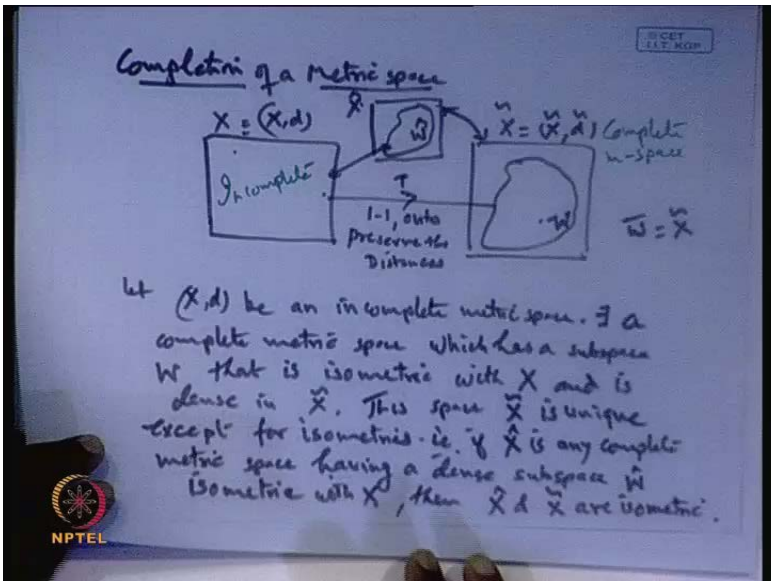


So, we go for the completion as completion of a metric space. Here, we will simply give a way how to add. Suppose, we have a metric (X, d), (X, d) be a metric space, be an incomplete metric space, be an incomplete metric space, then corresponding to this exist, then there exist a complete metric space, complete metric space x delta, which is equal to say, x delta d delta. This is say x; they are corresponding to each incomplete metric space. There exist a complete metric space x delta, which has a subspace, which has a subspace W, that is, this is a subspace W, which has a subspace W, that is isometric, that is isometric with X, that is, we can find out a mapping from x to W, which is 1, 1 onto and preserve the distance, preserve the distances and isometric with X, which is isometric with X and each dense in X delta, that is, W bar is equal to X delta.

So, what is that? If X be an incomplete metric space, then one can always find out a complete metric of X such that there will be a subspace W available in this space X hat, which is isometric with X and this w is dense in X.

For example, if I take, replace X by q, let us say a separation number, then corresponding to that q we can always get a complete metric r, which has a subspace w, that is, q isometric with x, that is q itself and closure of q is r. Now, this is unique, corresponding to each x we get one and only one x except for isometry. It means, this is a, that is, this space, this space X hat, which you are getting is unique; corresponding to each incomplete, we get one and only one X hat except, except, except for isometrics.

What do you mean by this is, except for isometric means, that is, if we suppose there is another metric X hat, which is a completion of X, then it has a W hat isometric with this dense in it, then these 2 spaces X z and X will also be isometric. So, we can say, this space X hat is unique except for isometry, that is, if X hat is any complete metric space, complete metric space having a dense subspace, dense subspace w hat isometric with X, with X, then X hat and x delta are isometric. This is the meaning of this.

We repeat again what we get, that we are getting, suppose this is an incomplete metric space, this is an incomplete metric space. What this result says, that corresponding to each incomplete metric space we can always find a complete metric space, complete metric space in such a way, then it has a W, subspace W, which is dense in it and isometric with X. And this space X delta, which you get is unique except under isometry, that is, if there exist another metric X hat, which also, which is also a complete metric space corresponding to X, then these X hat and X delta must be isometric, must be isometric, so that we cannot get a different one because once you take, the 2 spaces are isometric, the elements may be of different nature, but their metric properties remains the same. So, that is why, it will be unique always; that is fine, clear. So, this (( )).

Now, this almost completing, we are completed almost all the concepts. Now, we will take a few here the problems, which gives you further emphasis on this concepts and one of them is that concept on separable metrics.

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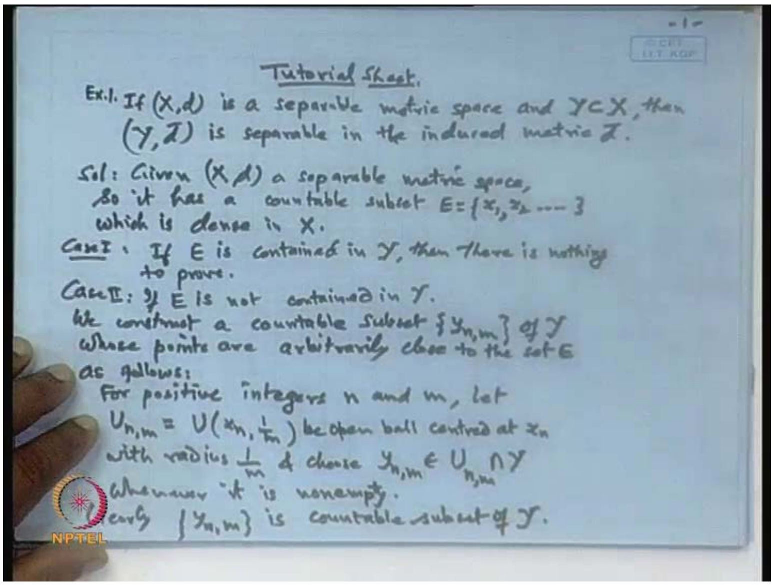

So, we have one problem here, what this problem says, that if X, d is a separable metric space and Y is a subspace of X, then Y d bar is separable in the induced metric d bar.

