Functional Analysis
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Lecture No. # 06
Examples of Complete and Incomplete Metric Spaces

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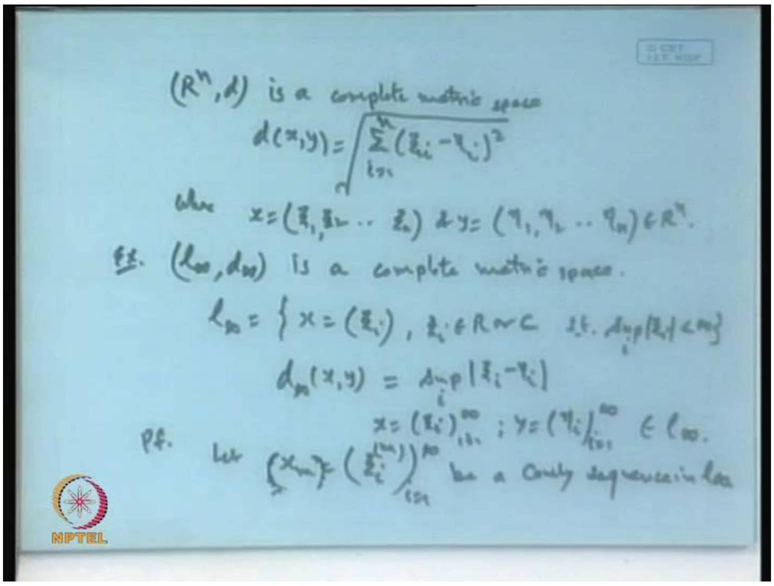

In the last lecture we have seen, that R n d is a complete metric space under the metric d defined as sigma mod xi i minus eta i whole square i is 1 to n underroot, where x, which is xi 1, xi 2, xi n and y, which is eta 1, eta 2, eta n are the points of R n.

We will go few more examples, where the metric space also comes out to be a complete one. So, another important example is l infinity space under the metric d infinity is a complete metric space, where l infinity, as we know, is the collection of those infinites you can see is xi i, such that supremum of mod xi i is finite, that is, set of all boundaries you can see is of real or complex number and the notion of the metric, which we have introduced as d infinity (x, y) is the supremum over i mod xi i minus eta i, where y is a sequence by eta 1, eta 2, eta n and, and so on, up to infinity x is xi i, i is 1 to infinity, y is eta i, i is 1 to infinity, both are the elements of l infinity, we claim, that this is a complete metric space.

So, it means, if we prove, that every cauchy sequence in l infinity is a convergent one, limit point belongs to the l infinity and the convergence is in the metric of d infinity, then it is a complete metric space. So, the proof goes like this. Let us consider an arbitrary sequence x n, which is cos c. So, let x m, which is, say xi m i is 1 to infinity be a cauchy sequence be a, is sequence, do not put this bracket, be a cauchy sequence in l infinity; be a cauchy sequence is l infinity.

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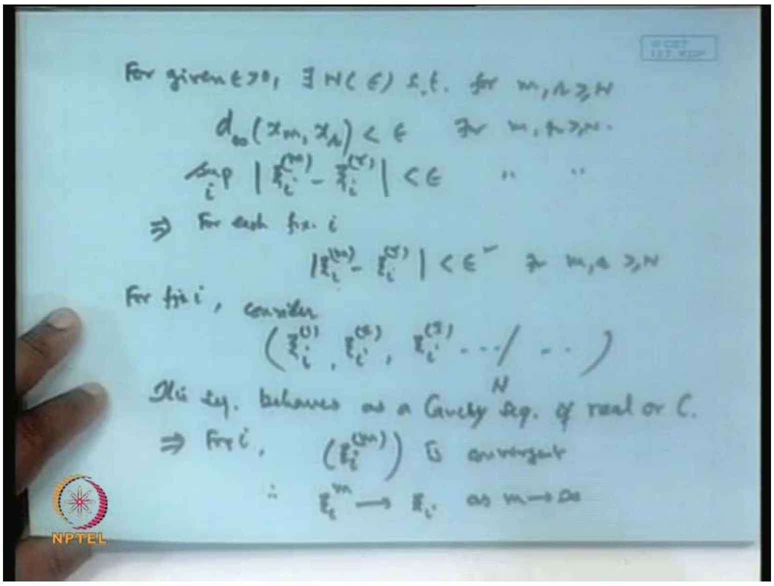

So, by definition of cauchy sequence we get, for given epsilon greater than 0, there exist in N depending on epsilon such that, for m r greater than equal to, say capital N, the d of d infinity of x m, x r is less than epsilon for m r is greater than equal to N. This we have mentioned, it means the supremum over i mod of xi i m minus xi i r is less than epsilon for all m r greater than equal to N. This implies, that for each I, for each fixed I, xi i m minus xi r is remains less than epsilon, clear. And this is true for all m r greater than equal to N.

It means, if I consider the sequence like this, for fixed i, if I consider a sequence like this xi 1, xi 2, xi 3 and so on, now this sequence has a character. The difference between any two arbitrary terms of the sequence, after a certain stage capital N onward remains less than epsilon, is it not, because here it will be the term xi to the power m, xi to the power m plus N plus 1 and so on and so forth, but for which this is true.

So, it means, this is a sequence, which behaves as a cauchy sequence of real or complex number for each i. So, for fixed i, if this sequence, this sequence behaves as a cauchy sequence of real or complex numbers and any sequence, which is cauchy real sequence or complex sequence, which is cauchy is a convergent one. So, this sequence for each i, the sequence xi i m converges, is a convergent one, clear, is convergent. Therefore, we can say, xi i m tends to xi i as m tends to infinity.