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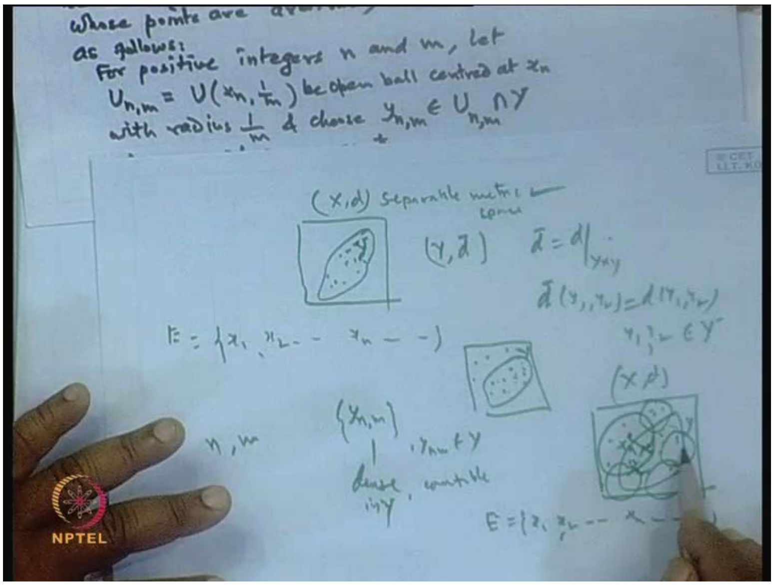


What this problem is, that suppose, we have X, d, which is a given to be a separable metric space. Now, if I take a subset y, subspace y of x, then what is the guarantee, that this subspace under the restricted metric is also separable? But what this problem says, that if we have a subspace, then this subspace has to be separable under the induced metric d bar. Induced metric means, that if n by restrict, that d on y cross y, then it must be d bar, that is, the distance of (y 1, y 2) is the same as y 1 and y 2. So, far y 1 and y 2 belongs to capital Y, if the points are in y 1, points are in y, then what these metric d bar and d will give? The same distance; but if the points are in x, then these values may not be same.

So, what it says is, if x be a complete, is a separable metric space, then its subspace under the induced metric will also be separable. So, what is required to prove is that this y under the metric d bar must have a countable subset which is dense in y. So, let us see the proof of it.

It is given, that x d be a separable metric space, it means there will be a set E, it has a countable subset E, which is dense in this, say suppose x 1, x 2, x n etcetera. This is the countable subsets, countable subsets of x, which it dense in x by definition. Now, if this set E is contained inside completely y, then our problem is solved because then E is also countable, as well as dense in y. So, there is nothing to prove, y will itself be a separable.

But if the elements of E partly lies outside of the y, that is, some points are here, some are insides, then you cannot say, that E is, can be treated as a countable dense subset of y. So, what we have to do is, we have to find out the points in y, a sequence of the points in y, which is a dense, which forms a dense subspace of y. So, our aim is to find a sequence y n, m of the elements of y, such that this is dense and countable; it is countable and dense in y. So, that is why, how to get this one.

Let us suppose, this is our x d and here it is y, these are the points of x 1, x 2, x n here, this is e, which is x 1, x 2, x n and so on. Now, for a positive integer n, for a positive integer n, n, m, let us find out a ball centered at x, n and with a radius say, 1 by n; this is 1 by n radius. So, we get a set U n, m centered at x n and radius 1 by n, around each point you are getting like this of x 1, x 2, x n. Now, some of them will intersect y and some of them may not intersect also because, because all the points are not inside y, even they are not a boundary point also. So, with suitable n, m you can get some of these, say, balls will definitely intersect. So, once they intersect, you pick up the point from here, say y 1, y 2, y n, which is the common point of this ball, as well as y.

So, what we did is that around the point x n, we have drawn the ball with center 1 by n and this ball, when it intersects with y, then you can pick up a point y n, m. So, with x 1 you are drawing a ball with a radius 1 by n, pickup a point y 1 m with x to you, draw the ball 1 by m, pick up a point y 2 m.

Now, this y 1 m, y 2 m, all may not be available. So, some may be empty also, said…

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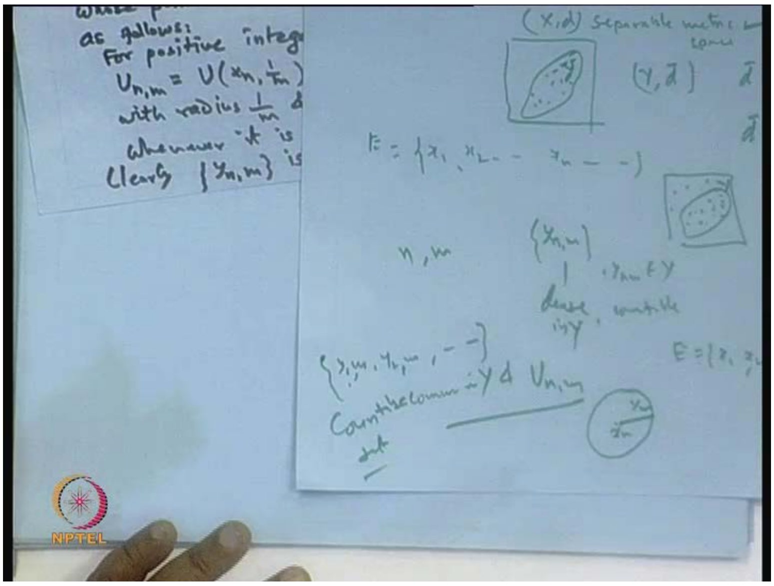

It means, which all, so in that case you are not able to get, but at least, you can get a sequence y n 1, y 1 n, y 2 m etcetera, which you get a sequence like this y 1 m, y 2 m and so on. These are the points common in y as well as U n, m, is it correct, this.

Now, these points if you remember, these are basically the points in this ball centered at x n with a radius 1 by n and x 1, x 2, x n is already given to be countable. It means, these balls are countable in number. Now, some of the balls, does, do not intersect with y. So, remove those balls, remaining ball will remain still countable. Therefore, the y 1 y 1 m, y 2 m, etcetera, this form a countable set, agreed, and these are the points of y, is it correct.

So, we get like this that first draw the ball, then find out its intersection with y. If it is nonempty, then pick up the point and then form a sequence and this sequence obviously, be a countable set of y. So, this…

Now, we are now interested to show, that this sequence is not only countable in y, but also dense in y if I…

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(( )) this is also dense in y; we wanted to show this is y dense in y.

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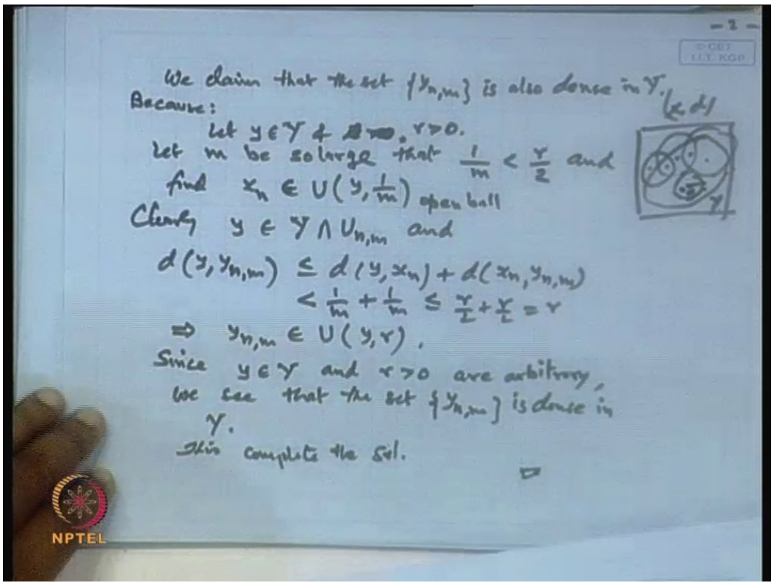


So, how to prove the dense in y means, if this is a set y, this is our x and here, this is y and these are the balls, these are the balls like this, where this y 1, y 2, y n, (y 1, n), (y n, m) are situated.

Now, if we want this to be this points, set is a dense in y, it means, you take any arbitrary point y. So, if we take any arbitrary point y, in y if we draw a ball around the point y with a radius, say r, then this ball must contain, this ball must contain, at least some point of (y n, m), then it becomes dense. So, let us take, take a point y and a suitable radius r, then draw the ball and pickup m so large, so that this will satisfy the condition.

So, if I take a ball around the point y with a radius 1 by n, then since x 1, x 2, x n, these are dense in x, so they are also scattered in y. So, some of the xi's will be available in this ball. Let one of these xi, say x n call it that point to be x n. So, let x n be a point available here with lies inside a ball centered y and radius 1 by n. So, x n will be this, but x n centered with radius 1 by n is a U m n, U n m. So, basically, the intersection of y n m in U m is the element of y. So, y belongs to this class, as well as, x n, this (( )).

Now, let us take the distance from y to a point (y n, m). So, this will be equal to d of y x n plus d of x n, x m by just angular inequality. Now, x n belongs to this ball, it means, its distance from y, center y, any arbitrary point will remain less than 1 by m. So, this is 1 by m, this is also 1 by m, so total becomes… And since m is so large, it is r by 2. So, it becomes r, it means, if we draw the ball around the point y with the radius r, then it includes the point y. Therefore, y n, m will be dense in this or you can say, otherwise if we take y, y 1, y 2, y n, draw the ball, you can get a point, arbitrary point y, clear. And this is arbitrary, therefore y n m belongs to this ball centered y radius r, hence this will be a dense. So, any subset, subspace of a separable metric space is also separable under the induced metric, clear. I think it is clear or you need (( )).

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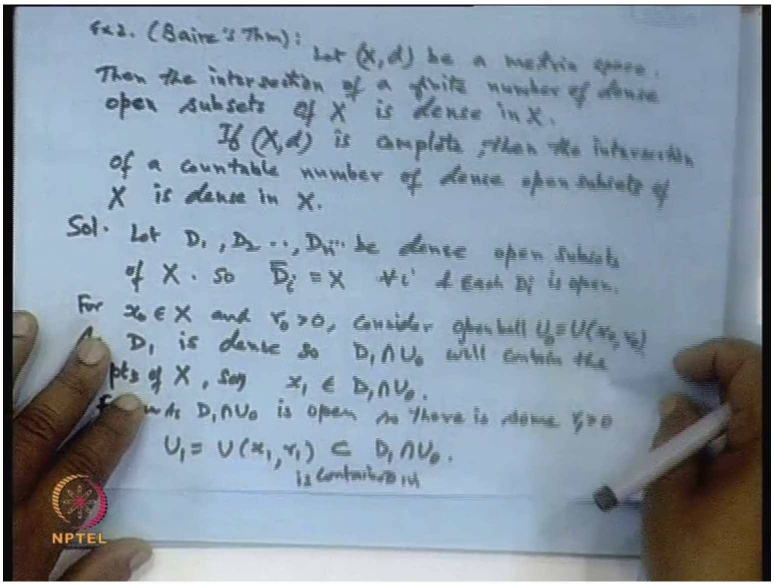


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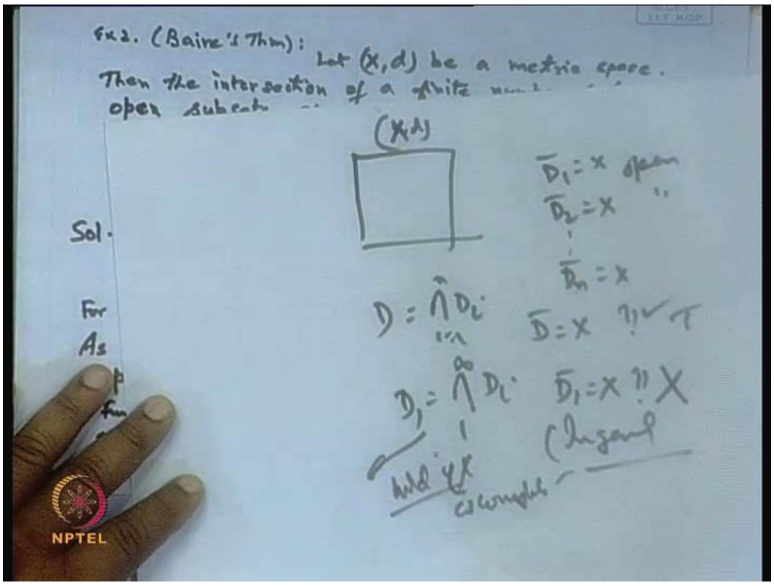

Now, another result, which also important and that is, that is known as the Baire's theorem. What this result says is that if suppose we have a metric space X d, this is our metric space X d and suppose, they are all finite number of dense open subsets, means suppose D 1 is dense, that is, D 1 closure is X; D 2 is also dense; D n is also dense, means, there are so many dense space, subspaces of X can be obtained. Now, what about the intersection of D i, i is 1 to… Will this D also it dense in x, this is the question; one thing.

2nd one, if we take a countable number of dense sets of x and then, find the countable intersection, whether this countable intersection, say D 1, whether this will also be a dense set in x?

So, these 2 questions are answered by Baire's. So, what Baire's told, that whether x is a complete metric space or incomplete metric space, if there are finite number of the dense subspaces of x, then the finite intersection of those dense subspace will also be a dense subset and these are open, dense open subset; these are dense open subsets. Remember, openness is also important.