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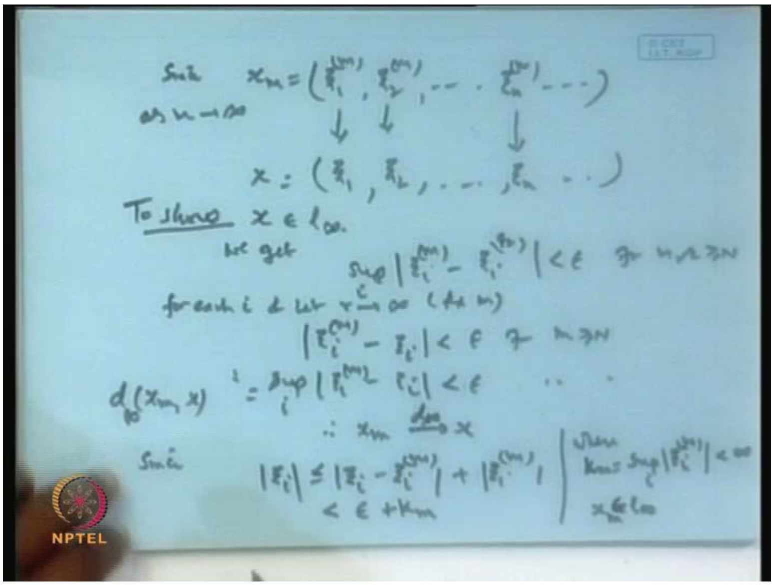

Now, let us pick up this again from this result, mod of. So, since x m we have taken as xi y n m, xi 2 m, xi n m and so on. As m tends to infinity, xi 1 m will go to xi 1, xi 2 will go to xi 2, xi n will go to xi n, so on. So, correspondingly, we get a point x, clear. Once we get the point x, then this will be the limit point of this sequence x m, which tends to x when m tends to infinity.

Now, if this point x belongs to l infinity and this convergence is in the metric of l infinity, then l infinity will be complete. So, first to show, that x m, this x belongs to l infinity. Now, we get by previous (( )), supremum over i mod xi i m minus xi i r, this is less than epsilon for m r greater than equal to N, clear. Therefore, for fixed for each i, xi i m for each i and let r tends to infinity fix m.

Then what happen is, xi i m and xi r xi i r will go to xi i, will remain less than epsilon for m greater than equal to m, for those m, which are greater than equal to m. Now, this is true for each i. So, supremum of this thing will remain less than epsilon for m greater than this is i. But this supremum, this is nothing but the d infinity x m x. Therefore, this x m converges to x in the metric of d infinity, is it ok?

Now, if x belongs to this class also, that is fine. So, to show, the x belongs to this class because if we get here, since xi i, we can write it as xi i minus xi i m plus xi i m. So, mod of this is less than equal to mod of this plus mod of this. Now, this part is 10 less than epsilon and this sequence x m is in the point of, this is less than epsilon plus K m, where K m is thus supremum of mod xi i m over i because x belongs to x m. This belongs to l infinity, therefore this must be finite.

So, once this is finite, then we can say, that each term of this will remain less than equal to K, depending on m K is sufficient. So, this is constant, this is also constant and it is independent of i; right hand side is independent of i.

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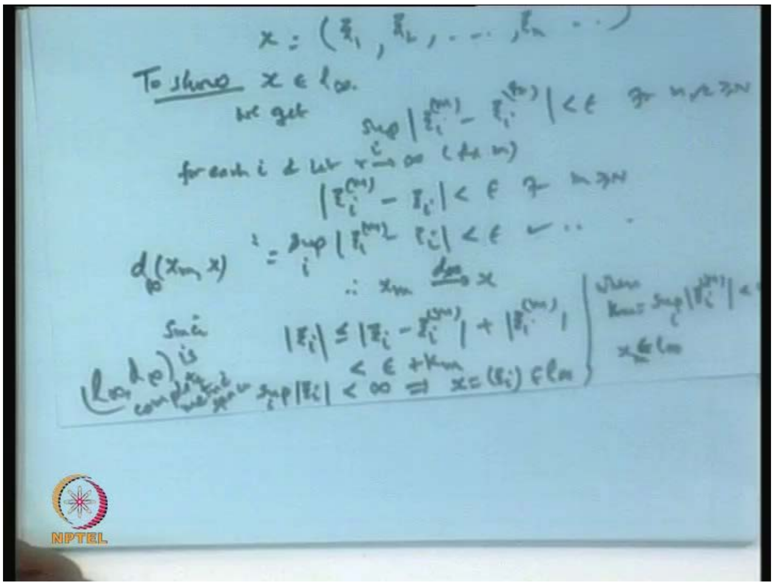

So, take the supremum and we get from here is a supremum mod xi i is finite. This implies our sequence x, which is xi i is in l infinity. So, the x m converges to x, the limit point x belongs to l infinity and the convergence is in the metric of l infinity. Therefore, l infinity d infinity is a complete metric space; is a complete metric space. So, this is very easy to show the completeness. Sometimes you have to use different lines of proof to show the completeness and that gives a simpler way, method.