However, if I do not take x to be a complete metric space, then this result in general, not true; this is true in general. But if we take an arbitrary intersection of the dense open subset of a metric space x, then the arbitrary intersection of the dense open subset need not be at dense in x, but if x to be a complete metric space, then this result will hold if x is complete. So, that is very interesting result, in the sense, that one can, without any each, take the sequence of the denseness, find out the intersection and it will guarantee, that there will be a dense subspace available, that is, it will be nonempty, we can always get, the closure will be entire space x. So, that is one.

The proof runs like this; let us take the finite first case when x is incomplete metric space and D 1, D 2, D n be a dense open subset of x.

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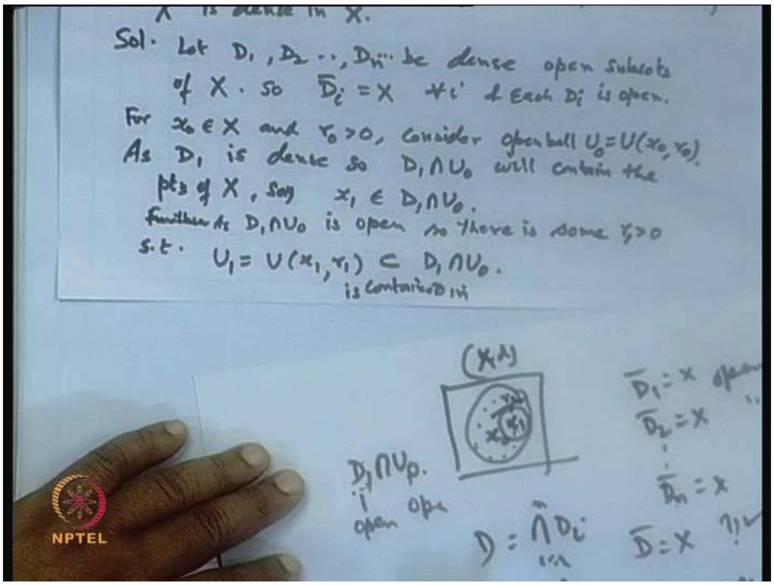


So, by the property, each 1 closure will be x. By this, now let us take a point x, n are not a positive number. Draw the ball around the point x naught with a radius, say r naught since D 1 is dense in x. So, it means, the point of x, this ball and the point of D 1 are very close because it is dense in x. So, we can find out, identify a point, say x 1 inside this ball, which is the intersection of D 1 intersection U naught, which is available because D 1 is dense and U naught is an open ball. So, we can find a point x 1, which is common to D 1 and U, U naught.

Now, further, D 1 is open, U naught is also open. So, intersection of the open set is open. So, it means, we can draw the ball around the point x 1 with a radius, suitable radius, say r 1, which is totally contained inside it. So, we can get the ball x 1, y 1, which totally contained inside it, is it clear.

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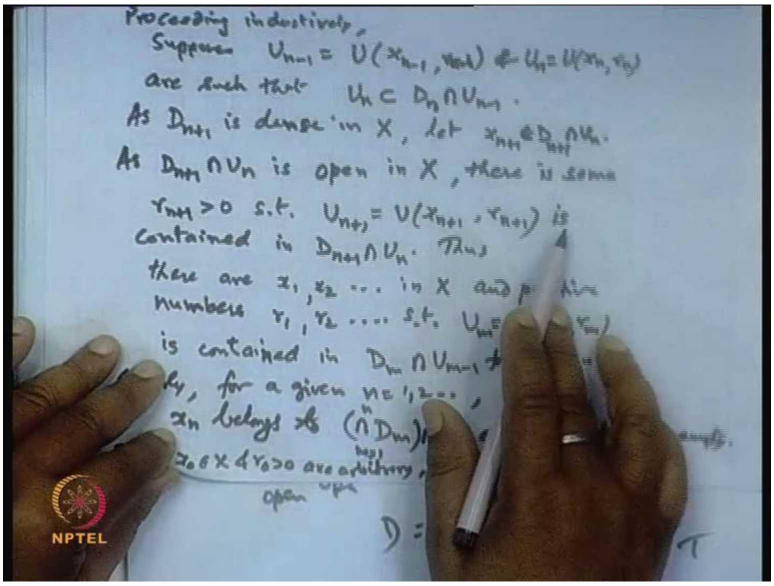

Once you have continued this process (( )) take another, so that at the nth stage, what you get it, you are drawing a ball centered at x n minus 1 with a radius r n minus 1 and this ball has and the D n, because it is dense. So, intersection part will be nonempty and it is open. So, it will get a one point, x, n will be available and a ball U n around this, which is totally contained inside it. So, continue this.

Now, let us take the x n plus 1 in these elements and then, corresponding ball we obtained U n plus 1 centered at x n plus 1 and so on. Now, since x 1, x 2, x n, these are the points in x, r 1, r 2, r n suitably in this. So, if you find the intersection of these balls, what happened? The intersection of these balls basically is contained in this a, this intersection x naught will contain the point x n and intersection will be nonempty because every time this ball, you are getting again intersection. So, you are getting again ball, which get the point x 2; similarly x n. So, when you take the intersection, this U naught, you will get a point x n available, that is, nonempty. So, whether x is complete or incomplete, unlimited, we get the intersection of the dense open sets will remain dense, that is, one clear always (( )).

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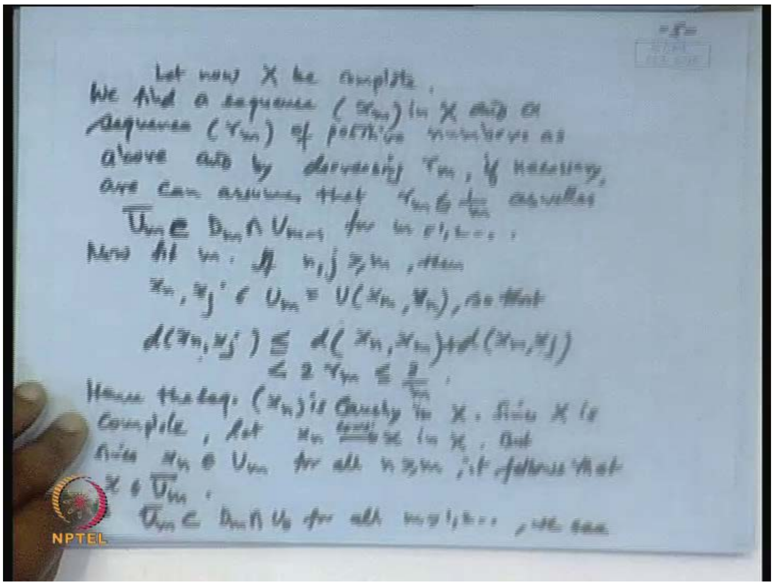


In case if it is complete, if suppose x is a complete space, then even arbitrary intersection of the dense open set will be dense. So, this we have seen. Suppose, it is complete, let us take a sequence x n and r n, the same procedure as we defined, but it is a decreasing nature and choose r m, such that it is less than 1 by m.

Now, let us pickup the 2 point in x n and x m in ball centered at x m and radius r n, then what is the distance? This x n, x m plus x m, x n. Now, this is less than r m, this is less than r m, so 2 times r m and this will be less than 2 by m.

So, when m is sufficiently large, this distance standing to 0, it means, the sequence x n is a cauchy sequence, but x is complete. So, it must converge and converge to the point x in x. So, we are getting a point x, which belongs to the closure.

So, this belongs to closure means, it is in the, contained in D m U ball and from here, one can say, it belongs to the intersection, hence it is nonempty. So, we get the intersection of this is dense, so is it. So, this part will be clear, I think it is. I will just put it, that this again, so that you can go through.

This is the first sheet, clear. Then, again balls are obtained, intersection these and this thing. So, finite case is over and then, for the, if x is complete sequence, find a sequence like previous way, R n previously, only restriction is this and then you can find out the points, so that this is a cauchy sequence and then limiting point is available. So, we get a sequence and once you get this, then it belongs to the intersection. So, it will arbitrary because x naught is arbitrary and r naught is also arbitrary. Therefore, this will be a dense at X.

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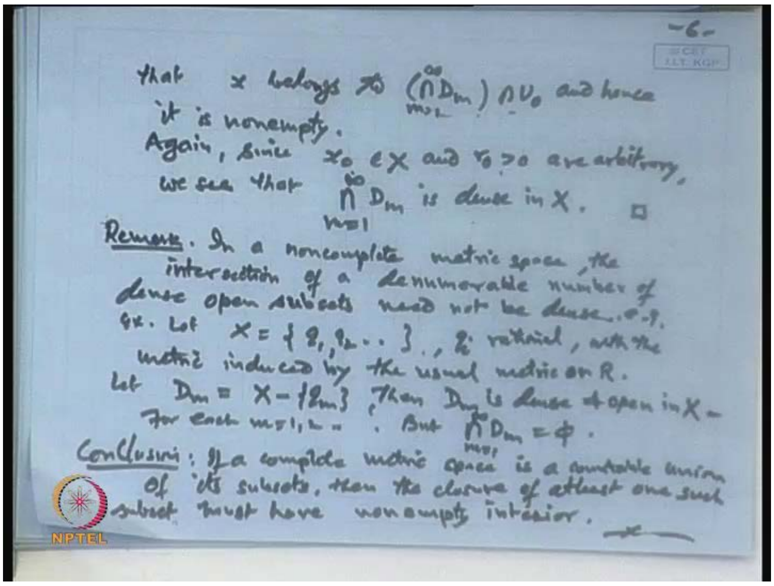


Now, what is if we are taking X to be an incomplete metric space, then this result does not hold good. It means, in case of an incomplete metric space, the arbitrary intersection of the dense set need not be dense. So, this is like this.

Suppose, we have a set X, which is, suppose I take the set q 1, q 2, q n of rational numbers X with metric induced by the usual metric or r set of rational number. So, it is an incomplete set.

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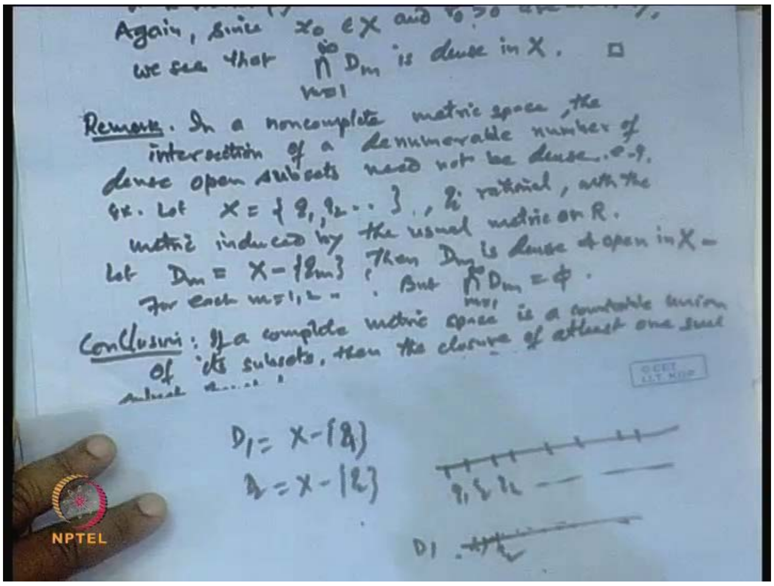


Now, let us find out a D 1, D 2, D n as x minus q m, that is, you are constructing D 1 as x minus q 1, D 2 as x minus q 2. So, basically, this is the real line and here are this q 1, q 2, q n and so on; these are rational numbers and completed.

Now, what we are doing is we are taking a D 1, where this portion is removed and rest of the things I have taken, including all irrational points also because x minus q 1, x is r, x, sorry, x minus x is q 1, q 2, q n, x minus this. Now, this will be dense in open in x and it will be, in open in x, and each intersection will be empty, why empty? Because e l is q 1, is not available here, q 2 is not available. So, when you take the intersection part, it will come out to be the empty set. So, this is the arbitrary intersection, comes out to be an empty, nonempty set.

So, we can conclude, that in a complete metric space, in a, if a complete metric space is a countable union of its subset, then the closure of at least one such subsets must have nonempty interior, that is what this. So, these 2 examples, which though it is given in the form of the example, but both are very interesting results and it will be used, particularly the Baire’s theorem, it will be used.