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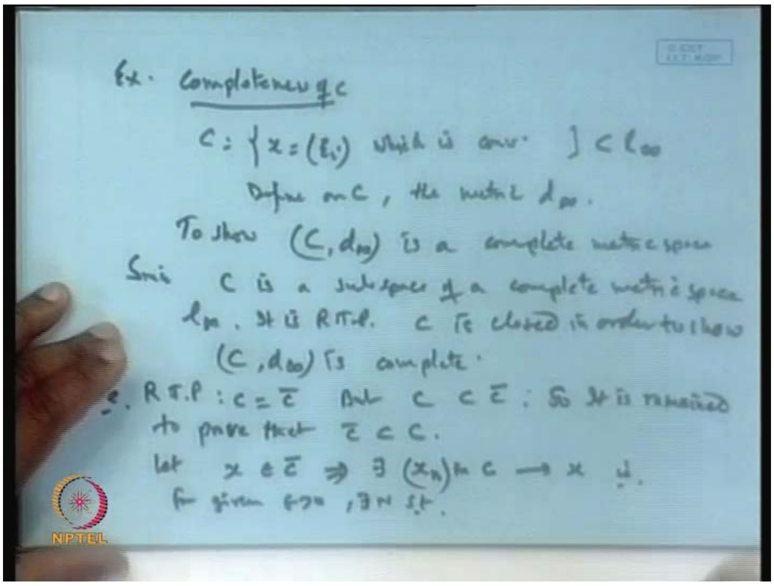

For example, if we want you to show completeness of C, what is C? C is the set of all sequences x, say xi i, which are, which is convergent; set of all sequence, which are convergent sequence. So, all convergent sequences are there, every convergent sequence is a bounded sequence. So, obviously, it is a subset of l infinity and the same metric we can introduce on C, so we can define on C the metric d infinity; same metric we can introduce it.

Now, our aim is to show, that C under this d infinity is a complete metric space; complete metric space, clear. Now, here we will use a different trick, different line of proofs to show this is complete. The advantage in this problem is, because it is given, that C is a subset of l infinity, it is known l infinity is a complete metric space. So, since C is a subset or subspace of a complete metric space, of complete metric space l infinity.

So, there is a lemma, which we have proved, that n is subspace of a complete metric space is complete if and only if it is closed. So, C is subspace of complete metric space. So, it is required to prove, it is required to prove only, that C is closed; C is closed. In order to show, in order to show C, d infinity is complete (( )), C is close means, required to prove C is equal to C bar. C is close means, it must be equal to its closure, but C is always be a subset of C bar. So, it is remained to, so remained to prove, that C bar is a subset of C.

So, entire thing is now reduced to only one part, that if I prove, that C bar is a subset of C, then C will be a complete metric space. So, in order to show C bar is a subset of C, we will pick up an arbitrary point in C bar and then, we say that point belongs to C, then our result is (( )). So, let x belongs to C bar, but C bar is a closure of this set C. It means, either x will be a point of C or x will be a limit point of C. If x is a point of C, then nothing to prove because it is only…

So, if x belongs to C bar, it means, there exist a sequence, say x n x n in C, which converges to x, is it not; there exists, it can, x n in C, which converges to x.

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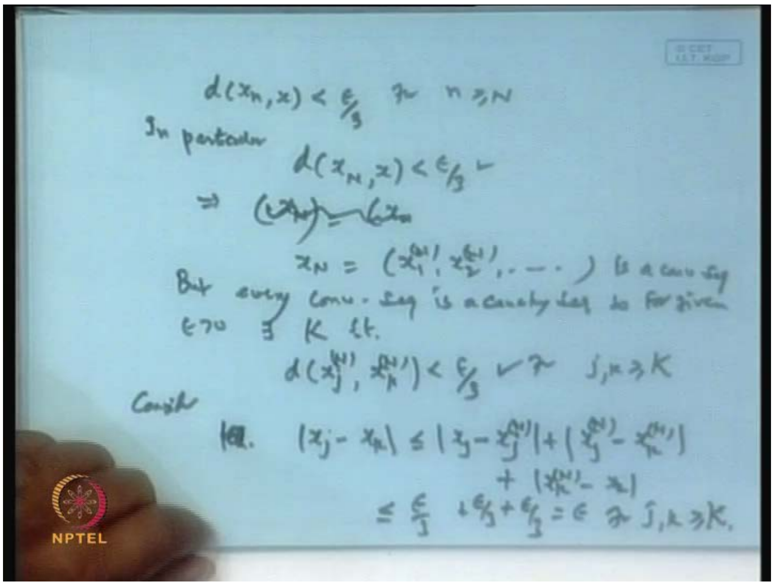

So, that is, what we get is, that for given epsilon greater than 0, there exist an n, such that d of x n, x remain less than epsilon for all n greater than equal to capital N, is it not? Let it be this in place of epsilon; let it be epsilon by 3, now this is true, for all n belongs to greater than equal to N. So, in particular, d of x N, x will remain less than epsilon by 3. It means, the sequence x N, that is equal to x n x 1, x n, which is x 1 n, x 2 n and so on, is a convergent sequence, is it not. Because it converges to x, it is a convergence sequence.

Every convergence sequence is a cauchy sequence, but every convergent sequence is a cauchy sequence, so this has also be a cauchy sequence. So, for given epsilon greater than 0, there exist some number capital K, such that d of x, this is a convergent sequence. So, we can say, d of x j N, d of x k N remain less than epsilon for all j k greater than equal to K, is it not; for all j k greater than equal to capital K. So, let it also be this epsilon by 3, is it ok.

Now, now consider the point. We wanted the limit point x x N converges to x, it means, that x, this becomes x, which is must belongs to the class C, that is, x must be a convergence sequence. So, if I prove, that x is a cauchy sequence of real complex number, then it has to be convergent. So, start with, xi, x j, I am taking x j, is it not, so it start with x j minus x k.

Now, this part can be written as x j minus x j N plus x j N, let it bracket use, let it. This is x N minus x k m and then plus x k N minus x k. Now, x j is a cauchy sequence. So, from this, the middle portion becomes epsilon by 3, the 1st portion becomes less than epsilon by 3 because of this, last portion also become epsilon by 3 because of this. So, this is less than equal to epsilon by 3 plus epsilon by 3 plus epsilon and this is true for all j k greater than equal to K.