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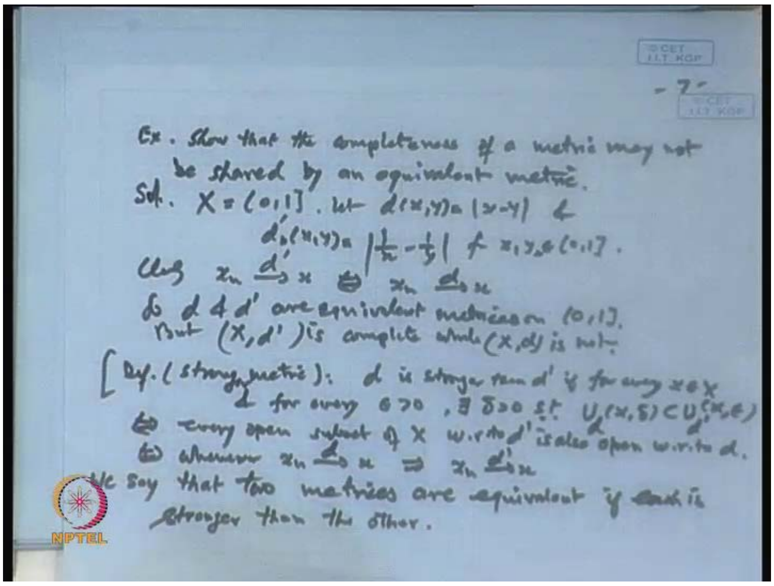


Now, let us take few problems now, where we can give, what is the equivalent metrics, how to define the metrics? Suppose, there are 2 sets are, x is a one metric space over which the 2 metrics are defined, d 1 and d 2. Then, we have to compare these metrics suppose, then how will you say that one with one metric x d 1 and with another metric x d 2, what is the difference? Whether any sequence, if we take any sequence converging in with respect to one metric, whether it converges to the other metric or not and vice versa or one way it is true or both way is true?

So, this brings the concept of the stronger and weaker metric spaces because once a set is fixed, but the 2 different metrics are given, then we can compare their properties by saying, that whether it is stronger metric or weaker metric. And we define, the stronger metric is a metric, d is stronger than d dash if for any x and for every epsilon greater than 0, there exist a delta such that the open ball centered at x and radius delta is under the metric d, is contained in this. What is the meaning of this, let us see.

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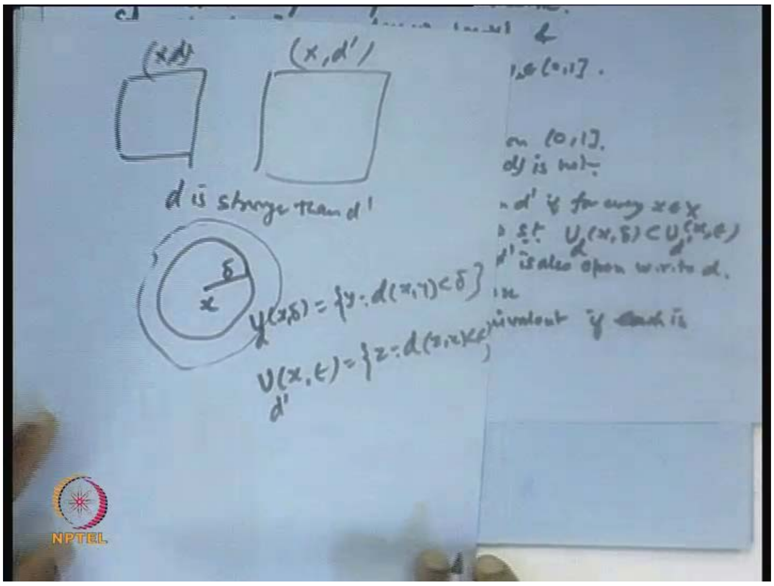


Suppose this is a metric x with a metric d, this is a metric x with metric d dash, metric space. We say d is stronger than d dash, d dash is stronger d, it means, if I take a point x belonging to x and draw the ball centered at x and radius delta, open ball centered at x and radius delta under the metric d. So, basically, this is the element y whose distance from x, y is less than delta. This is the open ball and then, draws the ball centered at x and with the radius epsilon under the metric d dash. So, this is the z, such that d of z x is less than epsilon.

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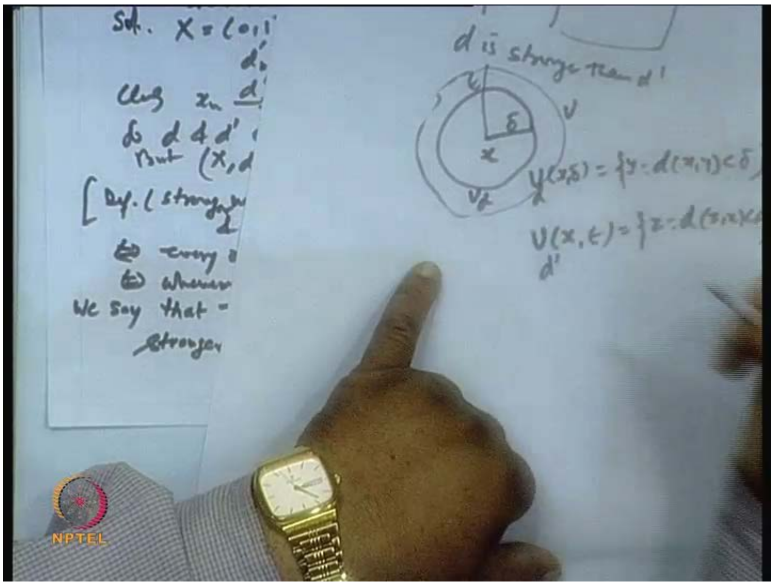

Now, these 2 balls are there centered at x and one is this, another one is, this is radius. Now, what he says is that if this ball is totally contained inside this ball, epsilon ball, this is the U, d; this is the U, d dash, then we say, this metric is stronger than this, why? Because a ball, this ball contains this much, it means, this behave as a neighborhood of this. So, if any sequence converges in this metric, then that sequence has to converge under this metric.

So, we can say, in case of the stronger convergence, if x n converges to x, then x n has to converge in the x. So, in rough sense you can say, the 2 metrics are given, then d is said to b stronger than d dash. If a sequence x n converges in d must imply x n converges to d dash, and strong convergence implies the weak; if a sequence converges in the strong metric, it has to converge in the weak metric. So, that is…

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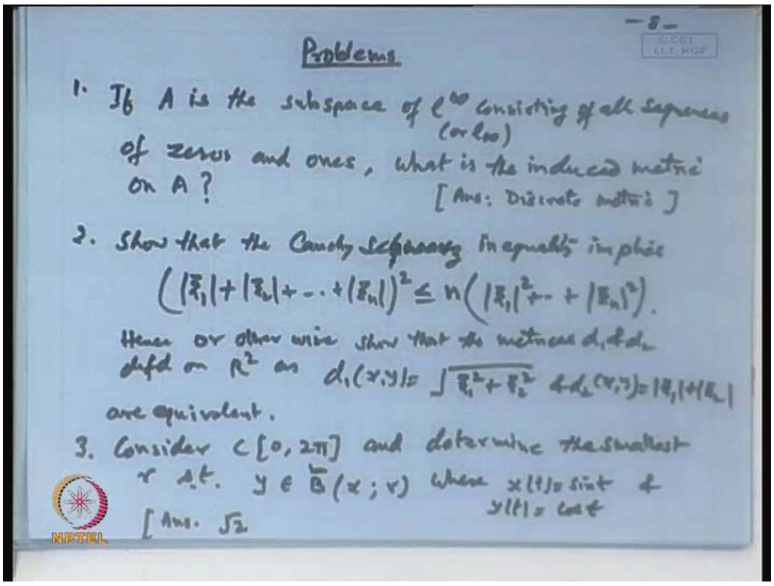


Now, let us see these problems. This, the problem 1st is, if A is a subspace of l infinity consisting of all sequences of 0s and 1, what is the induced metric? What is the induced metric?

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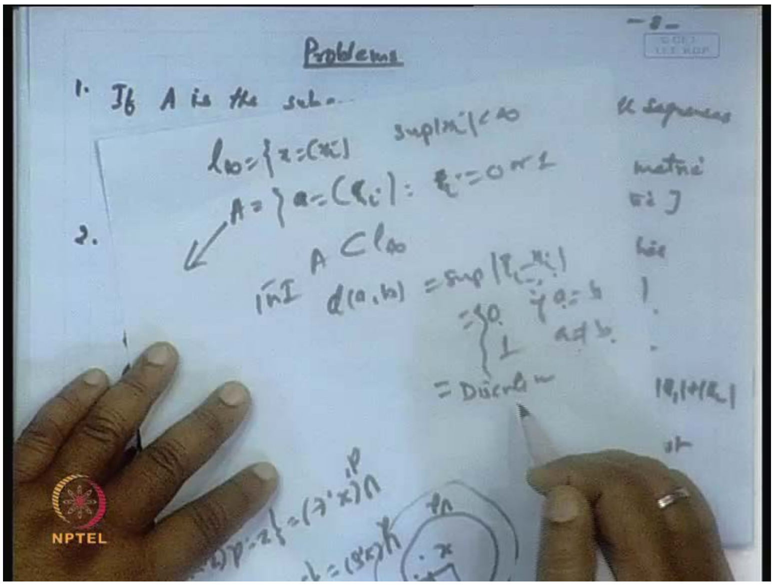


l infinity is the set of those sequences x, which are bounded, is it not, which are bounding and what is A? A is the sequence, which are either 0 or 1, where xi's are 0 or 1.

So, basically, A is a subset of l infinity. What is the distance between this, say suppose I take a, here then, what is the distance between a, b and A, that is, the supremum of xi i minus eta i and that will be 0, if a is equal to b and 1 if a is not equal to b, is it not.

So, because if, that terms of the sequence in A either 0 or 1, so supremum value will be either 0 if all terms are equal, or if any at one point it differs, the value will be 1. So, is it not a discrete metric; what is the discrete metric? If a is equal to b, the value of the discrete give 0; if a differs b, the under discrete, it is 1. So, basically, the over a, if I introduced a metric a, discrete metric, it will coincide with the induced metric of l infinity, does it not. So, the, what is the induced metric, is that discrete metric, clear.

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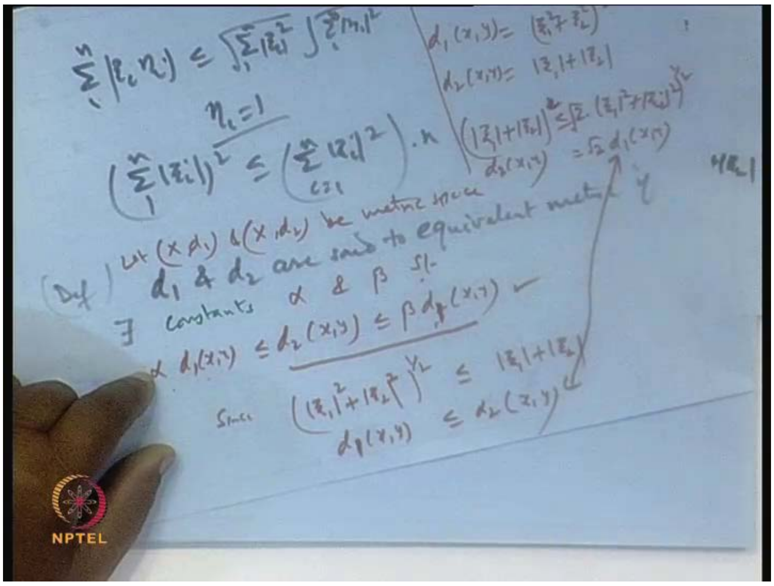


Then, 2nd example is, show, that the Cauchy Schwartz inequality implies this. What is the Cauchy Schwartz inequality? Will you remember sigma xi i eta i, i is 1 to n, suppose, is less than equal to sigma xi i square under root sigma mod eta i square under root, is it not clear.

Now, if I take eta i to be 1, then what is this? Is it not the left hand side sigma mod xi i is square (( )) 1 to n take the square. So, what is the right hand side? This is sigma xi i square i is 1 to n. And what is this? Each 1 is 1, so is it not n, so you are getting this, clear. So, we are getting this.

Now, what he says is, hence or otherwise, so the metric space defined on this are equivalent metric. The two metrics d 1 and d 2 are said to be equivalent metrics if there exist, if there exist constants, if there exist constants alpha and beta such that d 2 x, y is greater than equal to alpha times d 1 x, y is less than equal to beta time d 2 x, y.

Let x d be a metric space, x, d 1 and x, d 2 be the metric spaces, means d 1 and d 2 are the 2 metrics defined on x. We said, d 1 and d 2 are equivalent metrics, stronger and weaker is different, is equivalent metric, means, this condition has to be satisfied, that is, there exist constant such that d 2 x, y lies between alpha times d 1 and beta times of d 1, sorry, this is d 1, clear.