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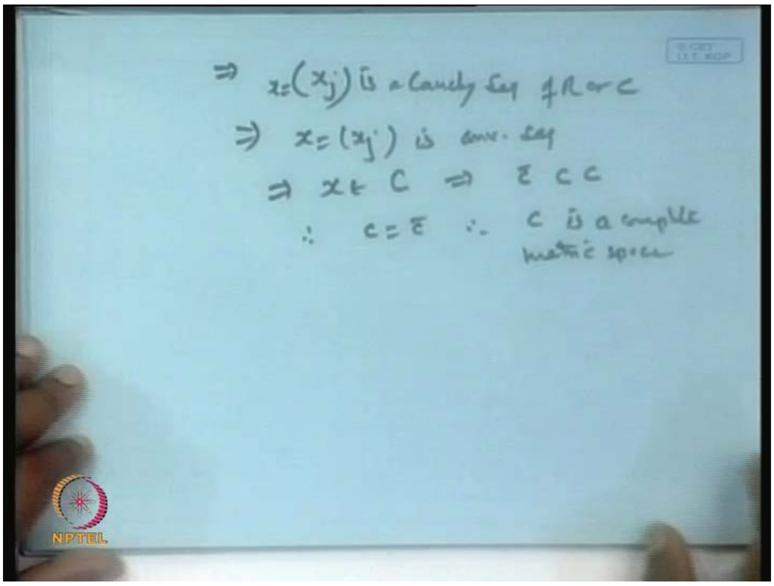

So, the sequence x j, so the sequence x j, which is x, is a cauchy sequence of real or complex numbers, but it is convergent. Therefore, x, which is x j is a convergent sequence, therefore x belongs to C.

So, x we have started in a point C bar and shown that x belongs to C. It means, C bar is a subset of C. So, this implies, C is equal to C bar, therefore C is complete metric space. So, this is the different way we have shown, that the advantage was here because the, say space, which we have chosen is basically comes out to be a subspace of a complete metric space. If l infinity could not be a complete, then we cannot use these criteria.

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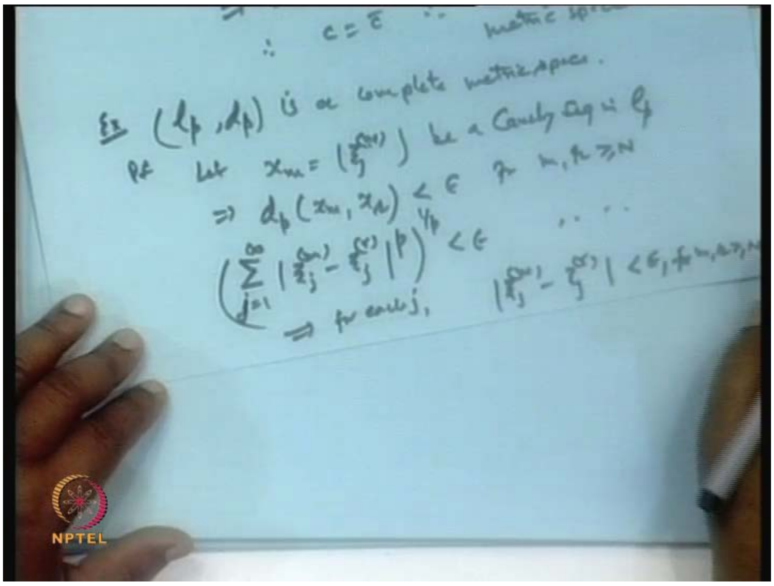

Another example, which we frequently use, the l p space, the l p space under the metric d p is a complete metric space; is a complete metric space. The line of proof is almost same, so I will just skip the few parts, only the portion from where it is required we will show.

So, what is the proof is let x m, which is xi m, be a cauchy sequence in l p. So, once you get this, it means the d p x m, x r remain less than epsilon for m, r greater than equal to N, just for a given epsilon we can find this. But how to define, that d p, d p if you remember, this is i equal to 1 to infinity mod of xi j or j, j is 1 to infinity xi j m minus xi i j r, xi j r power p power 1 by p is less than epsilon. This is the metric for all m, r.

So, from here if I remove the sum, then for each j, this total sum less than epsilon and again, this is a sequence of real or complex number, which is cauchy. Therefore, it must be, for each fix j it is a cauchy sequence. So, we get from here is, for each j xi j m minus xi i j r, this will remain less than, say epsilon 1 for all m, r greater than equal to N. So, this is a cauchy sequence.

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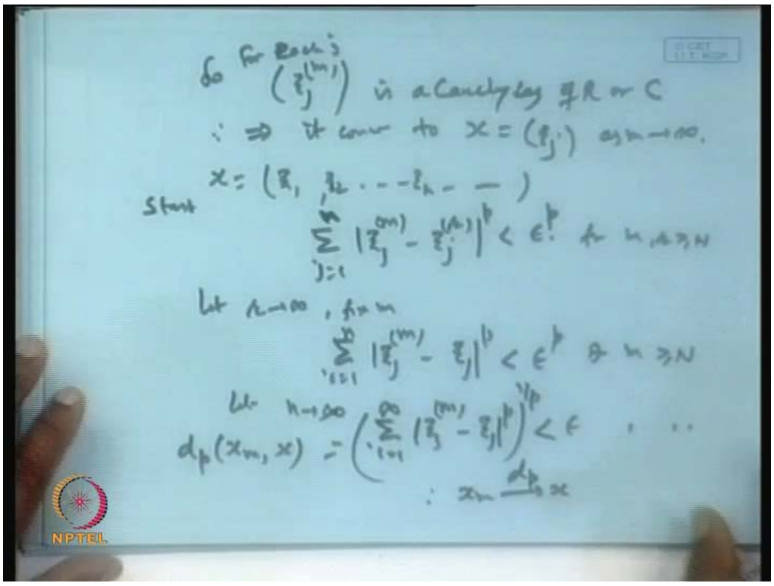

So, this sequence xi j m for fix j, for each j or for fix j, is a cauchy sequence of real or complex numbers that is important, therefore it is convergent. So, it converges to a point x, which xi j as m tends to infinity, clear. So, we are getting a point x as xi 1, xi 2, xi n and so on, this is the point. Now, whether this point belongs to l infinity or not that is important. If it belongs to l infinity and convergence is in the metric of d p, then again l, sorry, it belongs to l p and under the metric d p, this convergence is there, then we say the l p d p is a complete metric.