So, what it says is that if we take x to be r 2, x is equal to r 2, two-dimensional space and define, that is, the elements is (x 1, x 2) and define a metric d 1 x, y as one metric is on here xi 1, xi 2. So, let it be xi 1, xi 2. So, one way metric is defined as xi 1 square xi square raise to the power half; another metric is defined as the mod xi 1 plus mod xi 2. What he says is these 2 metrics will be equivalent metrics. It means, if we are able to find alpha and beta, where d 1 and d 2 satisfy this condition, then our problem is solved.

Now, by the previous result Cauchy's inequality if I take n equal to 2, so what we get mod xi 1 plus mod xi 2 square is mod xi 1 plus mod xi 2 square is less than equal to 2 mod xi 1 square plus mod xi 2 square is it not. So, take the power half both side. So, what you get if we take the power half this will go and this, this power will come.

Now, is it not this one is under root 2 d 1 x y, is it not. And what is, this is equal to d 2. So, one way it is true, beta comes out to root 2, other way round is obviously, true other way since mod xi 1 plus mod xi 2 this square, this square power half, is it not, always less than equal to this, clear. Because if I square both sides, then this side will be more. So, basically this is equal to what? This is equal to d 1 x, y and this is equal to…

So, combine these 2, so what we get? We get the d 2 lies between d 1 and d 2 with a suitable constant. Therefore, this metric, these 2 metrics are equivalent metrics and in fact, we can define any other metric on r 2 or r n, all will be equivalent, that we will see, clear. So, this is another example (( )).

Then, 3rd example is, denote the smallest r, where b belongs to closure of this end here. We have to find the maximum value, what is the, determine the small r, so that y belongs to closure. What is the closure property?

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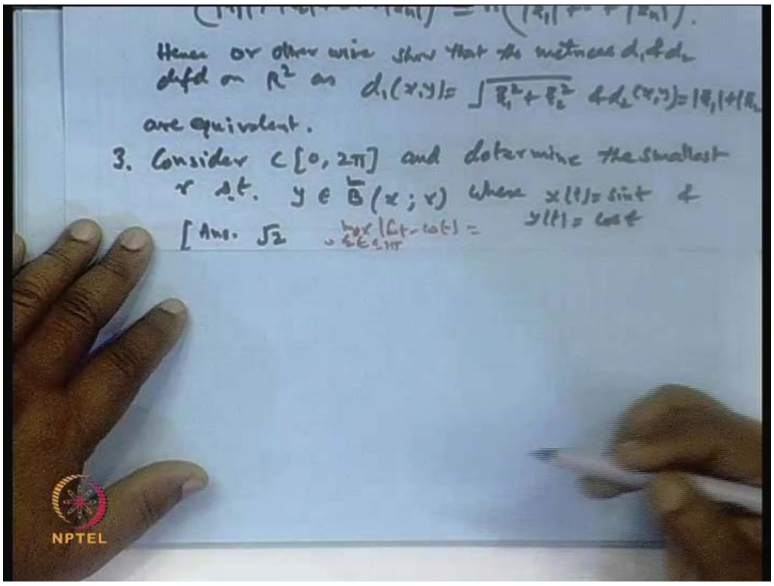

The closure ball set of this, such that distance between less than equal to r, so you find the maximum because it will give the maximum of this sine t minus cos t and t belongs to over this interval. So, we find the maximum value and maximum value will come out to be root 2. So, this thing, I think this.

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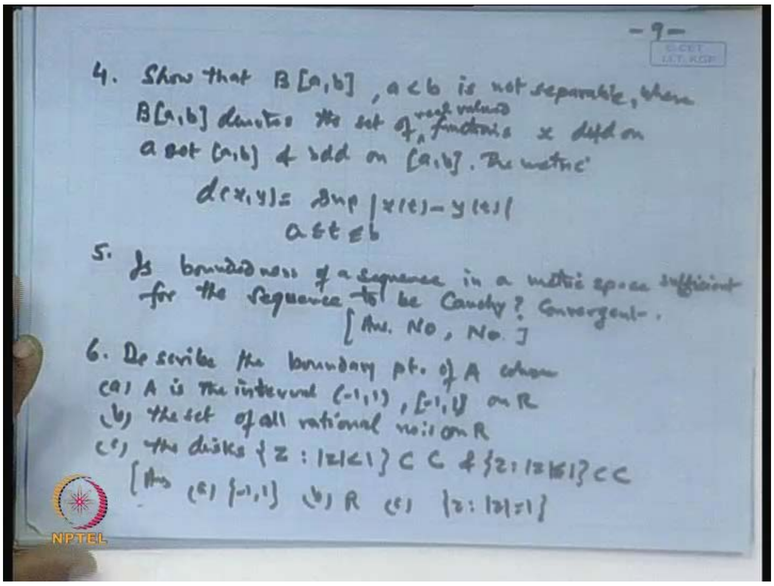

Now, there are few more problems, which I have written, you can just have a look and that do it later on this. B a b is not separable, this is one problem. Then, another problem is, if the boundedness of a sequence does not imply the cauchyness and convergence and boundary of the point like this. So, these problems you can just have a look.

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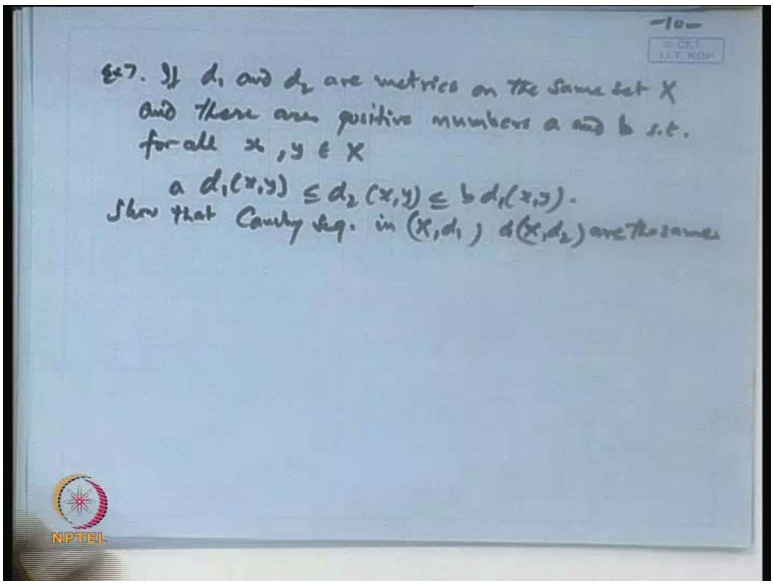


And then, last problem is this, that is, definition, which I have already discussed.

Thank you. Thanks.