So, to show this is l p and the convergence in d p, we start like this, sigma, just a previous one, this j is 1 to infinity, this is less than epsilon. So, if I truncate, say it start with j equal to 1 to m or 1 to say N, then obviously, this remain less than epsilon. So, start with sigma j equal to 1 to n, not m, 1 to n mod of xi j m minus xi j r, this remain less than epsilon to the power p and here is p.

Let us fix m and let r tends to infinity fix m. So, and this is true for m and r greater than equal to N. So, I am not touching m; let r goes to infinity. So, this xi i j, all will go to x i j. So, we get i is 1 to n xi i j m minus xi i j power p is less than epsilon power p for m greater than equal to…

Now, let n tends to infinity now. So, when n tends to infinity, it is the, i is 1 to infinity mod xi j m minus xi j power p is less to the power 1 by p is less than epsilon for m greater, but this is nothing, but the d p x m x definition. So, we get x m converges to x in the metric of d p.

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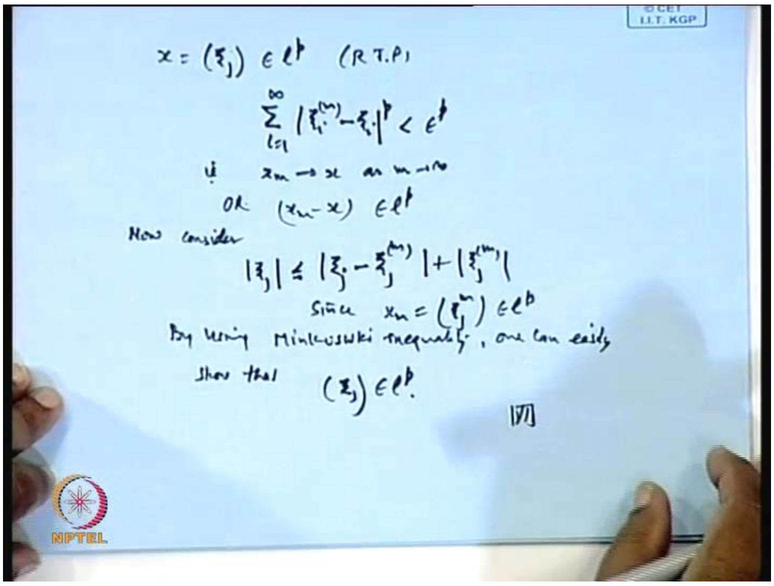

Now, to show x belongs to l p we now start. So, now, what we want is the limit point x belongs to, this belongs to l p, this we wanted to show. We have already seen, that this sigma, say i is 1 to infinity, i is 1 to infinity, mod of xi m minus xi I, say, raise to the power p is less than epsilon to power p, that is a sequence, x n m, that is a sequence. x m goes to x as m tends to infinity, this we have seen already.

So, using this, not only x m, this x m minus x, this will be a point in l p because the power sigma of this thing, power p is bounded, this is what, so this belongs to p. Now consider mod xi j, which is mod of xi j minus xi j m plus mod of xi j m. So, it becomes less than equal to xi.

Now, since x m, which is a sequence xi j m, this is in l p, this sequence is also in l p, therefore, by and l p is a linear express. So, by using the Minkowski inequality, Minkowski inequality, one can easily show that this sequence xi j is in l p because of this.

So, if you apply them, Minkowski inequality over this, take the power p summation and then, apply the Minkowski, we get, this is it and this proves the result of that l p under this metric d p is complete as same space may be complete in one metric. But if I change the metric, the same class of elements, same class may not behave as a complete metric space.

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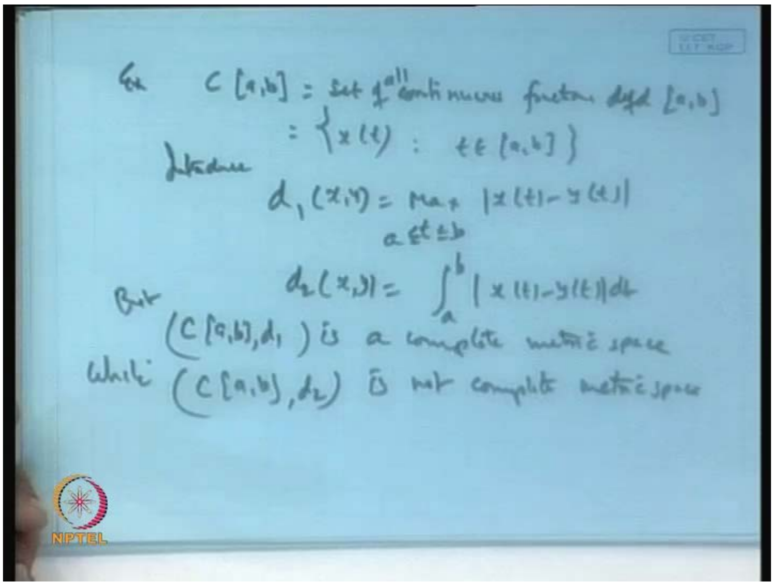

For example, if we take the C a, b, set of all continuous functions, set of all continuous functions, say (x, t), defined over the interval, this is our C a, b.

The metric, we can introduce the metric in 2 ways, one is d 1 x, y as the maximum of mod x t minus y t, t ranges from this. And another way metric is introduced like this, integral a to b mod x t minus y t d t. Both are well defined because the function is continuous on a close interval, so it will attain its maximum value. So, therefore, this is well defined.

Every integrate function, which is continuous is an integrable function on a finite interval, so it is also well defined. So, both the metrics, all well defined on C a, b and they satisfy all the property of the metric space, but this C a, b under the metric d 1 is a complete metric space, while C a, b under the metric d 2 is not complete, is not complete metric space. Class, I am not changing the class, just I am changing the metrics. With one metric it forms a complete metric spread, while with other metric it forms an incomplete metrics. So, the completeness of a space depends totally on the metric. If I change the metric, the completeness may be loosed.

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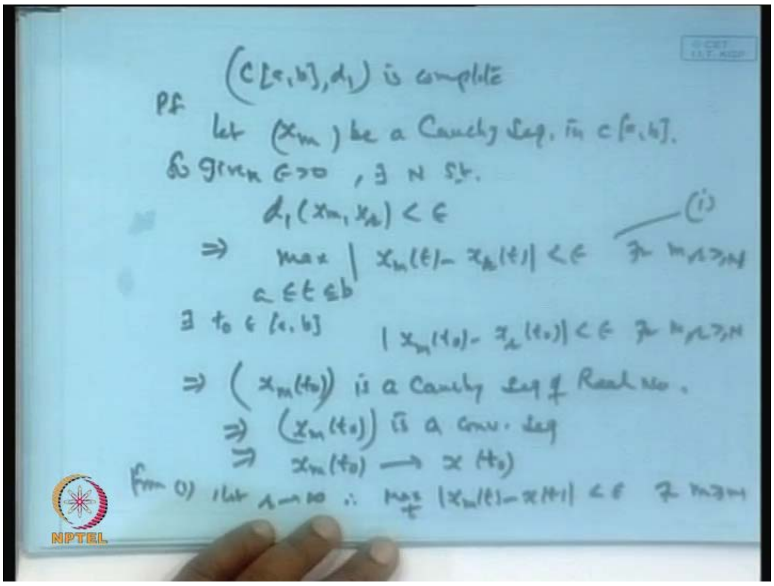

So, that let us see the proof. I think, 1st part proof does not require much, that C a, b under d 1 is complete, this we want. So, start with a cauchy sequence, let x m be a cauchy sequence in C a, b. So, by definition, so for given epsilon, there exist, say N, such that d 1 x m, x r is less than epsilon, but by definition this is the maximum of x m t minus x r t is less than epsilon, where t varies from a to b.

Now, maximum value we are taking, since function x m and x r both are continuous, difference of this will remain continuous, so maximum value we attain. So, there exist a t naught, we show there exist a t naught in the interval (a, b), where the maximum value will be attained, is it clear? And this is true for what? For all m r greater than equal to N, for all m r greater than equal to… It means, this sequence, x t naught, x m t naught this is a sequence.

Now, when you are taking x m t naught, it means, the x m is a function value, the value of the function is taking at a point t naught. So, it is a real number because C a, b is the set of all continuous real valued continuous functions, so it is a real number. Now, this real number satisfies the criteria, the difference between any 2 terms of the sequence after certain inter, after certain step remains less than epsilon. It means, this is a cauchy sequence of real numbers, agreed.

Once it is cauchy, so it must be convergent. So, x m t naught is convergent and it will converge to a point, say x t naught; let converge to x t naught. Now, whether this x, whether this is a continuous function or not? If it is continuous it will belongs to C a, b if it is not continuous cannot belongs to, but we claim, that this will be a continuous function, why? If we look the 1, the 1 is given the maximum of x m t minus x r t is less than epsilon. It means, for all t differences of these is less than epsilon, clear. So, it means, this less than equal to its true for all t.

Now, if I take m fix n all tends to infinity, then you are getting here is x m t minus x t is less than epsilon, but this is true for all t and it means x m converges to x uniformly from here from 1, 1; let r tends to infinity. So, what we get is the maximum of t is less than epsilon for m greater than equal to N.

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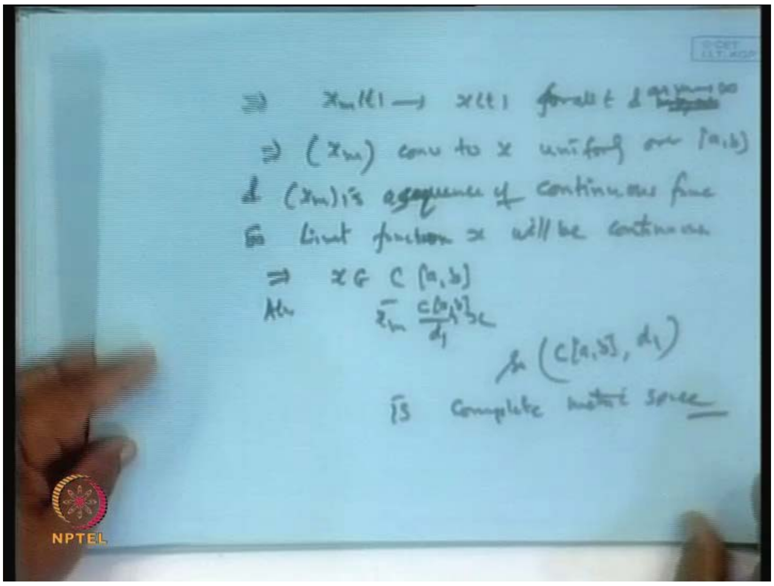

Now, does it not imply, that from here we can say, that x m t converge to x t as for all t all t and m is greater than equal to r as m tends to infinity, agreed or not? It means that it is independent of t. So, the sequence x m, t x m converges to x uniformly over the interval, it does not depend on t. Though it is from here we have seen, it is point wise convergent, but basically because of the maximum it converges for every t. So, you can say it is uniformly convergent.

Now, there is a result, if a sequence x n is a sequence of continuous function and it converge uniformly, then limit point has to be a continuous function. So, x m converges to x uniformly and x m is a continuous function, is a sequence of continuous function. So, the limit function x, x will be continuous, that is, result is standard in the real energies. So, if you want, I can give the proof where we need, otherwise it is available in the book, that if a sequence x n, which converge uniformly and the sequence are of continuous function, then the limit function has to be a continuous function.

So, this implies, x is a continuous function. So, x must be in C and it converges to and also this, and also x m converges to x in the metric of C a, b, that is d 1, clear. So, C a, b under d 1 is complete, complete metric space.

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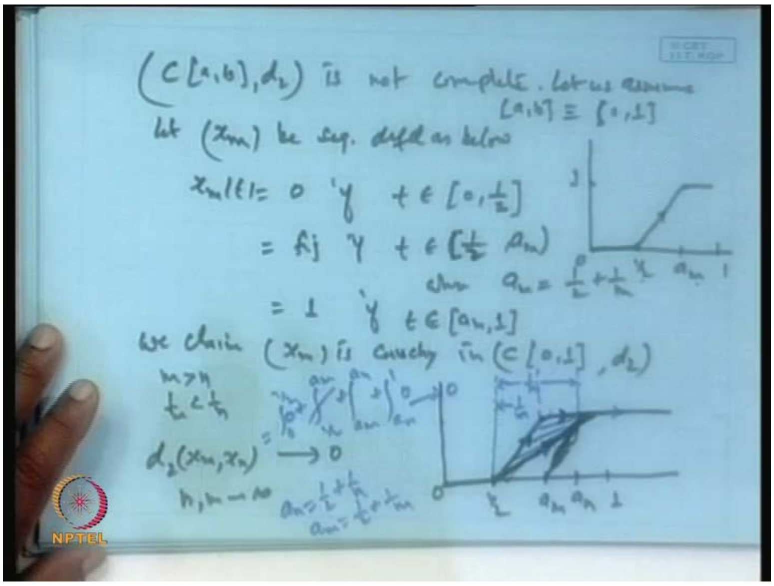

Now, see the 2nd part, that C a, b, C a, b under d 2 is not complete, is not complete. It means that every cauchy sequence in this class metric is not a convergent one. So, first we have to identify a sequence, which is cauchy, but it does not converge to a point of C a, b; that is, the limit point comes out to be a discontinuous function.

So, let us see this C a, b. So, let x n be a sequence or x m be a sequence defined as below, defined as below. We define x m t equal to 0, now, is not come. For the sake of proof, let us assume interval (a, b) is identical (0, 1). Suppose, I prove this result for (0, 1), then it can be extended to (a, b).

So, let us introduce this as 0. If t belongs to the interval 0 to half and equal to, as in the figure, if it is t belongs to half, close half and a m point a m, where a m is equal to half plus 1 by n. If this is in figure, figure is like this, that suppose this is 0, this is half, this is a m, this one is 1, here it is 1. So, function is going like this, it is correct. So, between half to a m, the curve is this and then from a m, a m, to 1, again the curve is the value of x m t is 1. I am introduced like, I have introduced the x m x.

Now, this x m is a continuous function, there no point of discontinuity at all, clear. We claim this x m is a cauchy sequence, is cauchy in the interval C 0 1 under d 2. D 2 is a metric. Why it is so? Because if you look, this again 0, here it is half, this is a m, here it is say 1.

Suppose m is greater than n, so 1 by m will be less than 1 by n. It means, a m will be less than a n, here it is 1. So, this will be this, this and then this; for a n it will be like this and so on, is it not. This is our a m, a n, is it ok, clear. So, basically, no this 1 by, if this is, this will not be there because half to this, the figure is this, a n will be from start from here, a m will start from here and we will go like this a and this, is it not. This is our a m and this will be our a n, clear. So, so this portion is there.

Now, when you take the d of d 2 of x m, x n, what happens? 0 to half, both are equal. So, integral 0 to half will be 0, then from a n onward, again if both are same, only the problem will come between half n and a m, is it not.

Now, h n, n m, both goes to infinity. When both goes to infinity, a m coincide with the half because a m is half, a n will also coincide with half. So, basically, both these figures will coincide to half and there will be directly jump here and we get this, point clear.

So, basically, this tends to 0 as it will tends to, agreed. So, it is a cauchy sequence, is this clear or not. What is this? Let me see. Take this is our, then this will be and this will be this one. Now, is it not this one is 1 by m, this is 1 by m, 1 by m, this one will be 1 by n, is it ok or not, 1 by n clear.

Now, when you, d of this, what is this relation? 0 to half plus half to, half to, a m, a m, a m to a n, a n to 1; a n to 1, it is 0, difference will be 0. Here it is 0, now these two things are there.

Now, when n and m goes to infinity, a n is half plus 1 by n, a m is half plus 1 by m. So, when goes n to infinity, m goes to infinity, a n will coincide with half. So, this will part will be 0. When a n, n m goes to infinity, both is half, half. So, again this is 0. So, total will go to 0 clear.

So, that is why (( )) tends to, so it is a cauchy sequence, clear, clear.

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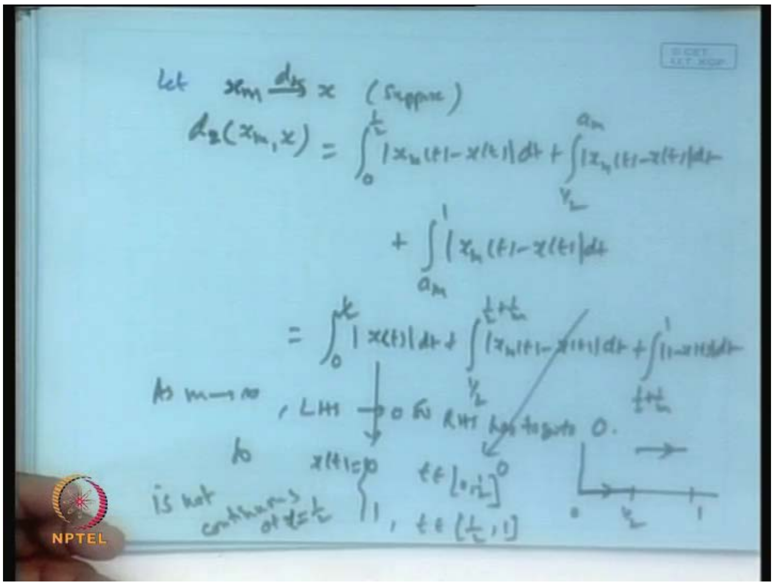

But it is not converging to its continuous function, why? Because if we look this, suppose, let x m, x m converges to a point x under the metric d 2 suppose this be our assumption, then what happen is d 2 x m, x, this is equal to 0 to half mod of x m t minus x t d t plus half to a m mod of x m t minus x t d t plus a m to 1 mod of x m t minus x t d t, is it correct.

Now, this is 0 to 1 mod, x m t, x m t d x t 0 to half, what is x m t? x m t between 0 to half is 0, so this is basically 0, and only thing left out is x t d t plus half to a m means half plus 1 by m. Let it be like this, x m t minus x t d t, I am not changing, and here is a m, a m will be half plus 1 by m n 1. The value of x m t is 1 minus x t d t, agreed.

Now, let m tends to infinity, the left hand side tends to 0 because we are assuming x n, x n or x m converges to x. So, left hand side will go to 0, so right hand side has to go to 0. So, right hand side has to go to 0, is it not? Now, in order to get 0, the middle part is already 0.

When m tends to infinity, this middle portion is already 0 because both upper and lower limit coincide. So, whatever the integrant may be, it comes out to be 0. Now, 0 and half, both the positive limits and integrant is also positive. So, in order to get this to be 0, what is required is, here the x t must be 0 when t lying between t belongs to 0 and half, is it not? And here again, this is nonnegative, limits are nonnegative, so x t must be equal to 1 if t belongs to half and 1.

So, what you get? The x t comes out to 0 between 0 and half and 1 when half n a. So, it is like this, that this is half, this is 1, the function is coming like this and all of sudden is at a jump. So, there is a point of discontinuity at half therefore, x t is discontinuous is not continuous at x equal to at t equal to half.

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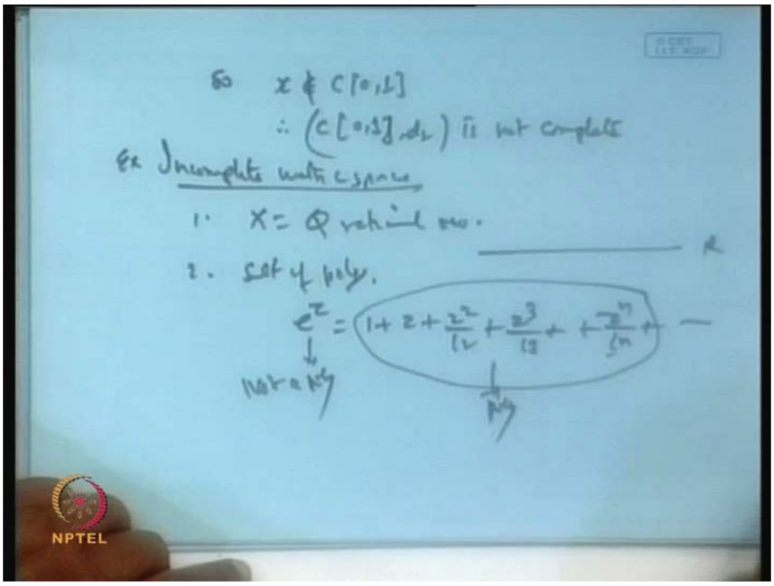

So, if it, so, so x cannot belong to C 0 1, therefore C 0 1, so x, so x does not belongs to C 0 1 therefore, C 0 1 under metric d 2 is not complete. So, this completeness is also a property will depends on the metric.

Now, there are another metrics, another, there are many spaces, which are incomplete metric space. So, there are few more example of the incomplete metric space, let me see incomplete metric spaces. The 1st one is rational number, this is incomplete because you can find the real number R, any, on a real line. If I take the limit of the sequence of rational comes out to be real, which is not a valid one. So, it is incomplete.

2nd example, set of all polynomials, polynomials of degree, say n, this is also incomplete, why? Because the limit point of this set of polynomial may be an entire function, like for example, e to the power z, we know the series is 1 plus z plus z square by factorial 2 z cube by factorial 3 z n y factorial and so on.

So, if we take this as a polynomial, what is the limit point? Limit point comes out e to the power z, which is not a polynomial, which is not a, it is not a polynomial. The set of all polynomials, which are continuous functions, does not form a complete metric space. So, there are any many or more examples, you can find out which are incomplete.

So, next class we will see, since the completeness also plays the role and in development of this functional analytics. So, can it be possible to find out a structural, which gives you a complete metric space out of incomplete one. So, the, we will discuss those things, how to find out a completion of an incomplete metric space. Thank you.